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This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

ON THE DEPENDENCE ON PARAMETERS FOR SECOND ORDER DISCRETE BOUNDARY VALUE PROBLEMS WITH THE $p(k)$-LAPLACIAN

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Abstract. In this paper we study the existence and the nonexistence of solutions for the boundary value problems of a class of nonlinear second-order discrete equations depending on a parameter. Variational (the mountain pass technique) and non-variational methods are applied.

Keywords: discrete boundary value problems, variational methods, mountain pass theorem.

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1. INTRODUCTION

In this note we will consider the following discrete boundary value problem which is subject to a certain parameter, namely

$$\begin{cases} \Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)+f(k,u(k),w)=0, & k\in\mathbb{N}(1,T),\\ u(0)=u(T+1)=0, \end{cases} \tag{1.1}$$

where $T\geq 2$ is an integer, Δ is the forward difference operator defined by $\Delta u(k)=u(k+1)-u(k)$, $u\,(k)\in\mathbb{R}$ for all $k\in\mathbb{N}(1,T)$, for fixed a,b such that $a<b<\infty$, $a\in\mathbb{N}\cup\{0\}$, $b\in\mathbb{N}$ we denote $\mathbb{N}(a,b)=\{a,a+1,\dots,b-1,b\}$, $f:\mathbb{N}(1,T)\times\mathbb{R}\times\mathbb{W}\to\mathbb{R}$, $w\in\mathbb{W}$, the space $\mathbb{W}$ is some topological space (in most applications one takes $\mathbb{R}$ in place of $\mathbb{W}$), $p:\mathbb{N}(0,T)\to[2,+\infty)$.

Put

$$p^-=\min_{k\in\mathbb{N}(0,T)}p\,(k)\,,\quad p^+=\max_{k\in\mathbb{N}(0,T)}p\,(k)\,.$$

The boundary value problem for a discrete anisotropic equation has been a very active area of research recently, we refer to the references by [2–4, 6, 10, 16, 17]. The authors have studied the boundary value problems with the Dirichlet, the Neumann or periodic conditions using critical point theory.

Our goal is to find conditions under which problem (1.1) has or has not solutions with respect to any parameter from some set (see for example [7, 11]). The approch relies on the application of the direct method of the calculus of variations and the mountain pass technique, which is a basic tool in critical point theory ([15, 18, 20]). Apart from that we investigate the continuous dependence on parameters, where we do not need to have the mountain pass geometry. The continuous dependence on parameters has been discussed for instance by [5, 7, 13, 21]. In this paper we consider the same boundary conditions as in [7], but the main operator involves a variable exponent, the $p(k)$-Laplacian being a generalization of the $p(x)$-Laplacian ([9]). Based on the results in the area of differential equations ([12, 19]) we examine some problem within a non-variational framework. The uniqueness of solutions is undertaken too.

The paper is organized as follows. Firstly, we provide some auxiliary materials. Then we give a variational formulation of the considered problem. The existence of nontrivial solutions and the continuous dependence on parameters are investigated in the next section. The example is also provided. Afterwards we focus on finding conditions under which the examined equation has an unique solution. Further, we examine some non-variational problem. The nonexistence of solutions is the subject of the last section.

Let $F(k,x,y)=\int\limits_0^x f(k,s,y)ds$ for all $(k,x,y)\in\mathbb{N}(1,T)\times\mathbb{R}\times\mathbb{W}$.

We assume that the nonlinear term satisfies:

(H0) $f\in C(\mathbb{N}(1,T)\times\mathbb{R}\times\mathbb{W};\mathbb{R})$;

(H1) there exist constants $c>0$ and $r>p^+$ such that

$$|f(k,x,y)|\le c(1+|x|^{r-1}) \text{ for all } k\in\mathbb{N}(1,T),(x,y)\in\mathbb{R}\times\mathbb{W};$$

(H2) $\lim_{x\to 0}\frac{f(k,x,y)}{|x|^{p^+-1}}=0$ uniformly for all $k\in\mathbb{N}(1,T)$, $y\in\mathbb{W}$;

(H3) there exists a constant $\mu>p^+$ such that

$$0<\mu F(k,x,y)\le xf(k,x,y) \text{ for all } k\in\mathbb{N}(1,T),x\in\mathbb{R}\setminus\{0\},y\in\mathbb{W};$$

(H4) there exist constants $c_1,c_2>0$ such that

$$F(k,x,y)\ge c_1|x|^{\mu}-c_2 \text{ for all } k\in\mathbb{N}(1,T),x\in\mathbb{R},\ y\in\mathbb{W}.$$

The condition $f\in C(\mathbb{N}(1,T)\times\mathbb{R}\times\mathbb{W};\mathbb{R})$ means that for each $k\in\mathbb{N}(1,T)$ the real valued function $f(k,\cdot,\cdot)$ is jointly continuous on $\mathbb{R}\times\mathbb{W}$.
By a solution to problem (1.1) we mean such a function $u:\mathbb{N}(0,T+1)\to\mathbb{R}$, which satisfies the given equation on $\mathbb{N}(1,T)$ and the boundary conditions. Solutions

are investigated in the space E of functions $u : \mathbb{N}(0, T+1) \to \mathbb{R}$ such that $u(0) = u(T+1) = 0$. We will consider the space E with the norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{\frac{1}{2}}.$$

In the space E we can also (like in [3]) introduce the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ v > 0 : \sum_{k=1}^{T+1} \left| \frac{\Delta u(k-1)}{v} \right|^{p(k-1)} \leq 1 \right\}.$$

Since E has a finite dimension these norms are equivalent, therefore there exist constants $L_1 > 0$, $L_2 > 1$ such that

$$L_1 \|u\|_{p(\cdot)} \leq \|u\| \leq L_2 \|u\|_{p(\cdot)}. \tag{1.2}$$

Now, let $\varphi : E \to \mathbb{R}$ be given by the formula

$$\varphi(u) = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)},$$

then inequalities

$$\|u\|_{p(\cdot)}^{p^+} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^-}, \text{ if } \|u\|_{p(\cdot)} < 1, \tag{1.3}$$

$$\|u\|_{p(\cdot)}^{p^-} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^+}, \text{ if } \|u\|_{p(\cdot)} > 1 \tag{1.4}$$

hold.

2. PRELIMINARY RESULTS

First, we recall some essential tools from critical point theory. Let E be a real reflexive Banach space and $J \in C^1(E, \mathbb{R})$. We say that J satisfies the Palais-Smale condition – the (PS) condition for short - if for any sequence $\{u_n\} \subset E$, such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence. This condition is needed for the mountain pass lemma.

Lemma 2.1 (Mountain Pass Lemma [20]). *Let E be a real reflexive Banach space. Assume that $J \in C^1(E, \mathbb{R})$ and J satisfies the* (PS) *condition. Suppose also that:*

1) $J(0) = 0$,
2) *there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*
3) *there exists u_1 in E with $\|u_1\| > \rho$ such that $J(u_1) < \alpha$.*

Then J has a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$\inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where $\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = u_1\}$.

The functional J is called anti-coercive if $\lim\limits_{\|x\|\to+\infty} J(x) = -\infty$. Directly from the definition of the (PS) condition and the notion of anti-coercivity we get the following lemma.

Lemma 2.2. *Let E be a finite dimensional Banach space and let $J \in C^1(E,\mathbb{R})$ be an anti-coercive functional. Then J satisfies the* (PS) *condition.*

Proof. Suppose to the contrary, i.e. suppose that in a finite dimensional Banach space the Gâteaux differentiable anti-coercive functional does not satisfy the (PS) condition. There exists an unbounded sequence $\{u_n\}$ such that $J^{'}(u_n) \to 0$ as $n \to \infty$ and sequence $\{J(u_n)\}$ is bounded. There exists a subsequence $\{u_{n_k}\}$ such that $\|u_{n_k}\| \to +\infty$ as $k \to \infty$ (since $\{u_n\}$ is unbounded) and by anticoercivity we get $J(u_{n_k}) \to -\infty$. The contradiction completes the proof. $\square$

Let us also recall the inequalities which we use throughout the paper (see [8]):

(A1) for every $u \in E$ and for every $m \geq 2$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq 2^m \sum_{k=1}^{T} |u(k)|^m ;$$

(A2) for every $u \in E$ and for every $m > 1$ we have

$$\sum_{k=1}^{T} |u(k)|^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k-1)|^m ;$$

(A3) for every $u \in E$ and for every $m \geq 1$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq (T+1) \|u\|^m ;$$

(A4) for every $u \in E$ and for every $m \geq 2$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \geq (T+1)^{\frac{2-m}{2}} \|u\|^m ;$$

(A5) for every $u \in E$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)\|u\|^{p^+} + (T+1).$$

3. VARIATIONAL SETTING

For a fixed parameter $w \in \mathbb{W}$ solutions to problem (1.1) correspond to the critical points of the following functional $J_w : E \to \mathbb{R}$:

$$J_w(u) = \sum_{k=1}^{T+1} \tfrac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T} F(k, u(k), w). \tag{3.1}$$

Lemma 3.1. *Assume that* (H0) *holds and let a parameter $w \in \mathbb{W}$ be fixed. Then $u \in E$ is a critical point of J_w if and only if u solves the problem* (1.1).

Proof. Let us fix u, $h \in E$. We consider a function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned}\psi(\varepsilon) &= J_w(u + \varepsilon h) \\ &= \sum_{k=1}^{T+1} \tfrac{1}{p(k-1)} |\Delta \left(u(k-1) + \varepsilon h(k-1)\right)|^{p(k-1)} - \sum_{k=1}^{T} F(k, u(k) + \varepsilon h(k), w).\end{aligned}$$

Recalling that $h(0) = h(T+1) = 0$ we obtain, what follows by summation by parts (see [1]),

$$\begin{aligned}\psi'(0) &= \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta h(k-1) - \sum_{k=1}^{T} f(k, u(k), w) h(k) \\ &= |\Delta u(T)|^{p(T)-2} \Delta u(T) \Delta h(T) + |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) h(k-1)|_1^{T+1} \\ &\quad - \sum_{k=1}^{T} \Delta \left(|\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1)\right) h(k) - \sum_{k=1}^{T} f(k, u(k), w) h(k) \\ &= - \sum_{k=1}^{T} \left(-\Delta \left(|\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1)\right) - f(k, u(k), w)\right) h(k).\end{aligned}$$

Since h was arbitrarily fixed, we arrive at the assertion. □

By the above lemma, it easy to obtain the following result.

Lemma 3.2. *Assume that* (H0) *holds. If u is a solution to problem* (1.1), *then*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta h(k-1) = \sum_{k=1}^{T} f\left(k, u(k), w\right) h(k) \quad \textit{for any } h \in E \tag{3.2}$$

and

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} = \sum_{k=1}^{T} f\left(k, u(k), w\right) u(k). \tag{3.3}$$

4. EXISTENCE OF NONTRIVIAL SOLUTIONS

In this section we consider the existence of solutions to equation (1.1). Note that the solution which we obtain need not be unique. We will study the question of uniqueness of solutions in the next section. However, solutions which we obtain are necessarily nontrivial. We also investigate the continuous dependence on parameters.

Theorem 4.1. *Assume that conditions* (H0), (H2), (H4) *hold. Then for any* $w \in \mathbb{W}$ *problem* (1.1) *has at least one nonzero solution.*

Proof. Let us fix $w \in \mathbb{W}$. We shall show that J_w defined by (3.1) satisfies the assumptions of Lemma 2.1. By (H4), (A5), (A1) and (A4), we obtain, for any $u \in E$,

$$\begin{aligned} J_w(u) &= \sum_{k=1}^{T+1} \tfrac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T} F(k, u(k), w) \\ &\le \tfrac{1}{p^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T} (c_1 |u(k)|^{\mu} - c_2) \\ &\le \tfrac{1}{p^-}(T+1)\|u\|^{p^+} + \tfrac{1}{p^-}(T+1) - c_1 \sum_{k=1}^{T} |u(k)|^{\mu} + c_2 T \\ &\le \tfrac{1}{p^-}(T+1)\|u\|^{p^+} - c_1 2^{-\mu} (T+1)^{\frac{2-\mu}{\mu}} \|u\|^{\mu} + \tfrac{1}{p^-}(T+1) + c_2 T \end{aligned} \tag{4.1}$$

and, as a consequence, $J_w(u) \to -\infty$ as $\|u\| \to +\infty$, since $\mu > p^+$. By Lemma 2.2, it follows that J_w satisfies the (PS) condition.

By (H2), for any given $0 < \varepsilon < \frac{(T+1)^{\frac{2-p^+}{2}}}{T(T+1)^{p^+}}$, there exists $\delta > 0$ such that for all $|x| \le \delta$ we have

$$|f(k,x,y)| \le \varepsilon |x|^{p^+-1} \quad \text{for all} \quad k \in \mathbb{N}(1,T), \ y \in \mathbb{W}.$$

For $0 < x \le \delta$ we observe that

$$\begin{aligned} |F(k,x,y)| &= \left| \int_0^x f(k,s,y) ds \right| \le \int_0^x |f(k,s,y)| \, ds \\ &\le \int_0^x \varepsilon \, |s|^{p^+-1} \, ds = \varepsilon \int_0^x s^{p^+-1} ds = \varepsilon \frac{s^{p^+}}{p^+} \Big|_0^x = \varepsilon \frac{x^{p^+}}{p^+} = \varepsilon \frac{|x|^{p^+}}{p^+} \end{aligned}$$

and for $-\delta \le x < 0$ we obtain

$$\begin{aligned} |F(k,x,y)| &= \left| \int_0^x f(k,s,y) ds \right| = \left| \int_x^0 -f(k,s,y) ds \right| \\ &\le \int_x^0 \varepsilon \, |s|^{p^+-1} \, ds = \varepsilon \int_x^0 (-s)^{p^+-1} ds = -\varepsilon \frac{(-s)^{p^+}}{p^+} \Big|_x^0 = \varepsilon \frac{|x|^{p^+}}{p^+}. \end{aligned}$$

Finally, for any $0<\varepsilon<\frac{(T+1)^{\frac{2-p^+}{2}}}{T(T+1)^{p^+}}$ there exists $\delta>0$ such that for all $|x|\le\delta$ we have

$$|F(k,x,y)|\le\varepsilon\frac{|x|^{p^+}}{p^+}\quad\text{for all } k\in\mathbb{N}(1,T),\ y\in\mathbb{W}. \tag{4.2}$$

Take $u\in E$ such that $|u(k)|\le\delta$, for any $k\in\mathbb{N}(1,T)$. Then, for any $k\in\mathbb{N}(1,T+1)$, we get $|\Delta u(k-1)|\le 2\delta$. Hence

$$\|u\|\le 2\left(\sum_{k=1}^{T+1}\delta^2\right)^{\frac{1}{2}}=2\delta\,(T+1)^{\frac{1}{2}}.$$

Let $u\in E$ with $\|u\|\le 1$. Then $|\Delta u(k-1)|\le 1$ for any $k\in\mathbb{N}(1,T+1)$, so by (A4) we obtain

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}\ge\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p^+}\ge(T+1)^{\frac{2-p^+}{2}}\|u\|^{p^+}. \tag{4.3}$$

Put $\eta=\min\left(2\delta\,(T+1)^{\frac{1}{2}},1\right)$. For $u\in E$ with $\|u\|\le\eta$, by (4.2), (4.3), (A2) and (A3) it follows that

$$\begin{aligned}J_w(u)&=\sum_{k=1}^{T+1}\tfrac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}-\sum_{k=1}^{T}F(k,u(k),w)\\&\ge\frac{1}{p^+}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}-\varepsilon\frac{1}{p^+}\sum_{k=1}^{T}|u(k)|^{p^+}\\&\ge\frac{1}{p^+}(T+1)^{\frac{2-p^+}{2}}\|u\|^{p^+}-\varepsilon\frac{1}{p^+}T\,(T+1)^{p^+}\,\|u\|^{p^+}\\&=\|u\|^{p^+}\frac{1}{p^+}\left((T+1)^{\frac{2-p^+}{2}}-\varepsilon T\,(T+1)^{p^+}\right).\end{aligned}$$

So, there exists positive numbers $0<\rho<\eta$ and

$$\alpha=\frac{\rho^{p^+}}{p^+}\left((T+1)^{\frac{2-p^+}{2}}-\varepsilon T\,(T+1)^{p^+}\right)$$

such that $J_w(u)\ge\alpha$ for all $u\in E$ with $\|u\|=\rho$. It is also obvious that $J_w(0)=0$. Since J_w is anti-coercive, there exists u_1 which satisfied condition 3 from the Mountain Pass Lemma. Therefore, the functional J_w has the mountain pass geometry.

By the Mountain Pass Lemma (see Lemma 2.1), functional J_w has a critical value $c^*>0$, i.e. there exists $u^*\in E$ such that $J_w(u^*)=c^*$ and $J_w'(u^*)=0$. It is obvious that $u^*\ne 0$, because $J_w(0)=0$. The critical value c^* can be characterized as

$$J_w(u^*)=\inf_{g\in\Gamma}\max_{t\in[0,1]}J_w(g(t)), \tag{4.4}$$

where $\Gamma=\{g\in C([0,1],E):g(0)=0,\ g(1)=u_1\}$. We have shown the existence of a solution to problem (1.1) for any parameter $w\in\mathbb{W}$. $\square$

Next, we prove that for a fixed parameter all solutions of (1.1) are bounded.

Theorem 4.2. *Assume that conditions* (H0)–(H4) *hold. Let a parameter* $w \in \mathbb{W}$ *be fixed. If* u *is a solution to problem* (1.1), *then there exists constants* C_1, $C_2 > 0$ *such that* $C_1 \leq \|u\| \leq C_2$.

Proof. We will distinguish two cases. First we will assume that u is a solution to (1.1) such that $\|u\|_{p(\cdot)} \leq 1$. Put $\gamma = \frac{1}{T(T+1)^{p^+} L_2^{p^+}}$. By (H2), for any given $0 < \varepsilon < \gamma$, there exists $\delta > 0$ such that for all $|x| \leq \delta$ we have

$$|f(k,x,y)| \leq \varepsilon |x|^{p^+-1} \quad \text{for all} \quad k \in \mathbb{N}(1,T), \ y \in \mathbb{W}.$$

By (H1), there exists a constant $c_\varepsilon > 0$ such that for all $|x| > \delta$ we have

$$|f(k,x,y)| \leq c_\varepsilon |x|^{r-1} \quad \text{for all} \quad k \in \mathbb{N}(1,T), \ y \in \mathbb{W}.$$

Indeed, if $|x| > \delta$ then

$$1 + |x|^{r-1} < \left(\frac{1}{\delta^{r-1}} + 1\right) |x|^{r-1},$$

so

$$|f(k,x,y)| \leq c(1 + |x|^{r-1}) \leq c_\varepsilon |x|^{r-1},$$

where $c_\varepsilon = \left(\frac{1}{\delta^{r-1}} + 1\right)$. Thus for any given $0 < \varepsilon < \gamma$, there exists a constant c_ε such that

$$|f(k,x,y)| \leq \varepsilon |x|^{p^+-1} + c_\varepsilon |x|^{r-1} \quad \text{for all} \quad k \in \mathbb{N}(1,T), \ y \in \mathbb{W}. \tag{4.5}$$

By (1.3), (3.3), (4.5), (A2), (A3) and (1.2), we get

$$\begin{aligned}
\|u\|_{p(\cdot)}^{p^+} &\leq \varphi(u) = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} = \sum_{k=1}^{T} f(k, u(k), w) \, u(k) \\
&\leq \varepsilon \sum_{k=1}^{T} |u(k)|^{p^+} + c_\varepsilon \sum_{k=1}^{T} |u(k)|^r \\
&\leq \varepsilon T (T+1)^{p^+} \|u\|^{p^+} + c_\varepsilon T (T+1)^r \|u\|^r \\
&\leq \varepsilon T (T+1)^{p^+} L_2^{p^+} \|u\|_{p(\cdot)}^{p^+} + c_\varepsilon T (T+1)^r L_2^r \|u\|_{p(\cdot)}^r .
\end{aligned}$$

Hence

$$\|u\|_{p(\cdot)} \geq \left(\frac{1 - \varepsilon T (T+1)^{p^+} L_2^{p^+}}{c_\varepsilon T (T+1)^r L_2^r}\right)^{\frac{1}{r-p^+}}.$$

Put

$$\widetilde{C}_1 = \left(\frac{1 - T (T+1)^{p^+} \varepsilon L_2^{p^+}}{T (T+1)^r c_\varepsilon L_2^r}\right)^{\frac{1}{r-p^+}}.$$

Notice that $0 < \widetilde{C}_1 < 1$. Indeed, we get

$$0 < 1 - \varepsilon T\left(T+1\right)^{p^+} L_2^{p^+} < 1,$$

because

$$0 < \varepsilon < \frac{1}{T(T+1)^{p^+} L_2^{p^+}}.$$

By definition, $c_\varepsilon > 1$ and $L_2 > 1$, therefore $c_\varepsilon T\left(T+1\right)^r L_2^r > 1$. Thus $0 < \widetilde{C}_1 < 1$.

In the second case, we will assume that u is a solution to (1.1), such that $\|u\|_{p(\cdot)} \geq 1$. Then, we can find constant $\widetilde{C}_2 > 1$ such that $\|u\|_{p(\cdot)} \leq \widetilde{C}_2$.

By (H3), (4.4), (4.1) and (1.2) we deduce that

$$\begin{aligned}
&\mu(J_w(u) + \sum_{k=1}^{T} F(k,u(k),w)) - \sum_{k=1}^{T} f(k,u(k),w)u(k)\\
&\leq \mu J_w(u) + \mu \sum_{k=1}^{T} \frac{1}{\mu} f(k,u(k),w))u(k) - \sum_{k=1}^{T} f(k,u(k),w)u(k)\\
&= \mu J_w(u) = \mu \inf_{g\in\Gamma} \max_{u\in g([0,1])} J_w(g(t)) \leq \mu \max_{t\in[0,1]} J_w(tu_1)\\
&= \mu \max_{t\in[0,1]} J_w\left(tu_1 \frac{\|u_1\|_{p(\cdot)}}{\|u_1\|_{p(\cdot)}}\right) \leq \mu \max_{t\geq 0} J_w\left(t\frac{u_1}{\|u_1\|_{p(\cdot)}}\right)\\
&\leq \mu \max_{t\geq 0}\left(\tfrac{1}{p^-}(T+1)L_2^{p^+} t^{p^+} - c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}} L_1^{\mu} t^{\mu} + Tc_2 + \tfrac{1}{p^-}(T+1)\right),
\end{aligned}$$

where u_1 is an element in the space E which satisfies condition 3 from the Mountain Pass Lemma. On the other hand, by (3.1) and (3.3), we get

$$\begin{aligned}
&\mu(J_w(u) + \sum_{k=1}^{T} F(k,u(k),w)) - \sum_{k=1}^{T} f(k,u(k),w)u(k)\\
&= \mu \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T} |\Delta u(k-1)|^{p(k-1)}\\
&\geq \mu \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)}\\
&= \left(\mu \frac{1}{p^+} - 1\right) \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)}.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\left(\mu \frac{1}{p^+} - 1\right) \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)}\\
&\leq \mu \max_{t\geq 0}\left(\tfrac{1}{p^-}(T+1)L_2^{p^+} t^{p^+} - c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}} L_1 t^{\mu} + Tc_2 + \tfrac{1}{p^-}(T+1)\right).
\end{aligned}$$

Bearing in mind (1.4) we have

$$\left(\mu\frac{1}{p^+}-1\right)\|u\|_{p(\cdot)}^{p^-}$$
$$\leq \mu\max_{t\geq 0}\left(\tfrac{1}{p^-}(T+1)L_2^{p^+}t^{p^+}-c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}}L_1^{\mu}t^{\mu}+Tc_2+\tfrac{1}{p^-}(T+1)\right).$$

The function

$$r\left(t\right)=\tfrac{1}{p^-}(T+1)L_2^{p^+}t^{p^+}-c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}}L_1^{\mu}t^{\mu}+Tc_2+\tfrac{1}{p^-}(T+1)$$

is continuous, $\lim\limits_{t\longrightarrow\infty} r(t)=-\infty$ and achieves its maximum at the point

$$t_0=\left(\frac{\frac{1}{p^-}(T+1)L_2^{p^+}p^+}{c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}}L_1^{\mu}\mu}\right)^{\frac{1}{\mu-p^+}}.$$

Therefore, we may put

$$\widetilde{C}_2=\left(\frac{\mu\left(\frac{1}{p^-}(T+1)L_2^{p^+}t_0^{p^+}-c_1 2^{-\mu}\left(T+1\right)^{\frac{2-\mu}{\mu}}L_1 t_0^{\mu}+Tc_2+\frac{1}{p^-}(T+1)\right)}{\mu\frac{1}{p^+}-1}\right)^{\frac{1}{p^-}}. \tag{4.6}$$

It easy to check that $\widetilde{C}_2>1$. Notice that $r(t_0)\geq Tc_2+\frac{1}{p^-}(T+1)$. Hence

$$\mu r(t_0)\geq \mu Tc_2+\tfrac{\mu}{p^-}(T+1)>\tfrac{\mu}{p^-}(T+1)>\tfrac{\mu}{p^-}\geq\tfrac{\mu}{p^+}>\tfrac{\mu}{p^+}-1,$$

since $\mu>p^+\geq 2$ and $p^+\geq p^-$. Consequently $\widetilde{C}_2>1$.

Thus we have shown that, if u is a solution to problem (1.1), then there exist constants $\widetilde{C}_1,\widetilde{C}_2>0$ such that

$$\widetilde{C}_1\leq\|u\|_{p(\cdot)}\leq\widetilde{C}_2.$$

Consequently, by (1.2), there exist constants $C_1=L_1\widetilde{C}_1, C_2=L_2\widetilde{C}_2$ such that

$$C_1\leq\|u\|\leq C_2. \qquad\square$$

The obtained result allows us to study the continuous dependence on parameters. Considering a sequence of parameters we get existence of a sequence of solutions (corresponding to parameters). Supposing that the sequence of parameters is convergent we arrive at the limit of a subsequence selected from a sequence of solutions. This limit is a solution to the considered problem and it corresponds to the limit of the sequence of parameters.

Theorem 4.3. *Let $\mathbb{W}$ be a Hausdorff space. Assume that conditions* (H0)–(H4) *are satisfied. Let $\{w_n\}_{n=1}^{\infty}\subset\mathbb{W}$ be a convergent sequence of parameters with $\lim_{n\to\infty}w_n=w\in\mathbb{W}$. For any sequence $\{u_n\}_{n=1}^{\infty}$ of nontrivial solutions to* (1.1) *corresponding to $\{w_n\}_{n=1}^{\infty}$, there exists a subsequence $\{u_{n_i}\}_{i=1}^{\infty}\subset E$ and an element $u\in E$ such that $\lim_{i\to\infty}u_{n_i}=u$ and that u satisfies problem* (1.1) *corresponding to w.*

Proof. We define a sequence $\{u_n\}_{n=1}^{\infty} \in E$ as follows: u_n is a solution to (1.1) with $w = w_n$. Thus, for $n = 1, 2, \ldots$

$$\begin{cases} \Delta\left(|\Delta u_n(k-1)|^{p(k-1)-2}\Delta u_n(k-1)\right) + f(k, u_n(k), w_n) = 0, \quad k \in \mathbb{N}(1,T), \\ u_n(0) = u_n(T+1) = 0. \end{cases} \tag{4.7}$$

By Theorem 4.2, there exist constants $C_1, C_2 > 0$ such that $C_1 \leq \|u_n\| \leq C_2$ for $n = 1, 2, \ldots$. Thus, the sequence $\{u_n\}_{n=1}^{\infty}$ can be assumed convergent, up to a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$. So there exists $u \in E$ such that $u_{n_i} \to u$ in E. Since $C_1 \leq \|u_{n_i}\| \leq C_2$, we know that $C_1 \leq \|u\| \leq C_2$ and thus $u \neq 0$. The assertion that u is a solution to problem (1.1) is equivalent to showing that for any $v \in E$

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) = \sum_{k=1}^{T} f(k, u(k), w))v(k). \tag{4.8}$$

For any $v \in E$ by continuity of f we get from (4.7)

$$\sum_{k=1}^{T} f(k, u_{n_i}(k), w_{n_i})v(k) \to \sum_{k=1}^{T} f(k, u(k), w)v(k)$$

and

$$\begin{aligned} &\sum_{k=1}^{T+1} |\Delta u_{n_i}(k-1)|^{p(k-1)-2}\Delta u_{n_i}(k-1)\Delta v(k-1) \\ &\to \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1). \end{aligned}$$

Hence (4.8) holds. Since v was arbitrarily fixed, summing by parts as we did in the proof of Lemma 3.1, we see that u is a solution to (1.1). □

Now, we will show an example of a function which satisfies conditions (H0)–(H4).

Example 4.4. Let us take a function $F : \mathbb{N}(1,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(k, x, y) = (3 + \sin k)\,|x|^4\left(1 + e^{-y^2}\right)$$

and $p : \mathbb{N}(0,T) \to [2, +\infty)$ given by

$$p(k) = 2 + \frac{1}{k+1}.$$

Then $p^+ = 3$ and conditions (H0)–(H4) are fulfilled with $\mu = r = 4$.

5. UNIQUENESS OF SOLUTIONS

In this section we will examine conditions under which problem (1.1) has a unique solution. Recall that, by the previous section, if u is a solution to (1.1), then there exists a constant $\widetilde{C}_2 > 1$ given by (4.6) such that $\|u\|_{p(\cdot)} \leq \widetilde{C}_2$.

Lemma 5.1 ([14]). *Assume that $p \geq 2$ and $c_p = \frac{2}{p(2^{p-1}-1)}$. Then*

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y) \geq c_p|x-y|^p \quad \textit{for all} \ \ x, y \in \mathbb{R}.$$

Put

$$c_{p(k)} = \frac{2}{p(k)(2^{p(k)-1}-1)} \text{ for all } k \in \mathbb{N}(0,T)$$

and

$$c_{p^+} = \frac{2}{p^+(2^{p^+-1}-1)} = \min_{k\in\mathbb{N}(0,T)} c_{p(k)}.$$

Let us introduce an additional condition:

(H5) there exists a constant

$$d \in \left(0, \frac{c_{p^+}}{T(T+1)^{p^+} L_2^{p^+}\left(2\widetilde{C}_2\right)^{p^+-p^-}}\right)$$

such that for any x, $\overline{x} \in \mathbb{R}$, $t \in \mathbb{N}(1,T)$ and $y \in \mathbb{W}$ we have

$$|f(t,x,y) - f(t,\overline{x},y)| \leq d|x-\overline{x}|^{p^+-1},$$

where $\widetilde{C}_2 > 1$ is given by (4.6).

Theorem 5.2. *Assume that conditions* (H0)-(H5) *hold. Then for every fixed $w \in \mathbb{W}$ there exists exactly one solution u to problem* (1.1) *satisfying $C_1 \leq \|u\| \leq C_2$ (for some $C_1, C_2 > 0$).*

Proof. The existence part follows by Theorem 4.1. Suppose that there exist two different functions u_1, u_2 satisfying (1.1). Then, by (3.2),

$$\sum_{k=1}^{T+1} |\Delta u_1(k-1)|^{p(k-1)-2}\Delta u_1(k-1)\Delta(u_2-u_1)(k-1) = \sum_{k=1}^{T} f(k,u_1(k),w)(u_2-u_1)(k) \tag{5.1}$$

and

$$\sum_{k=1}^{T+1} |\Delta u_2(k-1)|^{p(k-1)-2}\Delta u_2(k-1)\Delta(u_2-u_1)(k-1) = \sum_{k=1}^{T} f(k,u_2(k),w)(u_2-u_1)(k). \tag{5.2}$$

Using (1.3), Lemma 5.1, (5.1), (5.2), (H5), (A2), (A3) and (1.2), in case $\|u_2 - u_1\|_{p(\cdot)} \leq 1$, we deduce that

$$\begin{aligned}
&c_{p^+} \cdot \|u_2 - u_1\|_{p(\cdot)}^{p^+} \leq c_{p^+} \cdot \varphi(u_2 - u_1) \\
&\leq \sum_{k=1}^{T+1} c_{p(k-1)} |\Delta u_2(k-1) - \Delta u_1(k-1)|^{p(k-1)} \\
&\leq \sum_{k=1}^{T+1} \left(|\Delta u_2(k-1)|^{p(k-1)-2} \Delta u_2(k-1) - |\Delta u_1(k-1)|^{p(k-1)-2} \Delta u_1(k-1) \right) \\
&\quad (\Delta u_2(k-1) - \Delta u_1(k-1)) \\
&= \sum_{k=1}^{T} [f(k, u_2(k), w) - f(k, u_1(k), w)](u_2 - u_1)(k) \\
&\leq d \sum_{k=1}^{T} |u_2(k) - u_1(k)|^{p^+} \leq dT(T+1)^{p^+} \|u_2 - u_1\|^{p^+} \\
&\leq dT(T+1)^{p^+} L_2^{p^+} \|u_2 - u_1\|_{p(\cdot)}^{p^+}.
\end{aligned}$$

Thus

$$\left(c_{p^+} - dT(T+1)^{p^+} L_2^{p^+} \right) \|u_2 - u_1\|_{p(\cdot)}^{p^+} \leq 0,$$

that is, $u_1 = u_2$, since $d < \frac{c_{p^+}}{T(T+1)^{p^+} L_2^{p^+}}$.

In case $\|u_2 - u_1\|_{p(\cdot)} \geq 1$, the preceding arguments already imply (using (1.4) in place of (1.3)) that

$$\begin{aligned}
c_{p^+} \|u_2 - u_1\|_{p(\cdot)}^{p^-} \leq c_{p^+} \varphi(u_2 - u_1) &= c_{p^+} \sum_{k=1}^{T+1} |\Delta u_2(k-1) - \Delta u_1(k-1)|^{p(k-1)} \\
&\leq dT(T+1)^{p^+} L_2^{p^+} \|u_2 - u_1\|_{p(\cdot)}^{p^+}.
\end{aligned}$$

Hence

$$\|u_2 - u_1\|_{p(\cdot)}^{p^+ - p^-} \geq \frac{c_{p^+}}{dT(T+1)^{p^+} L_2^{p^+}},$$

so

$$\|u_2 - u_1\|_{p(\cdot)} \geq \left(\frac{c_{p^+}}{dT(T+1)^{p^+} L_2^{p^+}} \right)^{\frac{1}{p^+ - p^-}}.$$

Since u_1, u_2 are solutions to (1.1) and $d < \frac{c_{p^+}}{T(T+1)^{p^+} L_2^{p^+} \left(2\widetilde{C}_2\right)^{p^+ - p^-}}$, so

$$2\widetilde{C}_2 \geq \|u_2\|_{p(\cdot)} + \|u_1\|_{p(\cdot)} \geq \|u_2 - u_1\|_{p(\cdot)} > 2\widetilde{C}_2.$$

Contradiction, therefore $u_1 = u_2$. □

6. NON-VARIATIONAL PROBLEM

In this section we focus on the existence of solutions to the following Dirichlet boundary value problem

$$\begin{cases} \Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) + f(k, u(k), \Delta u(k-1)) = 0, \ k \in \mathbb{N}(1,T), \\ u(0) = u(T+1) = 0, \end{cases} \tag{6.1}$$

within a non-variational framework. Precisely, we discuss conditions under which the problem (6.1) has exactly one nonzero solution.

Recall, once again that if u is a solution to (1.1), then there exists a constant $\widetilde{C}_2 > 1$ given by (4.6) such that $\|u\|_{p(\cdot)} \le \widetilde{C}_2$.

Assume that f has the following property:

(H6) There exist constants $d_1 \in \left(0, \frac{c_{p^+}}{T(T+1)^{p^+} L_2^{p^+} \left(2\widetilde{C}_2\right)^{p^+-p^-}}\right)$, $d_2 > 0$ satisfying the inequality

$$d_1 T\,(T+1)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}}\,(T+1)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left(2\widetilde{C}_2\right)^{p^+-1} < \frac{c_{p^+}}{\left(2\widetilde{C}_2\right)^{p^+-p^-}} \tag{6.2}$$

for which

$$|f(t,x,y) - f(t,\overline{x},y)| \le d_1 |x-\overline{x}|^{p^+-1}$$

and

$$|f(t,x,y) - f(t,x,\overline{y})| \le d_2 |y-\overline{y}|^{p^+-1}$$

for all $x, \overline{x}, y, \overline{y} \in \mathbb{R}$ and $t \in \mathbb{N}(1,T)$.

Theorem 6.1. *Assume that conditions* (H0)–(H4) *and* (H6) *hold. Then the problem* (6.1) *has exactly one nonzero solution.*

Proof. Let $u_0 \in E$ be fixed. Putting $w = \Delta u_0(k-1)$ in (1.1) for every $k \in \mathbb{N}(1,T)$ it follows by Theorem 5.2 that the following problem

$$\begin{cases} \Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) + f(k, u(k), \Delta u_0(k-1)) = 0, \quad k \in \mathbb{N}(1,T), \\ u(0) = u(T+1) = 0 \end{cases} \tag{6.3}$$

has exactly one nonzero solution (namely u_1) since (H6) implies (H5).

Repeating the reasoning, we construct a sequence $\{u_n\}_{n=1}^{\infty}$ in E as a sequence of solutions to the following boundary value problems

$$\begin{cases} \Delta\left(|\Delta u_n(k-1)|^{p(k-1)-2}\Delta u_n(k-1)\right) + f(k, u_n(k), \Delta u_{n-1}(k-1)) = 0, \\ u_n(0) = u_n(T+1) = 0, \quad k \in \mathbb{N}(1,T), \ n \in \mathbb{N}. \end{cases} \tag{6.4}$$

By Theorem 4.2, there exist constants $C_1, C_2 > 0$ such that $C_1 \leq \|u_n\| \leq C_2$ for all $n \in \mathbb{N}$, so the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded. We will show that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose so for any $n \in \mathbb{N}$ we have $\|u_{n+1} - u_n\|_{p(\cdot)} \geq 1$. Then, by (1.4), Lemma 5.1 and (3.2), we get

$$
\begin{aligned}
& c_{p^+} \|u_{n+1} - u_n\|_{p(\cdot)}^{p^-} \leq c_{p^+} \varphi(u_{n+1} - u_n) \\
& \leq \sum_{k=1}^{T+1} c_{p(k-1)} |\Delta u_{n+1}(k-1) - \Delta u_n(k-1)|^{p(k-1)} \\
& \leq \sum_{k=1}^{T+1} \Big(|\Delta u_{n+1}(k-1)|^{p(k-1)-2} \Delta u_{n+1}(k-1) \\
& \quad - |\Delta u_n(k-1)|^{p(k-1)-2} \Delta u_n(k-1) \Big) \left(\Delta u_{n+1}(k-1) - \Delta u_n(k-1)\right) \\
& = \sum_{k=1}^{T} f(k, u_{n+1}(k), \Delta u_n(k-1))(u_{n+1} - u_n)(k) \\
& \quad - \sum_{k=1}^{T} f(k, u_n(k), \Delta u_{n-1}(k-1))(u_{n+1} - u_n)(k) \\
& = \sum_{k=1}^{T} \left(f(k, u_{n+1}(k), \Delta u_n(k-1)) - f(k, u_n(k), \Delta u_n(k-1))\right) (u_{n+1} - u_n)(k) \\
& + \sum_{k=1}^{T} \left(f(k, u_n(k), \Delta u_n(k-1)) - f(k, u_n(k), \Delta u_{n-1}(k-1))\right) (u_{n+1} - u_n)(k).
\end{aligned}
$$

Continuing, by (H6), (A2), (A5), the Hölder inequality, (1.2) and (A3), we have

$$
\begin{aligned}
& c_{p^+} \|u_{n+1} - u_n\|_{p(\cdot)}^{p^-} \\
& \leq d_1 \sum_{k=1}^{T} |(u_{n+1} - u_n)(k)|^{p^+} + d_2 \sum_{k=1}^{T} |(\Delta u_n - \Delta u_{n-1})(k-1)|^{p^+-1} (u_{n+1} - u_n)(k) \\
& \leq d_1 T (T+1)^{p^+} L_2^{p^+} \|u_{n+1} - u_n\|_{p(\cdot)}^{p^+} \\
& \quad + d_2 \left(\sum_{k=1}^{T+1} \left(|\Delta u_n(k-1) - \Delta u_{n-1}(k-1)|^{p^+-1} \right)^{\frac{p^+}{p^+-1}} \right)^{\frac{p^+-1}{p^+}} \\
& \quad \times \left(\sum_{k=1}^{T} |u_{n+1}(k) - u_n(k)|^{p^+} \right)^{\frac{1}{p^+}} \\
& \leq d_1 T (T+1)^{p^+} L_2^{p^+} \|u_{n+1} - u_n\|_{p(\cdot)}^{p^+} \\
& + d_2 \left((T+1) L_2^{p^+} \|u_n - u_{n-1}\|_{p(\cdot)}^{p^+} \right)^{\frac{p^+-1}{p^+}} \left(T (T+1)^{p^+} L_2^{p^+} \|u_{n+1} - u_n\|_{p(\cdot)}^{p^+} \right)^{\frac{1}{p^+}}.
\end{aligned}
$$

Bearing in mind that $\{u_n\}_{n=1}^{\infty}$ is a sequence of solutions to (6.4) we obtain

$$
\begin{aligned}
& c_{p^+} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^-} \le d_1 T \left(T+1\right)^{p^+} L_2^{p^+} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \\
& + d_2 \left(T+1\right)^{\frac{p^+-1}{p^+}} L_2^{p^+-1} \left(\left\| u_n \right\|_{p(\cdot)} + \left\| u_{n-1} \right\|_{p(\cdot)} \right)^{p^+-1} T^{\frac{1}{p^+}} \left(T+1\right) L_2 \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \\
& \le d_1 T \left(T+1\right)^{p^+} L_2^{p^+} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \\
& + d_2 T^{\frac{1}{p^+}} \left(T+1\right)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left(2\widetilde{C}_2 \right)^{p^+-1} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \\
& = \left(d_1 T \left(T+1\right)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}} \left(T+1\right)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left(2\widetilde{C}_2 \right)^{p^+-1} \right) \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+}.
\end{aligned}
$$

Hence

$$
\left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+-p^-} \ge \frac{c_{p^+}}{d_1 T \left(T+1\right)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}} \left(T+1\right)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left(2\widetilde{C}_2 \right)^{p^+-1}}.
$$

By (H6), it follows that

$$
\begin{aligned}
2\widetilde{C}_2 & \ge \left\| u_{n+1} \right\|_{p(\cdot)} + \left\| u_n \right\|_{p(\cdot)} \ge \left\| u_{n+1} - u_n \right\|_{p(\cdot)} \\
& \ge \left(\frac{c_{p^+}}{d_1 T \left(T+1\right)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}} \left(T+1\right)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left(2\widetilde{C}_2 \right)^{p^+-1}} \right)^{\frac{1}{p^+-p^-}} > 2\widetilde{C}_2.
\end{aligned}
$$

Contradiction. Thus for any $n \in \mathbb{N}$ we have $\left\| u_{n+1} - u_n \right\|_{p(\cdot)} < 1$. The preceding arguments already imply, using (1.3) in place of (1.4) that

$$
\begin{aligned}
& c_{p^+} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \le d_1 T \left(T+1\right)^{p^+} L_2^{p^+} \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+} \\
& + d_2 \left(T+1\right)^{\frac{p^+-1}{p^+}} L_2^{p^+-1} \left\| u_n - u_{n-1} \right\|_{p(\cdot)}^{p^+-1} T^{\frac{1}{p^+}} \left(T+1\right) L_2 \left\| u_{n+1} - u_n \right\|_{p(\cdot)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \left(c_{p^+} - d_1 T \left(T+1\right)^{p^+} L_2^{p^+} \right) \left\| u_{n+1} - u_n \right\|_{p(\cdot)}^{p^+-1} \\
& \le d_2 T^{\frac{1}{p^+}} \left(T+1\right)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} \left\| u_n - u_{n-1} \right\|_{p(\cdot)}^{p^+-1}.
\end{aligned}
$$

It is equivalent to

$$\|u_{n+1}-u_n\|_{p(\cdot)} \le \left(\frac{d_2 T^{\frac{1}{p^+}}(T+1)^{\frac{p^+-1}{p^+}+1} L_2^{p^+}}{c_{p^+}-d_1 T(T+1)^{p^+} L_2^{p^+}}\right)^{\frac{1}{p^+-1}} \|u_n-u_{n-1}\|_{p(\cdot)}.$$

By (H6), we deduce that

$$\begin{aligned}
&d_1 T\,(T+1)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}}(T+1)^{\frac{p^+-1}{p^+}+1} L_2^{p^+}\\
&< d_1 T\,(T+1)^{p^+} L_2^{p^+} + d_2 T^{\frac{1}{p^+}}(T+1)^{\frac{p^+-1}{p^+}+1} L_2^{p^+}\left(2\widetilde{C}_2\right)^{p^+-1}\\
&< \frac{c_{p^+}}{\left(2\widetilde{C}_2\right)^{p^+-p^-}} < c_{p^+}.
\end{aligned}$$

Finally

$$d_2 T^{\frac{1}{p^+}}(T+1)^{\frac{p^+-1}{p^+}+1} L_2^{p^+} < c_{p^+} - d_1 T\,(T+1)^{p^+} L_2^{p^+}.$$

It implies that the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Therefore, there exists $u \in E$ such that $u_n \to u$. Since $C_1 \le \|u_n\| \le C_2$ for all $n \in \mathbb{N}$, so $C_1 \le \|u\| \le C_2$. It implies that $u \ne 0$. For every $h \in E$, by continuity of f and (6.4), we have

$$\sum_{k=1}^{T} f(k,u_n(k),\Delta u_{n-1}(k-1))h(k) \to \sum_{k=1}^{T} f(k,u(k),\Delta u(k-1))h(k)$$

and

$$\begin{aligned}
&\sum_{k=1}^{T+1} |\Delta u_n(k-1)|^{p(k-1)-2}\Delta u_n(k-1)\Delta h(k-1)\\
\to\ &\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta h(k-1).
\end{aligned}$$

Eventually,

$$\sum_{k=1}^{T+1}\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)\Delta h(k-1) = \sum_{k=1}^{T} f\left(k,u(k),\Delta u(k-1)\right)h(k)$$

for any $h \in E$. Next, summing by parts we deduce that u is a solution to (6.1). The uniqueness of the solution can be demonstrated in the same way as in the proof of Theorem 5.2 (note that condition (H5) is satisfied). □

7. NONEXISTENCE OF SOLUTIONS

In this section we give a condition under which the considered problems (1.1) and (6.1) have no solutions.

Theorem 7.1. *Let* (H0) *hold. Suppose that*

$$xf(k,x,y)<0 \quad \text{for all} \quad k\in\mathbb{N}(1,T), x\in\mathbb{R}\setminus\{0\}, y\in\mathbb{W}. \tag{7.1}$$

Then for every $w\in\mathbb{W}$ *problem* (1.1) *has no nontrivial solutions.*

Proof. Assume that (1.1) has a nonzero solution. Then J_w has a nontrivial critical point $\widetilde{u}$. By Lemma 3.2,

$$0=\left(J_w^{'}(\widetilde{u}),\widetilde{u}\right)=\sum_{k=1}^{T+1}|\Delta\widetilde{u}(k-1)|^{p(k-1)}-\sum_{k=1}^{T}f(k,\widetilde{u}(k),w)\widetilde{u}(k),$$

so it holds

$$(f(k,\widetilde{u},w),\widetilde{u})=\sum_{k=1}^{T+1}|\Delta\widetilde{u}(k-1)|^{p(k-1)}\geq 0 \text{ for all } k\in\mathbb{N}(1,T).$$

On the other hand, from (7.1) we get

$$(f(k,\widetilde{u},w),\widetilde{u})=\sum_{k=1}^{T}\widetilde{u}(k)f(k,\widetilde{u}(k),w)<0 \text{ for all } k\in\mathbb{N}(1,T).$$

This contradicts our assumption, so for every $w\in\mathbb{W}$ problem (1.1) has no nontrivial solution. □

Example 7.2. Let us take a function $f:\mathbb{N}(1,T)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by

$$f(k,x,y)=-k\arctan x.$$

Then we see that assumptions of Theorem 7.1 are satisfied.

Remark 7.3. Under the assumptions of Theorem 7.1 it follows immediately that problem (6.1) has also no nontrivial solutions.

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