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EXISTENCE AND UNIQUENESS OF THE SOLUTIONS OF SOME DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper we are interested in the existence of solutions for the Dirichlet problem associated with degenerate nonlinear elliptic equations

$$-\sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{A}_j(x,u,\nabla u)\big] + b(x,u,\nabla u)\omega(x) + g(x)u(x) =$$
$$= f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \quad \text{on} \quad \Omega$$

in the setting of the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$.

Keywords: degenerate nonlinear elliptic equations, weighted Sobolev spaces.

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1. INTRODUCTION

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega)$ (see Definition 2.2) for the Dirichlet problem

$$\text{(P)} \quad \begin{cases} Lu(x) = f_0(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x) & \text{on} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{A}_j(x,u(x),\nabla u(x))\big] + b(x,u(x),\nabla u(x))\omega(x) + g(x)u(x),$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω is a weight function and the functions $\mathcal{A}_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ $(j = 1, \ldots, n)$, $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$ satisfy the following conditions:

(H1) $x \mapsto \mathcal{A}_j(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
$(\eta, \xi) \mapsto \mathcal{A}_j(x, \eta, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H2) There exist a constant $\theta_1 > 0$ such that

$$[\mathcal{A}(x, \eta, \xi) - \mathcal{A}(x, \eta', \xi')].(\xi - \xi') \geq \theta_1 |\xi - \xi'|^p,$$

whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where $\mathcal{A}(x, \eta, \xi) = (\mathcal{A}_1(x, \eta, \xi), \ldots, \mathcal{A}_n(x, \eta, \xi))$.

(H3)

$$\mathcal{A}(x, \eta, \xi).\xi \geq \lambda_1 |\xi|^p + \Lambda_1 |\eta|^p - g_1(x)|\eta|,$$

with $g_1 \in L^{p'}(\Omega, \omega)$, where λ_1 and Λ_1 are positive constants.

(H4)

$$|\mathcal{A}(x, \eta, \xi)| \leq K_1(x) + h_1(x)|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'},$$

where K_1, h_1 and h_2 are positive functions, with h_1 and $h_2 \in L^\infty(\Omega)$, and $K_1 \in L^{p'}(\Omega, \omega)$ (with $1/p + 1/p' = 1$).

(H5) $x \mapsto b(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
$(\eta, \xi) \mapsto b(x, \eta, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H6) There exists a constant $\theta_2 > 0$ such that

$$[b(x, \eta, \xi) - b(x, \eta', \xi')](\eta - \eta') \geq \theta_2 |\eta - \eta'|^p,$$

whenever $\eta, \eta' \in \mathbb{R}$, $\eta \neq \eta'$.

(H7)

$$b(x, \eta, \xi)\eta \geq \lambda_2 |\xi|^p + \Lambda_2 |\eta|^p - g_2(x)|\eta|,$$

with $g_2 \in L^{p'}(\Omega, \omega)$, where λ_2 and Λ_2 are positive constants.

(H8)

$$|b(x, \eta, \xi)| \leq K_2(x) + h_3(x)|\eta|^{p/p'} + h_4(x)|\xi|^{p/p'},$$

where K_2, h_3 and h_4 are positive functions, with $K_2 \in L^{p'}(\Omega, \omega)$, h_3 and $h_4 \in L^\infty(\Omega)$.

(H9) $g/\omega \in L^q(\Omega, \omega)$, where $1/q = 1/p' - 1/p$, and $g(x) \geq 0$ a.e. $x \in \Omega$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\, dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1,3] and [4]).

A class of weights, which is particulary well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [8]). These classes have found many usefull applications in harmonic analysis (see [9]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [7]). There are, in fact, many interesting examples of weights (see [6] for p-admissible weights).

The following theorem will be proved in section 3.

Theorem 1.1. *Assume* (H1)–(H9). *If* $\omega \in A_p$ *(with* $2 < p < \infty$*),* $f_0/\omega \in L^{p'}(\Omega, \omega)$*,* $f_j/\omega \in L^{p'}(\Omega, \omega)$ $(j = 1, \dots, n)$*, then the problem* (P) *has a unique solution* $u \in W_0^{1,p}(\Omega, \omega)$*. Moreover, we have*

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} \leq \frac{1}{\gamma^{p'/p}} \left(\sum_{j=0}^{n} \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} + \|g_1\|_{L^{p'}(\Omega,\omega)} + \|g_2/\omega\|_{L^{p'}(\Omega,\omega)} \right)^{p'/p},$$

where $\gamma = \min\{\lambda_1, \lambda_2, \Lambda_1, \Lambda_2\}$.

2. DEFINITIONS AND BASIC RESULTS

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x)dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx \right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [5,6] or [9] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x;r)) \leq C\mu(B(x;2r)),$$

for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [6]).

As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [9]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|} \right)^p \leq C \frac{\mu(E)}{\mu(B)}$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 the *strong doubling property* in [6]). Therefore, if $\mu(E) = 0$, then $|E| = 0$.

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx \right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [10]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let ω be a A_p-weight ($1 < p < \infty$), and let $\Omega \subset \mathbb{R}^n$ be open. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D_j u \in L^p(\Omega,\omega)$. The norm of u in $W^{1,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega(x)\, dx \right)^{1/p}. \tag{2.1}$$

We also define $W^{1,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega,\omega)}$.

If $\omega \in A_p$, then $W^{1,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [10]). The spaces $W^{1,p}(\Omega,\omega)$ and $W^{1,p}_0(\Omega,\omega)$ are Banach spaces.

It is evident that a weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), gives nothing new (the space $\mathrm{W}^{1,p}_0(\Omega,\omega)$ is then identical to the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following two theorems.

Theorem 2.3. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exists a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that:*

(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, *μ-a.e. on* Ω,

(ii) $|u_{m_k}(x)| \le \Phi(x)$, *μ-a.e. on* Ω

(*where* $\mu(E) = \int_E \omega(x)\, dx$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [2]. □

Theorem 2.4. *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist constants C_Ω and δ positive such that for all $u \in C^\infty_0(\Omega)$ and all k satisfying $1 \le k \le n/(n-1) + \delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}.$$

Proof. See Theorem 1.3 in [3]. □

Definition 2.5. We say that an element $u \in W^{1,p}_0(\Omega,\omega)$ is a (weak) solution of problem (P) if

$$\sum_{j=1}^n \int_\Omega \omega(x)\mathcal{A}_j(x,u(x),\nabla u(x)) D_j\varphi(x) dx + \int_\Omega b(x,u(x),\nabla u(x))\varphi(x)\omega(x)\, dx +$$
$$+ \int_\Omega g(x)\varphi(x)u(x)\, dx = \int_\Omega f_0(x)\varphi(x) dx + \sum_{j=1}^n \int_\Omega f_j(x) D_j\varphi(x) dx$$

for all $\varphi \in W^{1,p}_0(\Omega,\omega)$.

3. PROOF OF THEOREM 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) *for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$,*
(b) *if the operator A is strictly monotone, then equation $Au = T$ is uniquely solvable in X.*

Proof. See Theorem 26.A in [11]. □

To proof Theorem 1.1, we define $B, B_1, B_2 : W_0^{1,p}(\Omega,\omega) \times W_0^{1,p}(\Omega,\omega) \to \mathbb{R}$ and $T : W_0^{1,p}(\Omega,\omega) \to \mathbb{R}$ by

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi),$$

$$\begin{aligned} B_1(u,\varphi) &= \sum_{j=1}^{n} \int_\Omega \omega \mathcal{A}_j(x,u,\nabla u) D_j\varphi dx = \\ &= \int_\Omega \omega\, \mathcal{A}(x,u,\nabla u).\nabla\varphi\, dx, \end{aligned}$$

$$\begin{aligned} B_2(u,\varphi) &= \int_\Omega b(x,u(x),\nabla u(x))\varphi(x)\omega(x)\, dx + \\ &\quad + \int_\Omega g(x)\varphi(x)u(x) dx, \end{aligned}$$

$$\begin{aligned} T(\varphi) &= \int_\Omega f_0(x)\varphi(x)\, dx + \\ &\quad + \sum_{j=1}^{n} \int_\Omega f_j(x) D_j\varphi(x)\, dx. \end{aligned}$$

Then $u \in W_0^{1,p}(\Omega,\omega)$ is a (weak) solution to problem (P) if

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi),$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$.
Step 1. For $j = 1, ..., n$ we define the operator $F_j : W_0^{1,p}(\Omega,\omega) \to L^{p'}(\Omega,\omega)$ by

$$(F_j u)(x) = \mathcal{A}_j(x,u(x),\nabla u(x)).$$

We have that the operator F_j is bounded and continuous. In fact:
(i) Using (H4) we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u(x)|^{p'}\,\omega\,dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega\,dx \leq \\
&\leq \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'}\omega\,dx \leq \\
&\leq C_p \int_\Omega \Big[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega\Big]dx = \\
&= C_p\Big[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_1^{p'}|u|^p\omega\,dx + \int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx\Big],
\end{aligned} \tag{3.1}$$

where the constant C_p depends only on p.

We have

$$\int_\Omega h_1^{p'}|u|^p\omega\,dx \leq \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\omega\,dx \leq \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W^{1,p}_0(\Omega,\omega)}$$

and

$$\int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx \leq \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\omega\,dx \leq \|h_2\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W^{1,p}_0(\Omega,\omega)}.$$

Therefore, in (3.1) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega)} \leq C_p\bigg(\|K\|_{L^{p'}(\Omega,\omega)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p/p'}_{W^{1,p}_0(\Omega,\omega)}\bigg).$$

(ii) Let $u_m \to u$ in $W^{1,p}_0(\Omega,\omega)$ as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega)$.

If $u_m \to u$ in $W^{1,p}_0(\Omega,\omega)$, then $u_m \to u$ in $L^p(\Omega,\omega)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega)$. Using Theorem 2.3, there exists a subsequence $\{u_{m_k}\}$ and functions Φ_1 and Φ_2 in $L^p(\Omega,\omega)$ such that

$$\begin{aligned}
&u_{m_k}(x) \to u(x),\ \mu-\text{a.e. in } \Omega,\\
&|u_{m_k}(x)| \leq \Phi_1(x),\ \mu-\text{a.e. in } \Omega,\\
&|\nabla u_{m_k}(x)| \to |\nabla u(x)|,\ \mu-\text{a.e. in } \Omega,\\
&|\nabla u_{m_k}(x)| \leq \Phi_2(x),\ \mu-\text{a.e. in } \Omega.
\end{aligned}$$

Hence, using (H4), we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'} \omega\, dx = \\
&= \int_\Omega |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{p'} \omega\, dx \leq \\
&\leq C_p \int_\Omega \Big(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, u, \nabla u)|^{p'} \Big) \omega\, dx \leq \\
&\leq C_p \Bigg[\int_\Omega \Big(K_1 + h_1 |u_{m_k}|^{p/p'} + h_2 |\nabla u_{m_k}|^{p/p'} \Big)^{p'} \omega\, dx + \\
&\qquad + \int_\Omega \Big(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \Big)^{p'} \omega\, dx \Bigg] \leq \\
&\leq 2C_p \int_\Omega \Big(K_1 + h_1 \Phi_1^{p/p'} + h_2 \Phi_2^{p/p'} \Big)^{p'} \omega\, dx \leq \\
&\leq 2C_p \Bigg[\int_\Omega K_1^{p'} \omega\, dx + \int_\Omega h_1^{p'} \Phi_1^p \omega\, dx + \int_\Omega h_2^{p'} \Phi_2^p \omega\, dx \Bigg] \leq \\
&\leq 2\, C_p \Bigg[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^p \omega\, dx + \\
&\qquad + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_2^p \omega\, dx \Bigg] \leq \\
&\leq 2C_p \Bigg[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\ \|\Phi_1\|^p_{L^p(\Omega,\omega)} + \\
&\qquad + \|h_2\|^{p'}_{L^\infty(\Omega)} \|\Phi_2\|^p_{L^p(\Omega,\omega)} \Bigg].
\end{aligned}$$

By condition (H1), we have

$$F_j u_m(x) = \mathcal{A}_j(x, u_m(x), \nabla u_m(x)) \rightarrow \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),$$

as $m \to \infty$. Therefore, by the dominated convergence theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)} \to 0,$$

that is,

$$F_j u_{m_k} \to F_j u \ \text{ in } \ L^{p'}(\Omega, \omega).$$

By the convergence principle in Banach spaces, we have

$$F_j u_m \to F_j u \ \text{ in } \ L^{p'}(\Omega, \omega). \tag{3.2}$$

Step 2. We define the operator $G: W_0^{1,p}(\Omega,\omega) \to L^{p'}(\Omega,\omega)$ by

$$(Gu)(x) = b(x, u(x), \nabla u(x)).$$

We also have that the operator G is continuous and bounded. In fact:
(i) Using (H8) we obtain

$$\begin{aligned}
\|Gu\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |Gu|^{p'} \omega\, dx = \int_\Omega |b(x,u,\nabla u)|^{p'} \omega dx \le \\
&\le \int_\Omega \left(K_2 + h_3 |u|^{p/p'} + h_4 |\nabla u|^{p/p'} \right)^{p'} \omega\, dx \le \\
&\le C_p \int_\Omega \left[(K_2^{p'} + h_3^{p'} |u|^p + h_4^{p'} |\nabla u|^p)\omega \right] dx = \\
&= C_p \left[\int_\Omega K_2^{p'} \omega\, dx + \int_\Omega h_3^{p'} |u|^p \omega\, dx + \int_\Omega h_4^{p'} |\nabla u|^p \omega\, dx \right] \le \\
&\le C_p \Big(\|K_2\|^{p'}_{L^{p'}(\Omega,\omega)} + (\|h_3\|^{p'}_{L^\infty(\Omega)} + \|h_4\|^{p'}_{L^\infty(\Omega)})\, \|u\|^p_{W_0^{1,p}(\Omega,\omega)} \Big).
\end{aligned}$$

(ii) By the same argument used in Step 1(ii), we obtain analogously, if $u_m \to u$ in $W_0^{1,p}(\Omega,\omega)$, then

$$Gu_m \to Gu \quad \text{in} \quad L^{p'}(\Omega,\omega). \tag{3.3}$$

Step 3. We have

$$\begin{aligned}
|T(\varphi)| &\le \int_\Omega |f_0||\varphi|\, dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\, dx = \\
&= \int_\Omega \frac{|f_0|}{\omega} |\varphi| \omega\, dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega} |D_j\varphi| \omega\, dx \le \\
&\le \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} \|D_j\varphi\|_{L^p(\Omega,\omega)} \le \\
&\le \left(\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned}$$

Moreover, using (H4), (H8), (H9) and the generalized Hölder inequality, we also have

$$\begin{aligned}
|B(u,\varphi)| &\le |B_1(u,\varphi)| + |B_2(u,\varphi)| \le \\
&\le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\omega\, dx + \int_\Omega |b(x,u,\nabla u)|\, |\varphi|\omega\, dx + \\
&+ \int_\Omega |g|\, |u|\, |\varphi|\, dx.
\end{aligned} \tag{3.4}$$

In (3.4) we have

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega\,dx \le \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)|\nabla\varphi|\,\omega\,dx \le \\
&\le \|K_1\|^{p'}_{L^{p'}(\Omega,\omega)}\|\varphi\|^p_{L^p(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} + \\
&\quad + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} \le \\
&\le \Big(\|K_1\|_{L^{p'}(\Omega,\omega)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |b(x,u,\nabla u)|\,|\varphi|\,\omega\,dx \le \int_\Omega \Big(K_2 + h_3|u|^{p/p'} + h_4|\nabla u|^{p/p'}\Big)|\varphi|\,\omega\,dx \le \\
&\le \int_\Omega K_2\,|\varphi|\omega\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |u|^{p/p'}|\varphi|\,\omega\,dx + \|h_4\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{p/p'}|\varphi|\,\omega\,dx \le \\
&\le \Big(\|K_2\|_{L^{p'}(\Omega,\omega)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \|h_4\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}
\end{aligned}$$

and, since $1/q + 1/p + 1/p = 1$,

$$\begin{aligned}
\int_\Omega |g||u||\varphi|\,dx = \int_\Omega \frac{|g|}{\omega}|u||\varphi|\,\omega\,dx &\le \\
&\le \|g/\omega\|_{L^q(\Omega,\omega)}\|u\|_{L^p(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} \le \\
&\le \|g/\omega\|_{L^q(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned}$$

Hence, in (3.4) we obtain, for all $u,\varphi \in W_0^{1,p}(\Omega,\omega)$

$$\begin{aligned}
|B(u,\varphi)| \le \Big[&\|K_1\|_{L^{p'}(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \\
&+ \|K_2\|_{L^{p'}(\Omega,\omega)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \|h_4\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \\
&+ \|g/\omega\|_{L^q(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega)}\Big]\|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned}$$

Since $B(u,\cdot)$ is linear, for each $u \in W_0^{1,p}(\Omega,\omega)$, there exists a linear and continuous operator

$$A : W_0^{1,p}(\Omega,\omega) \to [W_0^{1,p}(\Omega,\omega)]^*$$

such that $\langle Au, \varphi \rangle = B(u,\varphi)$, for all $u,\ \varphi \in W_0^{1,p}(\Omega,\omega)$ (where $\langle f, x\rangle$ denotes the value of the linear functional f at the point x) and

$$\begin{aligned}\|Au\|_* \leq \|K_1\|_{L^{p'}(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}+\\ + \|K_2\|_{L^{p'}(\Omega,\omega)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)} + \|h_4\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega)}+\\ + \|g/\omega\|_{L^q(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T, \quad u \in W_0^{1,p}(\Omega,\omega).$$

Step 4. Using condition (H2), (H6) and (H9), we have

$$\begin{aligned}&\langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1, u_1-u_2) - B(u_2, u_1-u_2) =\\ &= \int_\Omega \omega\, \mathcal{A}(x,u_1,\nabla u_1).\nabla(u_1-u_2)\,dx + \int_\Omega b(x,u_1,\nabla u_1)(u_1-u_2)\,\omega\,dx+\\ &\quad + \int_\Omega (u_1-u_2) g\, u_1\,dx-\\ &\quad - \int_\Omega \omega\, \mathcal{A}(x,u_2,\nabla u_2).\nabla(u_1-u_2)\,dx - \int_\Omega b(x,u_2,\nabla u_2)(u_1-u_2)\,\omega\,dx-\\ &\quad - \int_\Omega g(u_1-u_2)u_2\,dx =\\ &= \int_\Omega \omega\left(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\right).\nabla(u_1-u_2)\,dx+\\ &\quad + \int_\Omega (b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2))(u_1-u_2)\omega\,dx + \int_\Omega g\,(u_1-u_2)^2\,dx \geq\\ &\geq \theta_1 \int_\Omega \omega\,|\nabla(u_1-u_2)|^p\,dx + \theta_2\int_\Omega |u_1-u_2|^p\omega\,dx \geq\\ &\geq \theta\,\|u_1-u_2\|^p_{W_0^{1,p}(\Omega,\omega)},\end{aligned}$$

where $\theta = \min\{\theta_1, \theta_2\}$.

Therefore, the operator A is strictly monotone. Moreover, using (H3), (H7) and (H9), we obtain

$$\begin{aligned}
\langle Au, u\rangle &= B(u,u) = B_1(u,u) + B_2(u,u) = \\
&= \int_\Omega \omega\, \mathcal{A}(x,u,\nabla u).\nabla u\, dx + \int_\Omega b(x,u,\nabla u)\, u\, \omega\, dx + \int_\Omega g\, u^2\, dx \geq \\
&\geq \int_\Omega \Big(\Lambda_1 |u|^p + \lambda_1 |\nabla u|^p - g_1|u|\Big)\, \omega\, dx + \\
&\quad + \int_\Omega \Big(\Lambda_2 |u|^p + \lambda_2 |\nabla u|^p - g_2|u|\Big)\, \omega\, dx \geq \\
&\geq \gamma\, \|u\|^p_{W^{1,p}_0(\Omega,\omega)} - (\|g_1\|_{L^{p'}(\Omega,\omega)} + \|g_2\|_{L^{p'}(\Omega,\omega)})\|u\|_{W^{1,p}_0(\Omega,\omega)},
\end{aligned}$$

where $\gamma = \min\{\lambda_1, \lambda_2, \Lambda_1, \Lambda_2\}$. Hence, since $p > 2$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_{W^{1,p}_0(\Omega,\omega)}} \to \infty, \text{ as } \|u\|_{W^{1,p}_0(\Omega,\omega)} \to \infty,$$

that is, A is coercive.
Step 5. We need to show that the operator A is continuous.

Let $u_m \to u$ in $W^{1,2}_0(\Omega,\omega)$ as $m \to \infty$. We have

$$\begin{aligned}
|B_1(u_m,\varphi) - B_1(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m) - \mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\omega\, dx = \\
&= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\omega\, dx \leq \\
&\leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} \|D_j\varphi\|_{L^p(\Omega,\omega)} \leq \\
&\leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{W^{1,p}_0(\Omega,\omega)}
\end{aligned}$$

and

$$\begin{aligned}
&|B_2(u_m,\varphi) - B_2(u,\varphi)| = \\
&= \left| \int_\Omega (b(x,u_m,\nabla u_m) - b(x,u,\nabla u))\, \varphi\omega\, dx + \int_\Omega g\, \varphi\, (u_m - u)\, dx \right| \leq \\
&\leq \int_\Omega |Gu_m - Gu||\varphi|\, \omega\, dx + \int_\Omega |g|\, |\varphi|\, |u_m - u|\, dx \leq \\
&\leq \|Gu_m - Gu\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} + \|g/\omega\|_{L^q(\Omega,\omega)} \|\varphi\|_{L^p(\Omega,\omega)} \|u_m - u\|_{L^p(\Omega,\omega)} \leq \\
&\leq \|Gu_m - Gu\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_{W^{1,p}_0(\Omega,\omega)} + \|g/\omega\|_{L^q(\Omega,\omega)} \|\varphi\|_{W^{1,p}_0(\Omega,\omega)} \|u_m - u\|_{W^{1,p}_0(\Omega,\omega)}
\end{aligned}$$

for all $\varphi \in W_0^{1,2}(\Omega,\omega)$. Hence,

$$|B(u_m,\varphi)-B(u,\varphi)| \leq |B_1(u_m,\varphi)-B_1(u,\varphi)| + |B_2(u_m,\varphi)-B_2(u,\varphi)| \leq$$
$$\leq \Bigg[\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} +$$
$$+ \|Gu_m - Gu\|_{L^{p'}(\Omega,\omega)} + \|g/\omega\|_{L^q(\Omega,\omega)} \|u_m - \|_{W_0^{1,p}(\Omega,\omega)} \Bigg] \|\varphi\|_{W_0^{1,p}(\Omega,\omega)}.$$

Then we obtain

$$\|Au_m - Au\|_* \leq \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} +$$
$$+ \|Gu_m - Gu\|_{L^{p'}(\Omega,\omega)} + \|g/\omega\|_{L^q(\Omega,\omega)} \|u_m - u\|_{W_0^{1,p}(\Omega,\omega)}.$$

Therefore, using (3.2) and (3.3) we have $\|Au_m - Au\|_* \to 0$ as $m \to \infty$, that is, A is continuous (and this implies that A is hemicontinuous).

Therefore, by Theorem 3.1, the operator equation $Au = T$ has a unique solution $u \in W_0^{1,p}(\Omega,\omega)$ and it is the unique solution for problem (P).
Step 6. In particular, by setting $\varphi = u$ in Definition 2.5, we have

$$B(u,u) = B_1(u,u) + B_2(u,u) = T(u). \tag{3.5}$$

Hence, using (H3), (H7), (H9) and $\gamma = \min\{\lambda_1, \lambda_2, \Lambda_1, \Lambda_2\}$, we obtain

$$B_1(u,u) + B_2(u,u) = \int_\Omega \omega\, \mathcal{A}(x,u,\nabla u).\nabla u\, dx + \int_\Omega b(x,u,\nabla u)\, u\omega\, dx + \int_\Omega g\, u^2\, dx \geq$$
$$\geq \int_\Omega \Big(\Lambda_1|u|^p + \lambda_1|\nabla u|^p - g_1|u|\Big)\, \omega\, dx +$$
$$+ \int_\Omega \Big(\Lambda_2|u|^p + \lambda_2|\nabla u\|^p - g_2|u|\Big)\, \omega\, dx \geq$$
$$\geq \gamma \|u\|^p_{W_0^{1,p}(\Omega,\omega)} - \left(\|g_1\|_{L^{p'}(\Omega,\omega)} + \|g_2\|_{L^{p'}(\Omega,\omega)}\right) \|u\|_{W_0^{1,p}(\Omega,\omega)}$$

and

$$T(u) = \int_\Omega f_0\, u\, dx + \sum_{j=1}^{n} \int_\Omega f_j\, D_j u\, dx \leq$$
$$\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} \|u\|_{L^p(\Omega,\omega)} + \sum_{j=1}^{n} \|f_j/\omega|_{L^{p'}(\Omega)} \|D_j u\|_{L^p(\Omega,\omega)} \leq$$
$$\leq \left(\sum_{j=0}^{n} \|f_j/\omega\|_{L^{p'}(\Omega)} \right) \|u\|_{W_0^{1,p}(\Omega,\omega)}.$$

Therefore, in (3.5), we obtain

$$\gamma \|u\|^p_{W^{1,p}_0(\Omega,\omega)} - \left(\|g_1\|_{L^{p'}(\Omega,\omega)} + \|g_2\|_{L^{p'}(\Omega,\omega)}\right) \|u\|_{W^{1,p}_0(\Omega,\omega)} \leq$$
$$\leq \left(\sum_{j=0}^{n} \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} \right) \|u\|_{W^{1,p}_0(\Omega,\omega)},$$

and we obtain

$$\|u\|_{W^{1,p}_0(\Omega,\omega)} \leq \frac{1}{\gamma^{p'/p}} \left(\sum_{j=0}^{n} \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} + \|g_1\|_{L^{p'}(\Omega,\omega)} + \|g_2\|_{L^{p'}(\Omega,\omega)} \right)^{p'/p}.$$

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