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Analysis of system operation process influence on its reliability

Keywords

system reliability, system operation process, analytical approach, Monte Carlo simulation

Abstract

The paper presents analytical and Monte Carlo simulation methods applied to the reliability evaluation of a system operating in two different operation states. A semi-Markov process is applied to construct the system operation model and its main characteristics are determined. Analytical linking of this operation model with the system reliability model is proposed to get a general reliability model of the system operating at two varying in time operation conditions and to find its reliability characteristics. The application of Monte Carlo simulation based on this general model to the reliability evaluation of this system is proposed as well. The exemplary results obtained from those two considered methods are illustrated.

1. Introduction

The reliability analysis of a system undergoing time-dependent operation process very often leads to complicated calculations and therefore it is difficult to implement analytical modeling, prediction and optimization, especially in the case when we assume the system multistate reliability model and the multistate model of its operation process [1]-[4]. On the other hand, the complexity of the systems' operation processes and their influence on changing in time the systems' reliability parameters are very often met in real practice [3]-[4], [9]. Thus, the practical importance of an approach linking the system reliability models and the system operation processes models into an integrated general model in reliability assessment of real technical systems is evident. The Monte Carlo simulation method [8], [12] is a tool that sometimes allows to simplify solving this problem [3], [8]. All above-mentioned publications present general results obtained under a strong assumption that the system components have exponential conditional reliability functions at different operation states. To omit this assumption that narrows the investigation down and to get general solutions of the problem, at the beginning, we deal with the two-state reliability model of the system and two-state model of its operation process. The analytical approach to the reliability analysis of two-state systems subjected to two-state operation processes is presented and next the computer simulation modeling method for such systems reliability assessment is proposed.

2. System operation process

We assume that a system during its operation at the fixed moment t, $t \in \langle 0,+\infty \rangle$, may be at one of two different operations states z_b, $b = 1,2$. Consequently, we mark by $Z(t)$, $t \in \langle 0,+\infty \rangle$, the system operation process, that is a function of a continuous variable t, taking discrete values at the set $\{z_1, z_2\}$ of the system operation states. We assume a semi-Markov model [2], [4] of the system operation process $Z(t)$ and we mark by Θ_{bl} its random conditional sojourn times at the operation states z_b, when its next operation state is z_l, $b, l = 1,2$, $b \neq l$.

Consequently, the operation process may be described by the following parameters [4]:

- the vector $[p_b(0)]_{1\times 2}$, $b = 1,2$, of the initial probabilities of the system operation process $Z(t)$ staying at the particular operation states at the moment $t = 0$;
- the matrix $[p_{bl}]_{2\times 2}$ of the probabilities of the system operation process $Z(t)$ transitions between the operation states z_b and z_l, $b, l = 1,2$, $b \neq l$;

- the matrix $[H_{bl}(t)]_{2\times 2}$ of the conditional distribution functions of the system operation process $Z(t)$ conditional sojourn times Θ_{bl} at the operation states, $b,l=1,2,\ b\neq l$.

We mark by

$$\phi_1^{(n)}(t)=P(\Theta_1^{(n)}<t),\ t\in\langle 0,\infty),\ n=1,2,...,$$

the distribution functions of the random variables

$$\Theta_1^{(n)}=\Theta_{12}^{(1)}+\Theta_{12}^{(2)}+...+\Theta_{12}^{(n)},\ n=1,2,...,$$

where the variables $\Theta_{12}^{(i)},\ i=1,2,...,n,$ are independent random variables having identical distribution functions with the distribution of the sojourn time Θ_{12}, i.e.

$$P(\Theta_{12}^{(i)}<t)=P(\Theta_{12}<t)=H_{12}(t),\ \ i=1,2,...,n,$$

and by

$$\phi_2^{(n)}(t)=P(\Theta_2^{(n)}<t),\ t\in\langle 0,\infty),\ n=1,2,...,$$

the distribution functions of the random variables

$$\Theta_2^{(n)}=\Theta_{21}^{(1)}+\Theta_{21}^{(2)}+...+\Theta_{21}^{(n)},\ n=1,2,...,$$

where the variables $\Theta_{21}^{(i)},\ i=1,2,...,n,$ are independent random variables having identical distribution functions with the distribution of the sojourn time Θ_{21}, i.e.

$$P(\Theta_{21}^{(i)}<t)=P(\Theta_{21}<t)=H_{21}(t),\ \ i=1,2,...,n.$$

Realizations $\theta_{12}^{(i)}$ and $\theta_{21}^{(i)}$ of the random variables $\Theta_{12}^{(i)}$ and $\Theta_{21}^{(i)},\ i=1,2,...,$ are illustrated in *Figure 1*.

Consequently, we get

$$\phi_1^{(1)}(t)=H_{12}(t),$$
$$\phi_1^{(n)}(t)=\int_0^t\phi_1^{(n-1)}(t-u)dH_{12}(u),\ n=2,3,...,$$
$$\phi_2^{(1)}(t)=H_{21}(t),$$
$$\phi_2^{(n)}(t)=\int_0^t\phi_2^{(n-1)}(t-u)dH_{21}(u),\ n=2,3,....$$

The following auxiliary theorems are obvious [8].

Lemma 1. If the distribution of sojourn times Θ_{bl}, $b,l=1,2,\ b\neq l$, is exponential of the form

$$H_{bl}(t)=1-\exp[-\alpha_{bl}t],\ t\in\langle 0,\infty), \tag{1}$$

then the random variable $\Theta_b^{(n)}$ have Erlang distribution of order *n*, i.e.

$$\phi_b^{(n)}(t)=\int_0^t\frac{\alpha_{bl}^n t^{n-1}}{(n-1)!}\exp[-\alpha_{bl}t]\,dt$$
$$=1-\sum_{i=0}^{n-1}\frac{\alpha_{bl}^i t^i}{(i)!}\exp[-\alpha_{bl}t],\ t\in\langle 0,\infty), \tag{2}$$
$$b,l=1,2,\ \ b\neq l.$$

Lemma 2. If the distribution of sojourn times Θ_{bl} is normal with the parameters m_{bl},σ_{bl}, $b,l=1,2,$ $b\neq l$, i.e.

$$H_{bl}(t)=F_{N(m_{bl},\sigma_{bl})}(t)=\frac{1}{\sigma_{bl}\sqrt{2\pi}}\int_0^t e^{-\frac{(t-m_{bl})^2}{2\sigma_{bl}}}dt,$$
$$t\in\langle 0,\infty), \tag{3}$$

then the random variable $\Theta_b^{(n)}$ have normal distribution with parameters $n\cdot m_{bl},\sqrt{n}\cdot\sigma_{bl}$, i.e.

$$\phi_b^{(n)}(t)=F_{N(n\cdot m_{bl},\sqrt{n}\cdot\sigma_{bl})}(t),\ t\in\langle 0,\infty). \tag{4}$$

Lemma 3. If the distribution of sojourn times Θ_{bl} is uniform with the parameters x_{bl},y_{bl}, $b,l=1,2,$ $b\neq l$, i.e.

$$H_{bl}(t)=\frac{t-x_{bl}}{y_{bl}-x_{bl}},\ t\in\langle x_{bl},y_{bl}\rangle, \tag{5}$$

then the random variable $\Theta_b^{(n)}$ have the distribution given below [11]

$$\phi_b^{(n)}(t)=\xi(n)\int_0^t\sum_{i=0}^{\tilde{n}(n,t)}(-1)^i\binom{n}{i}\left(t-n\cdot x_{bl}-i(y_{bl}-x_{bl})\right)^{n-1}dt,$$
$$t\in\langle x_{bl},y_{bl}\rangle, \tag{6}$$

where

$$\xi(n)=\frac{1}{(n-1)!(y_{bl}-x_{bl})^n},$$

$$\tilde{n}(n,t) := \left\lceil \frac{t-n}{y_{bl} - x_{bl}} \right\rceil,$$

whereas $\lceil \cdot \rceil$ denotes the ceiling function. Further, we mark by

$$\psi^{(n)}(t) = P(\Theta^{(n)} < t), t \in \langle 0,\infty), n = 1,2,...,$$

the distribution functions of the random variables

$$\Theta^{(n)} = \Theta_1^{(n)} + \Theta_2^{(n)}, n = 1,2,... . \quad (7)$$

and we have

$$\psi^{(n)}(t) = \int_0^t \phi_1^{(n)}(t-u) d\phi_2^{(n)}(u), t \in \langle 0,\infty), \quad (8)$$
$$n = 1,2,... .$$

From *Lemmas 1 – 3* we get the following Theorems.

Theorem 1. If the distribution of sojourn times Θ_{12}, Θ_{21}, are exponential of the form (1), then the distribution function of the random variables $\Theta^{(n)}$ defined by (7) is i.e.

$$\psi^{(n)}(t) = \frac{e^{-(n-1)(\alpha_{12}+\alpha_{21})t} (\varsigma(t))^{n-2} \kappa(t)}{(\alpha_{12}\alpha_{21})^{n-1} (\alpha_{12} - \alpha_{21})^{n+1}}, t \in \langle 0,\infty), \quad (9)$$

where

$$\varsigma(t) = e^{\alpha_{12}t} \alpha_{12}^2 - e^{\alpha_{21}t} \alpha_{21}^2 + e^{(\alpha_{12}+\alpha_{21})t} (\alpha_{12} - \alpha_{21})(-\alpha_{21} + \alpha_{12}(\alpha_{21}t - 1)) \cdot$$

$$\kappa(t) = -e^{\alpha_{21}t} \alpha_{21}^3 (-\alpha_{12}^2 t + 2\alpha_{21} + \alpha_{12}(-4 + \alpha_{21}t)) + e^{(\alpha_{12}+\alpha_{21})t} (\alpha_{12} - \alpha_{21})^3 (-2\alpha_{21} + \alpha_{12}(\alpha_{21}t - 2)) + e^{\alpha_{12}t} \alpha_{12}^3 (\alpha_{12}(\alpha_{21}t + 2) - \alpha_{21}(\alpha_{21}t + 4)).$$

Proof. Since by the distribution functions $H_{12}(t)$ and $H_{21}(t)$ of Θ_{12} and Θ_{21} respectively are given by (1) then the convolution of $H_{12}(t)$ and $H_{21}(t)$ is given by

$$\int_0^t H_{12}(t-u) dH_{21}(u) = 1 + \frac{\alpha_{12} e^{-\alpha_{21}t} - \alpha_{21} e^{-\alpha_{12}t}}{\alpha_{21} - \alpha_{12}}.$$

Therefore, according to (8), the distribution function of $\Theta^{(n)}$ is the n^{th}-fold convolution of the form (9).

Theorem 2. If the distribution of sojourn times Θ_{12}, Θ_{21}, are normal of the form (3), then the distribution function of the random variables $\Theta^{(n)}$ defined by (7) is normal with parameters $n \cdot (m_{12} + m_{21})$, $\sqrt{n \cdot (\sigma_{12}^2 + \sigma_{21}^2)}$, i.e.

$$\psi^{(n)}(t) = F_{N(n(m_{12}+m_{21}), \sqrt{n(\sigma_{12}+\sigma_{21})})}(t), t \in \langle 0,\infty),$$

Theorem 3. If the distribution of sojourn times Θ_{12}, Θ_{21}, are uniform of the form (5), then the distribution function of the random variables $\Theta^{(n)}$ defined by (7) is given by (8), where $\phi_1^{(n)}(t)$ and $\phi_2^{(n)}(t)$ are defined by (6).

If we denote by $N(t)$ the number of changes of the system operation process' states before the moment t, by $N_b(t)$, $b = 1,2,$ the number of changes of the system operation process' states before the moment t when its operation process at the initial moment $t = 0$ was in the operation state z_b, $b = 1,2,$ for $t \in \langle 0, \infty),$ we immediately get the following results [8].

Theorem 4. The distribution of the number $N_b(t)$, $b = 1,2,$ $b \neq l$, of changes of the system operation process' states before the moment t, $t \in \langle 0, \infty)$, is given by

$$P(N_1(t) = 2n) = \psi^{(n)}(t) \left(1 - \int_0^t \psi^{(n)}(t-u) dH_{12}(u)\right), \quad (10)$$

$$P(N_1(t) = 2n+1) = \int_0^t \psi^{(n)}(t-u) dH_{12}(u) \cdot \left(1 - \int_0^t \psi^{(n+1)}(t-u) dH_{21}(u)\right) \quad (11)$$

$$P(N_2(t) = 2n) = \psi^{(n)}(t) \cdot \left(1 - \int_0^t \psi^{(n)}(t-u) dH_{21}(u)\right), \quad (12)$$

$$P(N_2(t) = 2n+1) = \int_0^t \psi^{(n)}(t-u) dH_{21}(u) \cdot \left(1 - \int_0^t \psi^{(n+1)}(t-u) dH_{12}(u)\right), \quad (13)$$

for $t \in \langle 0,\infty)$, $n = 0,1,2,...,$ where $\psi^{(0)}(t) = 1$ and $\psi^{(n)}(t)$ for $n = 1,2,...,$ are determined by (7).

Proof. For $n = 0$, we get

$$P(N_1(t)=0)=P(\Theta_{12}^{(1)}\ge t)=1-P(\Theta_{12}^{(1)}<t)$$
$$=1-H_{12}(t),$$

$$P(N_2(t)=0)=P(\Theta_{21}^{(1)}\ge t)=1-P(\Theta_{21}^{(1)}<t)$$
$$=1-H_{21}(t),$$

which after considering that $\psi^{(0)}(t)=1$, is consistent with (10) and (12) for $n=0$.
Generally, for $n=1,2,...$, we have

$$P(N_1(t)=2n)=P\{\Theta^{(n)}<t\cap\Theta^{(n)}+\Theta_{12}^{(2)}\ge t\}$$
$$=P(\Theta^{(n)}<t)\cdot P(\Theta^{(n)}+\Theta_{12}^{(2)}\ge t\,|\,\Theta^{(n)}<t)$$
$$=\psi^{(n)}(t)\left(1-\int_0^t\psi^{(n)}(t-u)dH_{12}(u)\right),$$

$$P(N_2(t)=2n)\ =P\{\Theta^{(n)}<t\cap\Theta^{(n)}+\Theta_{21}^{(n+1)}\ge t\}$$
$$=P(\Theta^{(n)}<t)\cdot P(\Theta^{(n)}+\Theta_{21}^{(n+1)}\ge t\,|\,\Theta^{(n)}<t)$$
$$=\psi^{(n)}(t)\cdot\left(1-\int_0^t\psi^{(n)}(t-u)dH_{21}(u)\right),$$

which also is consistent with (10) and (12).
Moreover, we get

$$P(N_1(t)=1)=P\{\Theta_{12}^{(1)}<t\cap\Theta_{12}^{(1)}+\Theta_{21}^{(1)}\ge t\}$$
$$=P(\Theta_{12}^{(1)}<t)\cdot P(\Theta_{12}^{(1)}+\Theta_{21}^{(1)}\ge t\,|\,\Theta_{12}^{(1)}<t)$$
$$=H_{12}(t)\cdot\left(1-\int_0^t\psi^{(1)}(t-u)dH_{21}(u)\right),$$

$$P(N_2(t)=1)=P\{\Theta_{21}^{(1)}<t\cap\Theta_{21}^{(1)}+\Theta_{12}^{(1)}\ge t\}$$
$$=P(\Theta_{21}^{(1)}<t)\cdot P(\Theta_{21}^{(1)}+\Theta_{12}^{(1)}\ge t\,|\,\Theta_{21}^{(1)}<t)$$
$$=H_{21}(t)\cdot\left(1-\int_0^t\psi^{(1)}(t-u)dH_{12}(u)\right)$$

which after considering that $\psi^{(0)}(t)=1$, is consistent with (11) and (13) for $n=0$.
Generally, for $n=1,2,...$, we have

$$P(N_1(t)=2n+1)\ =P\{\Theta^{(n)}+\Theta_{12}^{(2)}<t\cap\Theta^{(n+1)}\ge t\}$$
$$=P(\Theta^{(n)}+\Theta_{12}^{(n+1)}<t)\cdot P(\Theta^{(n+1)}\ge t\,|\,\Theta^{(n)}+\Theta_{12}^{(2)}<t)$$
$$=\int_0^t\psi^{(n)}(t-u)dH_{12}(u)\cdot\left(1-\int_0^t\psi^{(n+1)}(t-u)dH_{21}(u)\right)$$

$$P(N_2(t)=2n+1)=P\{\Theta^{(n)}+\Theta_{12}^{(n+1)}<t\cap\Theta^{(n+1)}\ge t\}$$
$$=P(\Theta^{(n)}+\Theta_{21}^{(n+1)}<t)\cdot P(\Theta^{(n+1)}\ge t\,|\,\Theta^{(n)}+\Theta_{21}^{(2)}<t)$$
$$=\int_0^t\psi^{(n)}(t-u)dH_{21}(u)\cdot\left(1-\int_0^t\psi^{(n+1)}(t-u)dH_{12}(u)\right)$$

which is consistent with (11) and (13). This completes the proof. □

From *Theorem 4*, we get the following result.

Corollary 1. The distribution of the number $N(t)$ of changes of the system operation process' states before the moment t, $t\in\langle 0,\infty)$, are given by

$$P(N(t)=2n)=$$
$$\psi^{(n)}(t)\Big[p_1(0)\cdot\left(1-\int_0^t\psi^{(n)}(t-u)dH_{12}(u)\right)$$
$$+p_2(0)\cdot\left(1-\int_0^t\psi^{(n)}(t-u)dH_{21}(u)\right)\Big],\quad(14)$$

$$P(N(t)=2n+1)=p_1(0)\cdot\int_0^t\psi^{(n)}(t-u)dH_{12}(u)$$
$$\cdot\left(1-\int_0^t\psi^{(n+1)}(t-u)dH_{21}(u)\right)$$
$$+p_2(0)\cdot\int_0^t\psi^{(n)}(t-u)dH_{21}(u)$$
$$\cdot\left(1-\int_0^t\psi^{(n+1)}(t-u)dH_{12}(u)\right)\quad(15)$$

for $t\in\langle 0,\infty)$, $n=0,1,2,...$, where $\psi^{(0)}(t)=1$ and $\psi^{(n)}(t)$ for $n=1,2,...$, are determined by (7).

3. Reliability of system undergoing a two-state operation process

We assume that the considered two-state system reliability depends on its operation state it is operating and on the number of changes of the operation process states. We define the system conditional reliability function at the operation state z_b, $b=1,2$, after k, $k=0,1,...$, changes of its operation process states

$$R_k^{(b)}(t)=P(T_k^{(b)}>t),\quad(16)$$

for $t\in\langle 0,\infty)$, where $T_k^{(b)}, b=1,2$, $k=0,1,...$, is the lifetimes of the system at the operation state z_b, after k changes of its operation process states with the conditional distribution functions

$$F_k^{(b)}(t)=P(T_k^{(b)}\le t)\ =1-R_k^{(b)}(t),\quad(17)$$

for $t \in \langle 0, \infty)$. Under those assumptions, we want to find the unconditional reliability function of the system subjected to two-state operation process

$$\boldsymbol{R}(t) = P(T > t),\ t \in \langle 0, \infty),$$

where T is the unconditional lifetime of the system with the unconditional distribution function

$$\boldsymbol{F}(t) = P(T \le t),\ t \in \langle 0, \infty).$$

3.1. Analytical approach to system reliability evaluation

The application of the formula for total probability and *Corollary 1* results in the following proposition.

Theorem 5. The unconditional reliability function of the system subjected to two-state operation process is given by

$$\boldsymbol{R}(t) = \sum_{k=0}^{\infty} P(N(t) = k) R_k^{(b)}(t),$$

for $t \in \langle 0, \infty)$, $k = 0,1,...,$ $b = 1,2,$ where the distribution $P(N(t) = k)$, $t \in \langle 0, \infty)$, $k = 0,1,...,$ is determined by (14)-(15) and $R_k^{(b)}(t)$, $t \in \langle 0, \infty)$, $b = 1,2,$ $k = 0,1,...,$ are the conditional reliability functions of the system defined by (16).

Its particular case for the Weibull conditional reliability functions is as follows.

Corollary 2. If the conditional reliability functions of the system subjected to two-state operation process are

$$R_k^{(b)}(t) = \exp[-\lambda_k^{(b)} t^{\beta_k^{(b)}}], \tag{24}$$

for $t \in \langle 0, \infty)$, $k = 0,1,...,$ $b = 1,2,$ then the unconditional reliability function of the system subjected to two-state operation process is given by

$$\boldsymbol{R}(t) = \sum_{k=0}^{\infty} P(N(t) = k) \exp[-\lambda_k^{(b)} t^{\beta_k^{(b)}}],$$

for $t \in \langle 0, \infty)$, $k = 0,1,...,$ $b = 1,2,$ where the distribution $P(N(t) = k)$, $t \in \langle 0, \infty)$, $k = 0,1,...,$ is determined by (14)-(15).

Remark 1. If we put $\beta_k^{(b)} = 1$, $k = 0,1,..., b = 1,2,$ in (24), we get appropriate result for the exponential distribution.

3.2. Monte Carlo approach to system reliability evaluation

We denote by $\theta_{bl}^{(i)}$, $b, l = 1,2, b \ne l$, $i = 1,2,...,\frac{N}{2}$, the realizations of the conditional sojourn times Θ_{bl} of the object operation process generated from the distribution H_{bl}, where N is the even number of the system lifetime realizations. Those realizations can be generated according to the formulae

$$\theta_{bl}^{(i)} = H_{bl}^{-1}(h),\ b, l = 1,2, b \ne l,\ i = 1,2,...,\frac{N}{2}, \tag{19}$$

where $H_{bl}^{-1}(h)$ is the inverse function of the distribution function $H_{bl}(t)$ and h is a randomly generated number from the uniform distribution on the interval $\langle 0, 1 \rangle$.
For the considered distributions (1), (5), the formula (19) takes respectively the following forms:
- for the exponential distribution

$$\theta_{bl}^{(i)} = -\frac{1}{\alpha_{bl}} \ln(1-h),\ b, l = 1,2, b \ne l, i = 1,2,...,\frac{N}{2};$$

- for the uniform distribution

$$\theta_{bl}^{(i)} = h(b_{bl} - a_{bl}) + a_{bl},\ b, l = 1,2, b \ne l,$$
$$i = 1,2,...,\frac{N}{2}.$$

In the case of normal distribution defined by (3) instead of using (19), we generate two random numbers h_1 and h_2 from the uniform distribution on the unit interval and call one of the predefined functions [2]

$$\theta_{bl}^{(i)} = \sigma_{bl} \sin(2\pi h_2)\sqrt{-2\ln(1-h_1)} + m_{bl},$$
$$b, l = 1,2, b \ne l, i = 1,2,...,\frac{N}{2}.$$

Realizations of the considered system operation process are presented in *Figure 1*.

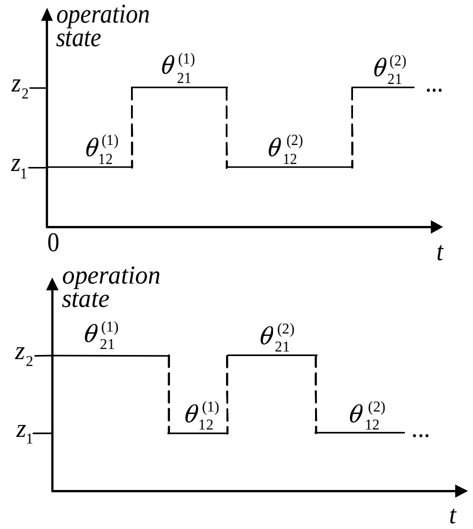

Figure 1. Realizations of the system operation process

The realizations of the object conditional lifetimes $t_k^{(b)}$, $b=1,2$, $k=0,1,...$, are generated according to the distribution (17), i.e. they are generated by the sampling formula

$$t_k^{(b)} = \left(F_k^{(b)}(f)\right)^{-1} = \left(1-R_k^{(b)}(f)\right)^{-1},$$

where $\left(F_k^{(b)}(f)\right)^{-1}$ is the inverse function of the distribution function $F_k^{(b)}(t)$ of the object conditional lifetime $T_k^{(b)}, b=1,2,\ k=0,1,...,$ defined by (17).

In the case of Weibull distribution, according to (24), we have

$$F_k^{(b)}(t) = 1-\exp[-\lambda_k^{(b)} t^{\beta_k^{(b)}}], \quad (20)$$

for $t \geq 0$, $b=1,2,\ k=0,1,...$, and the realizations of the system conditional lifetimes take the form

$$t_k^{(b)} = \left(-\frac{1}{\lambda_k^{(b)}}\ln(1-f)\right)^{\frac{1}{\beta_k^{(b)}}},\ b=1,2,\ k=0,1,... . \quad (21)$$

where $\lambda_k^{(b)}$, $\beta_k^{(b)}$, $b=1,2,\ k=0,1,...$, are the Weibull distribution parameters existing in (24) and (20) and f is a randomly generated number from the uniform distribution on the interval $\langle 0,1 \rangle$.

In the case of exponential distribution, the formula (20) takes the form

$$F_k^{(b)}(t) = 1-\exp[-\lambda_k^{(b)} t], \quad (22)$$

for $t \geq 0$, $b=1,2,\ k=0,1,...$, and the realizations of the system conditional lifetimes take the form

$$t_k^{(b)} = -\frac{1}{\lambda_k^{(b)}}\ln(1-f),\ b=1,2,\ k=0,1,... .$$

where $\lambda_k^{(b)}$, $b=1,2,\ k=0,1,...$, are the failure rates existing in (22) and f is a randomly generated number from the uniform distribution on the interval $\langle 0,1 \rangle$.

3.2.1. The procedure of Monte Carlo simulation application to system reliability characteristics determination

We can apply the Monte Carlo simulation method based on the result of *Corollary 2*, according to a general Monte Carlo simulation scheme presented in *Figure 2*.

At the beginning, we fix the following parameters:

- the number $N \in \boldsymbol{N} \setminus \{0\}$ of iterations (runs of the simulation);
- the vector of the initial probabilities $[p_b(0)]$, $b=1,2,$ of the system operation process $Z(t)$ states at the moment $t=0$ defined in Section 2;
- the matrix of the probabilities $[p_{bl}]$, $b,l=1,2,$ $b \neq l$ of the system operation process $Z(t)$ transitions between the various system operation states defined in Section 2;

Next, in the program, we define functions $\theta_{bl}^{(i)}$ and $t_k^{(b)}$, $b,l=1,2, b\neq l, i=1,2,...,\frac{N}{2}$, according to (6)-(11).

Further we introduce:

- $j \in \boldsymbol{N}$ as the subsequent iteration in the main loop and set $j=1$;
- $k \in \boldsymbol{N}$ as the number of system operation process states changes and set $k=0$;
- $t_j \in \langle 0,\infty)$, $j=1,2,...,N$ as the system unconditional lifetime realization and set $t_j=0$.

As the algorithm progresses, we draw a random number q from the uniform distribution on the interval $\langle 0,1 \rangle$. Based on this random value, the realization $z_b(q)$, $b=1,2,$ of the system operation process initial operation state at the moment $t=0$ is generated according to the formula

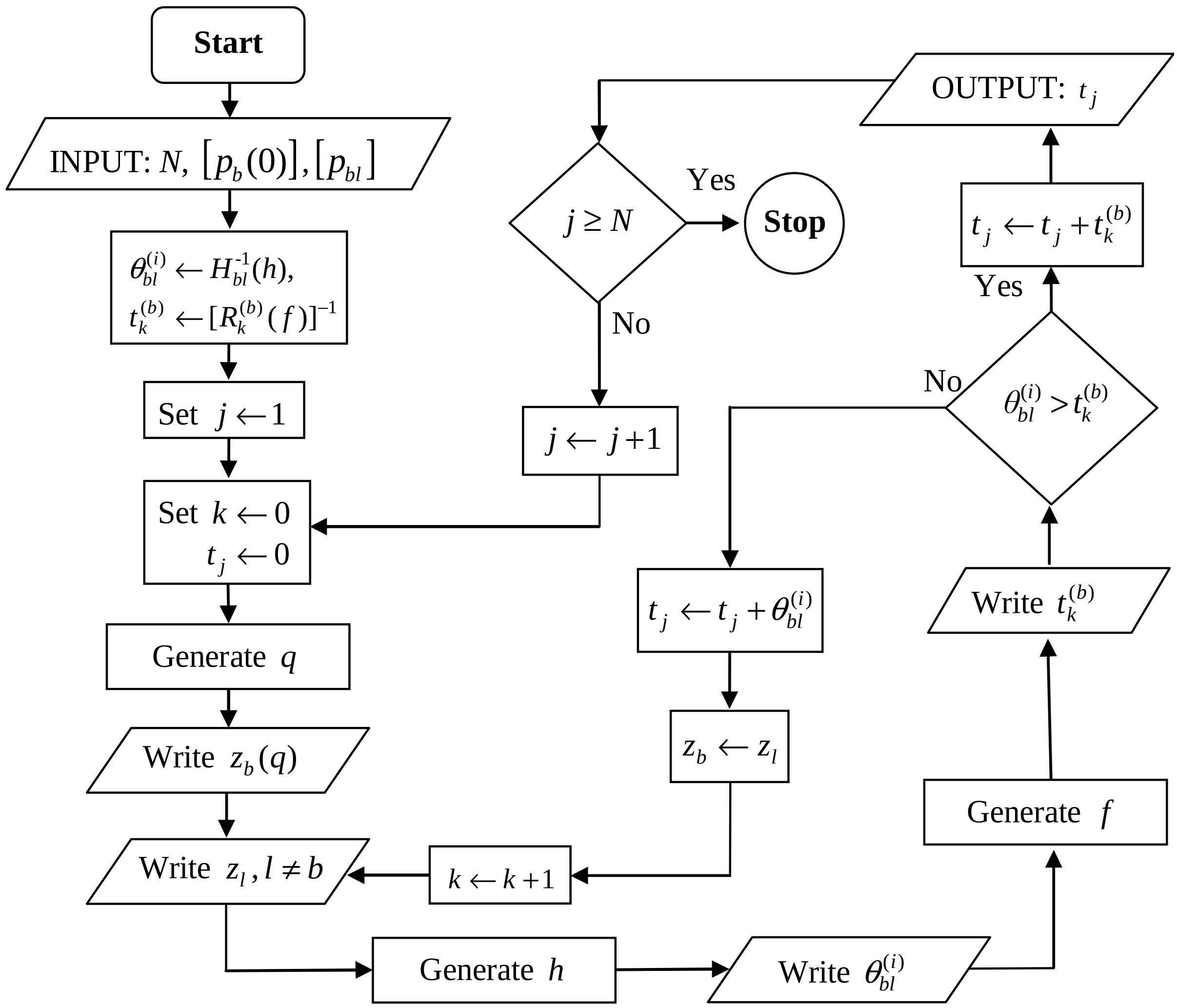

Figure 2. Monte Carlo algorithm for a system reliability evaluation

$$z_b(q) = \begin{cases} z_1, & 0 \le q < p_1(0), \\ z_2, & p_1(0) \le q < 1. \end{cases}.$$

In the next step, the realization z_l, $l = 1,2, l \neq b$, of the system operation process consecutive operation state is generated. If $b = 1$, then $z_l = z_2$, else $z_l = z_1$.

Further, we generate a random number h from the uniform distribution on the interval $\langle 0,1 \rangle$, on which we put into the formula (6) obtaining the realization $\theta_{bl}^{(i)}$, $b, l = 1,2, b \neq l, i = 1,2,..., n,$. Subsequently, we generate a random number f uniformly distributed on the interval $\langle 0,1 \rangle$, which we put into the formula (11) obtaining the realization $t_k^{(b)}$, $b = 1,2$, $k = 0,1,...$.

If the realization of the empirical conditional sojourn time is not greater than the realization of the system conditional lifetime, we add to the system unconditional lifetime realization t_j the value $\theta_{bl}^{(i)}$. The realization t_j is recorded, z_l is set as the initial operation state and $k = 1$.

We generate another random numbers g, h, f from the uniform distribution on the interval $\langle 0,1 \rangle$ obtaining the realizations z_l, $\theta_{bl}^{(i)}$ and $t_k^{(b)}$, $b, l = 1,2$, $b \neq l$. Each time we compare the realization of the conditional sojourn time $\theta_{bl}^{(i)}$ with the realization of the system conditional lifetime $t_k^{(b)}$. If $\theta_{bl}^{(i)}$ is greater than $t_k^{(b)}$, we add to the sum of the realizations of the conditional sojourn times $\theta_{bl}^{(i)}$ the realization $t_k^{(b)}$ and we obtain and record an system unconditional lifetime realization t_j. Thus, we can proceed replacing j with $j+1$ and shift into the next iteration in the loop if $j < N$. In the other case, we stop the procedure.

Using the above procedure, the histogram of the system unconditional lifetime can be found and the empirical mean value and the standard deviation of the system unconditional lifetime can be calculated according to the formulae

$$\bar{T} = \frac{1}{N}\sum_{j=1}^{N} t_j \,, \qquad (23)$$

$$\bar{\sigma} = \sqrt{\frac{\sum_{j=1}^{N}\left(t_j - \bar{T}\right)^2}{N}} \,, \qquad (24)$$

where N is the number of the system lifetime realizations and t_j, $j = 1,2,...,N$, are the object unconditional lifetime realizations.

The input data for the system operation process are:

- the vector of the initial probabilities of the system operation process $Z(t)$ staying at the particular operation states at the moment $t = 0$

$$[\,p_b(0)]_{1\times 2} = [0.4, 0.6]\,; \qquad (25)$$

- the matrix of the probabilities of the system operation process $Z(t)$ transitions between the operation states

$$[\,p_{bl}\,]_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

- the matrix of the conditional distribution functions of the system operation process $Z(t)$ sojourn times $\theta_{bl}^{(i)}$, $b, l = 1,2, b \neq l, i = 1,2,...,n$ at the operation states is given in *Table 1*.

The input data for the system reliability are given in *Table 2*.

For the cases considered in *Table 1* and *Table 2*, the realizations t_j, $j = 1,2,...,N$, for $N = 1000000$ of the system unconditional lifetimes *T* are illustrated in the form of the histograms are presented in *Figures 3-10* and their empirical mean values and standard deviations calculated according to (23) and (24) are given in *Table 3*.

Table 1. Exemplary conditional distribution functions of sojourn times of operation process

Case	Name	CDF	Parameters
$O1$	Exponential distribution	$H_{bl}(t) = 1 - \exp[-\alpha_{bl} t]$, $t \in \langle 0, \infty), b, l = 1, 2$	$\alpha_{12} = 290$, $\alpha_{21} = 71$
$O2$	Normal distribution	$H_{bl}(t) = F_{N(m_{bl}, \sigma_{bl})}(t)$, $t \in \langle 0, \infty), b, l = 1, 2$	$m_{12} = 290$, $\sigma_{12} = 10$, $m_{21} = 71$, $\sigma_{21} = 5$
$O3$	Focused uniform distribution	$H_{bl}(t) = \frac{t - x_{bl}}{y_{bl} - x_{bl}}$, $t \in \langle x_{bl}, y_{bl} \rangle, b, l = 1, 2$	$x_{12} = 270$, $y_{12} = 305$, $x_{21} = 62$, $y_{12} = 80$
$O4$	Stretch uniform distribution	$H_{bl}(t) = \frac{t - a_{bl}}{b_{bl} - a_{bl}}$, $t \in \langle x_{bl}, y_{bl} \rangle, b, l = 1, 2$	$x_{12} = 0$, $y_{12} = 575$, $x_{21} = 0$, $y_{12} = 142$

Table 2. Exemplary conditional reliability functions

Case	Name	Reliability functions	Parameters
$R1$	Exponential distribution	$R_k^{(b)}(t) = \exp[-\lambda_k^{(b)} t]$, $t \in \langle 0, \infty), b, l = 1, 2, \; k = 0,1,2,...$.	$\lambda_k^{(1)} = 0.00206667\,\frac{2k+1}{k+1}$, $\lambda_k^{(2)} = 0.00144001\frac{2k+1}{k+1}$
$R2$	Weibull distribution	$R_k^{(b)}(t) = \exp[-\lambda_k^{(b)} t^{\beta_k^{(b)}}]$, , $t \in \langle 0, \infty), b, l = 1, 2, \; k = 0,1,2,...$.	$\lambda_k^{(1)} = 0.00000335\left(\frac{2k+1}{k+1}\right)^2$, $\lambda_k^{(2)} = 0.00000163\left(\frac{2k+1}{k+1}\right)^2$, $\beta_k^{(1)} = \beta_k^{(2)} = 2$

On the basis of that results, it is possible to try to formulate and to verify the hypotheses on the forms of the system reliability functions. Unfortunately, fixing the system reliability functions in most cases is not successful.

4. Conclusion

The discussed problem seems to be justified by practice because of the natural omitting the usually used strong assumption on the exponentiality of the system reliability functions at the particular operation states. This change sensibility is evident and its influence on the results is clearly showed in the form of the histograms and numerical characteristics presented in the paper. Both the analytical method and the simulation method should be modified and developed in the direction of more than two-state system operation processes to get results better fitting to real technical systems.

Table 3. The mean values and the standard deviations of the system unconditional lifetime

Case	Mean value [days]	Standard deviation [days]
$O1$ and $R1$	$\overline{T} \approx 350.912$	$\sigma \approx 308.114$
$O2$ and $R1$	$\overline{T} \approx 355{,}432$	$\sigma \approx 305.037$
$O3$ and $R1$	$\overline{T} \approx 355.502$	$\sigma \approx 305.786$
$O4$ and $R1$	$\overline{T} \approx 353.673$	$\sigma \approx 306.396$
$O1$ and $R2$	$\overline{T} \approx 541.649$	$\sigma \approx 355{,}998$
$O2$ and $R2$	$\overline{T} \approx 569.715$	$\sigma \approx 403.316$
$O3$ and $R2$	$\overline{T} \approx 568.648$	$\sigma \approx 402.084$
$O4$ and $R2$	$\overline{T} \approx 532.467$	$\sigma \approx 349.989$

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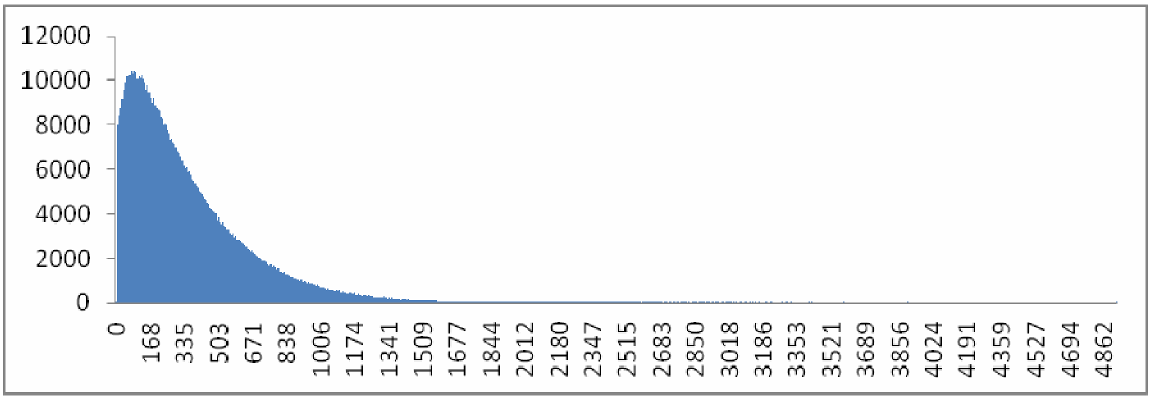

Figure 3. The graph of the histogram of the system lifetimes (in cases $O1$ and $R1$)

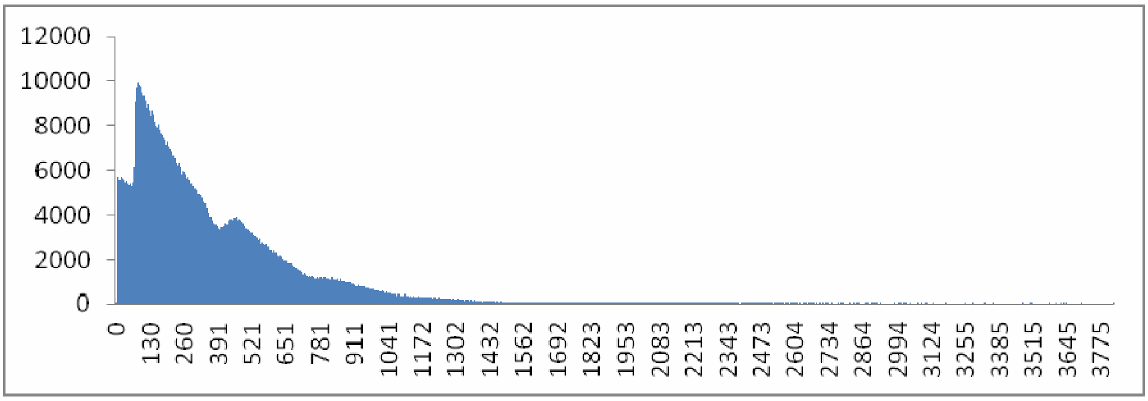

Figure 4. The graph of the histogram of the system lifetimes (in cases $O2$ and $R1$)

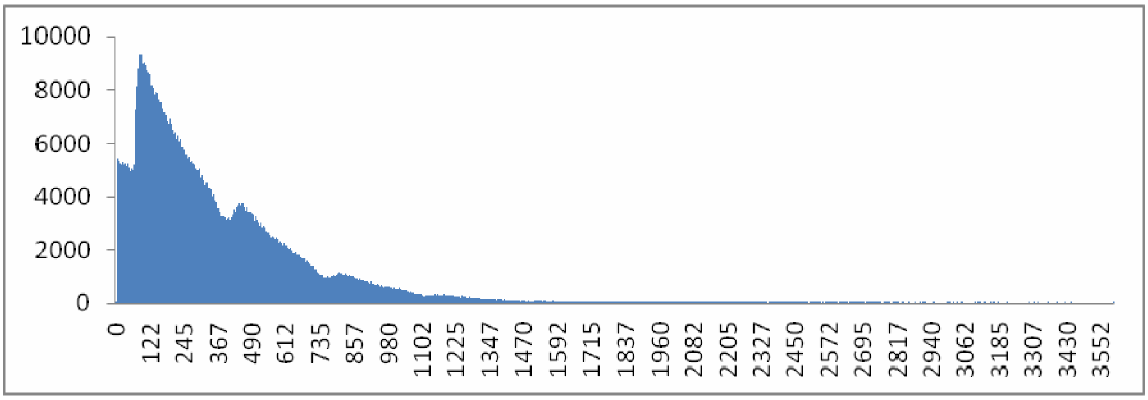

Figure 5. The graph of the histogram of the system lifetimes (in cases $O3$ and $R1$)

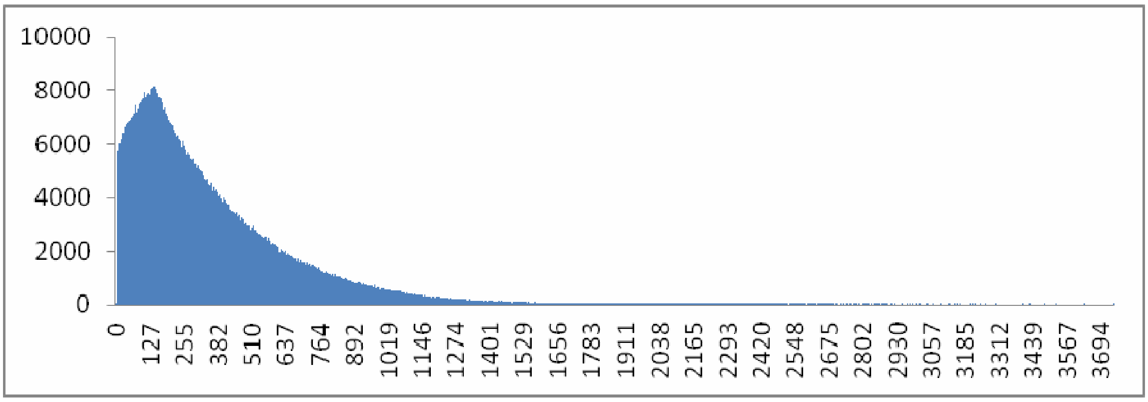

Figure 6. The graph of the histogram of the system lifetimes (in cases $O4$ and $R1$)

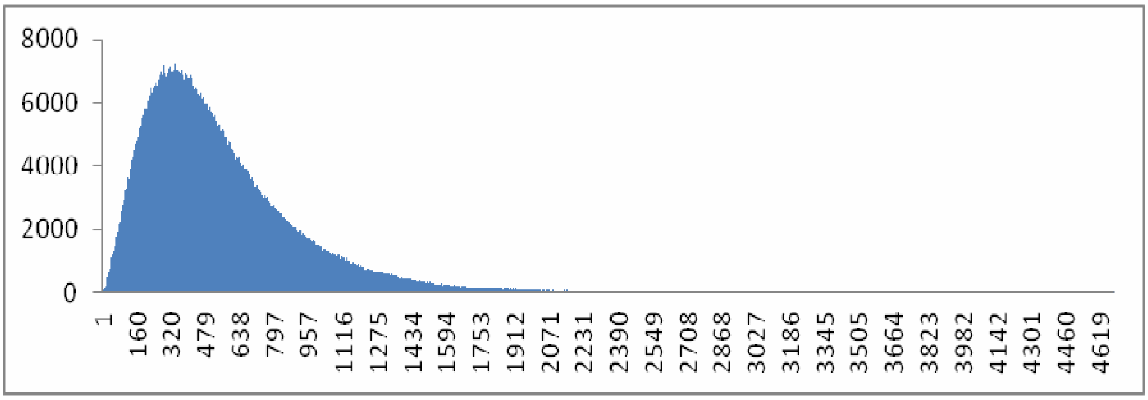

Figure 7. The graph of the histogram of the system lifetimes (in cases $O1$ and $R2$)

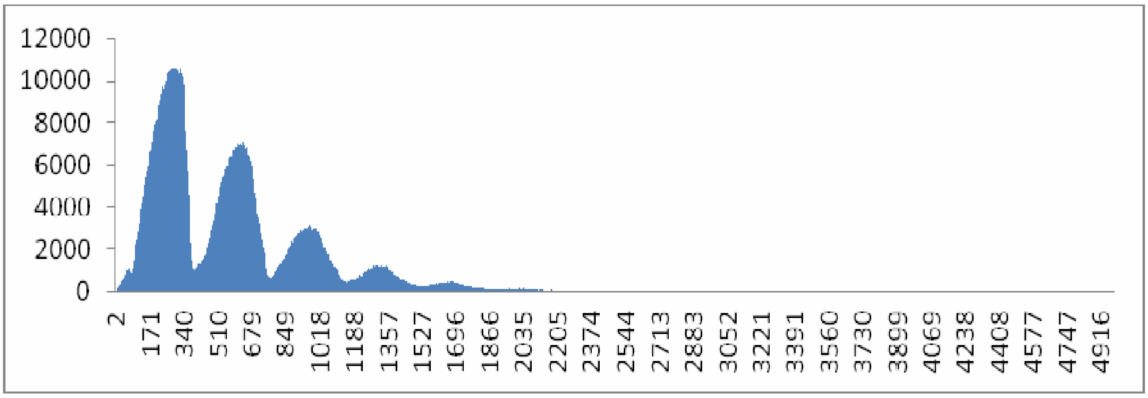

Figure 8. The graph of the histogram of the system lifetimes (in cases $O2$ and $R2$)

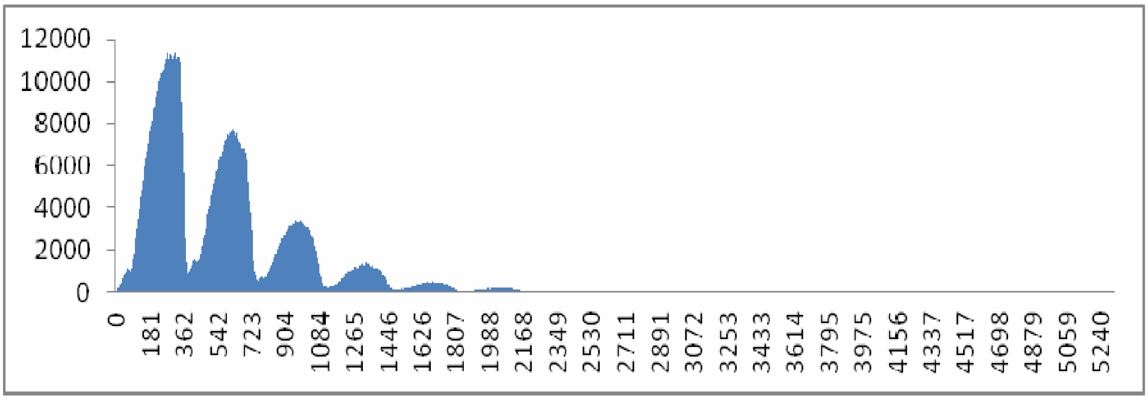

Figure 9. The graph of the histogram of the system lifetimes (in cases $O3$ and $R2$)

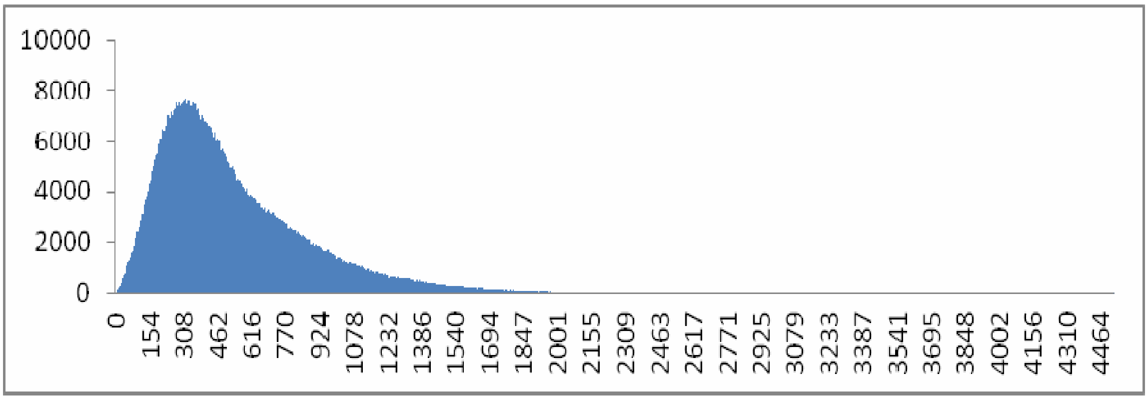

Figure 10. The graph of the histogram of the system lifetimes (in cases $O4$ and $R2$)