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# ALGEBRAIC FORMS OF SOLUTIONS OF LINEAR STATE EQUATIONS FOR DETERMINED CONTROL VECTORS

Despite intensive development of numerical methods of resolution of ordinary differential equations, we still seek analytical solutions of these equations in the algebraic form. The analytical solution in an algebraic form, is very convenient in case of investigation of the course of solution in a large interval of changes of parameters, it also gives more qualitative informations. This concerns first of all linear equations which are the most often met models of physical objects. Important advantage of these solutions is also their open form regarding established state of the system. The paper proves existence of algebraic forms of solutions of control vectors with components: exponential, sinusoidal with different circular frequencies and sinusoidal multiple. Also solutions for other kind of control vectors which are mentioned in the literature are presented.

KEYWORDS: state equation, solutions of state equations, algebraic forms

## 1. INTRODUCTION

In spite of considerable progress in the field of numerical methods of resolve differential equations, analytical methods are still developed and are essential in such fields as circuit theory, control theory or theoretical mechanics.

A large part of the basic theory of physical systems is based on foundation that such systems can be adequately described by means of linear differential equations with constant coefficients [1, 3, 4, 8]. This foundation is propper for many systems and leads usually to concepts and evaluations which at least qualitatively refer to the original, not reduced task. Major part of electrical circuit theories is based on this foundation.

Analytical methods give solutions in algebraic form without necessity of substitution numerical values of parameters in the course of calculations. Such solution makes possible to introduce numerical values and easily examine the effect of the change of value for some values. Consequently it is obvious that far more information about the system and in significantly shorter time can be obtained in case of analytical solution. Therefore the affords to obtain analytical solution are fully justified.

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Also for non-linear systems are developed methods of findings analytical solutions of equations describing these systems. Tools applied to this are theory-based methods of Lie groups transformation [2, 7]. This leads in many cases to a global decoupling and linearization of equations describing these systems.

The paper shows existence of algebraic forms of solutions for linear state equation for three control vectors with components: exponential, sinusoidal with different amplitudes, circular frequencies and phases, and also for sinusoidal multiple control for which every component of the control vector has sinusoidal components with different amplitudes, circular frequencies and phases. It presents also appearing in literature analytical solutions for control vectors with other components e.g. [1, 4, 6, 8]. The advantage of the algebraic forms is the open form of solution concerning the steady state of the system, what practically is not possible to obtain by the use of numerical methods, as well as the minimization of calculating errors.

## 2. ALGEBRAIC FORMS OF SOLUTIONS FOR DETERMINED CONTROL VECTORS

Solution of the linear state equation in the normal form [1, 4, 8]:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1}$$

for the control vector applied in the moment $t = t_0$ can be presented in a form of sum of solutions of the homogeneous equation (normal solution) $\boldsymbol{x}_S(t)$ independent from the extortion and solution coming from control vector (forced solution) $\boldsymbol{x}_w(t)$, i.e.:

$$\boldsymbol{x}(t) = \boldsymbol{x}_S(t) + \boldsymbol{x}_w(t) \tag{2}$$

where normal solution is determined as:

$$\boldsymbol{x}_S(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0 \tag{3}$$

and forced solution has a form:

$$\boldsymbol{x}_W(t) = \int_{t_0}^{t} \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\,\boldsymbol{u}(\tau)\,d\tau \tag{4}$$

Therefore it is essential to find analytical form of the forced component of solution of the equation state. Below we present the forced solutions for three kinds of control vector: exponential, sinusoidal with components having different amplitudes, circular frequencies and phases, and for sinusoidal of multiple for which every vector component of the control has sinusoidal components witht different amplitude, circular frequency and phase.

## 2.1. Exponential control

Vector of exponential control can be presented in the form:

$$\boldsymbol{u}(t) = [a_1\,\mathrm{e}^{b_1 t} \quad \cdots \quad a_i\,\mathrm{e}^{b_2 t} \quad \cdots \quad a_p\,\mathrm{e}^{b_p t}]^{\mathrm{T}} = \boldsymbol{a}\,\mathrm{e}^{\boldsymbol{b}t} \tag{5}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are $p$-dimensional vectors with real components $\boldsymbol{a} \in \Re^p$, $\boldsymbol{b} \in \Re^p$, in the form:

$$\boldsymbol{a} = diag[a_1, \ldots, a_i, \ldots, a_p], \quad \boldsymbol{b} = diag[b_1, \ldots, b_i, \ldots, b_p] \tag{6}$$

For the purpes of designation of the integral of solution of the state equation the control vector is recorded in form:

$$\boldsymbol{u}(t) = \sum_{i=1}^{p} u_i(t)\,\boldsymbol{n}_i \tag{7}$$

whereat $\boldsymbol{n}_i$ is an singular vector having $i$-th components equal one, and the remaining equal zero, and $u_i(t)$ is an expression

$$u_i(t) = a_i\,\mathrm{e}^{b_i t} \tag{8}$$

where $a_i, b_i$ are $i$-th components of vectors $\boldsymbol{a}$, $\boldsymbol{b}$. After substitution (5) and (6) to the general integral of solution we receive:

$$\mathrm{e}^{At}\int_{t_0}^{t} \mathrm{e}^{-A\tau}\boldsymbol{B}\left[\sum_{i=1}^{p} u_i(\tau)\,\boldsymbol{n}_i\right] d\tau = \mathrm{e}^{At}\sum_{i=1}^{p}\int_{t_0}^{t} \mathrm{e}^{-A\tau}\boldsymbol{B}\,\boldsymbol{n}_i\,u_i(\tau)\,d\tau \tag{9}$$

Easily we can prove equality:

$$\boldsymbol{B}\sum_{i=1}^{p} u_i(\tau)\,\boldsymbol{n}_i = \sum_{i=1}^{p} u_i(\tau)\,\boldsymbol{B}_i \tag{10}$$

where:

$$\boldsymbol{B} = [\boldsymbol{B}_1 | \ldots | \boldsymbol{B}_i | \ldots | \boldsymbol{B}_p] \tag{11}$$

Consequently expression (9) can be presented for $i$-th component in the form:

$$\mathrm{e}^{At}\int_{t_0}^{t} \mathrm{e}^{-A\tau}\boldsymbol{B}\,u_i(\tau)\,d\tau = \mathrm{e}^{At}\sum_{i=1}^{p}\left(\int_{t_0}^{t} \mathrm{e}^{-A\tau}u_i(\tau)\,d\tau\right)\boldsymbol{B}_i \tag{12}$$

As first we have to designate the integral in expression (12). Integrating by parts we receive:

$$\int_{t_0}^{t} \mathrm{e}^{-A\tau}u_i(\tau)\,d\tau = \int_{t_0}^{t} \mathrm{e}^{-A\tau}a_i\,\mathrm{e}^{b_i\tau}\,d\tau = a_i\,b_i^{-1}\,\mathrm{e}^{b_i\tau}\mathrm{e}^{-A\tau} + b_i^{-1}\boldsymbol{A}\int_{t_0}^{t} \mathrm{e}^{-A\tau}a_i\,\mathrm{e}^{b_i\tau}\,d\tau \tag{13}$$

Multiplying both sides (13) by $b_i$ and subtracting both integrals from each other in this expression we obtain:

$$[\boldsymbol{A}-\boldsymbol{I}b_i]\int_{t_0}^{t} \mathrm{e}^{-\boldsymbol{A}\tau} a_i\, \mathrm{e}^{b_i\,\tau}\, d\tau = -\,\mathrm{e}^{-\boldsymbol{A}\tau} a_i\, \mathrm{e}^{b_i\,\tau} \tag{14}$$

and consequently:

$$\int_{t_0}^{t} \mathrm{e}^{-\boldsymbol{A}\tau} a_i\, \mathrm{e}^{b_i\,\tau}\, d\tau = -\,a_i\, [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}\, \mathrm{e}^{-\boldsymbol{A}\tau}\, \mathrm{e}^{b_i\,\tau}\Big|_{t_0}^{t} \tag{15}$$

By putting the integral (15) to the expression (12), after executing the operation we receive:

$$\mathrm{e}^{\boldsymbol{A}t}\sum_{i=1}^{p}\left(\int_{t_0}^{t} \mathrm{e}^{-\boldsymbol{A}\tau} u_i(\tau)\, d\tau\right)\boldsymbol{B}_i = \mathrm{e}^{\boldsymbol{A}t}\sum_{i=1}^{p} a_i\, [\boldsymbol{A}-\boldsymbol{I}\,b_i]^{-1}\mathrm{e}^{-\boldsymbol{A}\tau}\, \mathrm{e}^{b_i\,\tau}\Big|_{t}^{t_0}\, \boldsymbol{B}_i =$$

$$= \mathrm{e}^{\boldsymbol{A}t}\sum_{i=1}^{p} a_i\, [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}(\mathrm{e}^{-\boldsymbol{A}t_0}\mathrm{e}^{b_i\,t_0} - \mathrm{e}^{-\boldsymbol{A}t}\mathrm{e}^{b_i\,t})\boldsymbol{B}_i \tag{16}$$

We can easily show the commutativity of the matrix [5]:

$$\mathrm{e}^{\boldsymbol{A}\tau}[\boldsymbol{A}-\boldsymbol{I}b_i]^{-1} = [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}\mathrm{e}^{-\boldsymbol{A}\tau} \tag{17}$$

therefore:

$$\mathrm{e}^{\boldsymbol{A}t}\sum_{i=1}^{p}\left(\int_{t_0}^{t} \mathrm{e}^{-\boldsymbol{A}\tau} u_i(\tau)\, d\tau\right)\boldsymbol{B}_i = \sum_{i=1}^{p} a_i\, [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}(\mathrm{e}^{\boldsymbol{A}(t-t_0)}\mathrm{e}^{b_i\,t_0} - \boldsymbol{I}\mathrm{e}^{b_i\,t})\boldsymbol{B}_i \tag{18}$$

Finally the forced component is in a form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\sum_{i=1}^{p} a_i\, [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}\boldsymbol{B}_i\, a_i\, \mathrm{e}^{b_i\,t_0} - \sum_{i=1}^{p}[\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}\boldsymbol{B}_i\, a_i\, \mathrm{e}^{b_i\,t} \tag{19}$$

The only condition of existence of the algebraic form of solution is, such that the matrix $[\boldsymbol{A}-\boldsymbol{I}\,b_i]^{-1}$ is nonsingular for each $i$-th component of control vector. This matrix will be always singular in case, when $i$-th exponent of the exponential control will be equal any value from among the eigenvalue of state matrices $\boldsymbol{A}$. Because $\boldsymbol{b}\in\Re^p$, therefore this concerns only real eigenvalues.

For the exponential excitation (5) the vector of forced solution can be presented in the form:

$$\boldsymbol{x}_w(t) = \sum_{i=1}^{p}\boldsymbol{x}_w^i(t) \tag{20}$$

where $i$-th forced component has a form:

$$\boldsymbol{x}_w^i(t) = a_i\, [\boldsymbol{A}-\boldsymbol{I}b_i]^{-1}(\mathrm{e}^{\boldsymbol{A}(t-t_0)}\mathrm{e}^{b_i\,t_0} - \boldsymbol{I}\mathrm{e}^{b_i\,t})\boldsymbol{B}_i \tag{21}$$

When $\det[\boldsymbol{A}-\boldsymbol{I}b_i]^{-1} = 0$, this expression this can be presented in the form:

$$\frac{\mathrm{adj}\,[\boldsymbol{A}-\boldsymbol{I}b_i]}{\det\,[\boldsymbol{A}-\boldsymbol{I}b_i]}(\mathrm{e}^{\boldsymbol{A}(t-t_0)}\mathrm{e}^{b_i\,t_0} - \boldsymbol{I}\mathrm{e}^{b_i\,t})\boldsymbol{B}_i \tag{22}$$

The expression in brackets can be transformed as follows:

$$\mathrm{e}^{\boldsymbol{A}(t-t_0)}\mathrm{e}^{b_i t_0} - \boldsymbol{I}\,\mathrm{e}^{b_i t} = (\mathrm{e}^{\boldsymbol{A}(t-t_0)}\mathrm{e}^{-b_i (t-t_0)} - \boldsymbol{I})\mathrm{e}^{b_i t} =$$
$$= (\mathrm{e}^{\boldsymbol{A}(t-t_0)}\,\mathrm{e}^{-\boldsymbol{I}b_i (t-t_0)} - \boldsymbol{I})\,\mathrm{e}^{b_i t} = (\mathrm{e}^{[\boldsymbol{A}-\boldsymbol{I}b_i](t-t_0)} - \boldsymbol{I})\,\mathrm{e}^{b_i t} \tag{23}$$

Let us consider the expression:

$$\operatorname{adj}\,[\boldsymbol{A}-\boldsymbol{I}b_i](\mathrm{e}^{[\boldsymbol{A}-\boldsymbol{I}b_i](t-t_0)} - \boldsymbol{I}) \tag{24}$$

Since the exponential function of the matrix can be presented in the form:

$$\mathrm{e}^{\boldsymbol{G}t} = \boldsymbol{I} + \sum_{k=1}^{\infty}\frac{1}{k!}\boldsymbol{G}^k\, t^k \tag{25}$$

we obtain:

$$\operatorname{adj}\,[\boldsymbol{A}-\boldsymbol{I}b_i]\sum_{k=1}^{\infty}\frac{1}{k!}[\boldsymbol{A}-\boldsymbol{I}b_i]^k\,(t-t_0)^k = \boldsymbol{F} \tag{26}$$

Let us accept the mark:

$$\boldsymbol{G} = [\boldsymbol{A}-\boldsymbol{I}b_i] \tag{27}$$

since:

$$[\operatorname{adj}\boldsymbol{G}]\ \boldsymbol{G} = \boldsymbol{I}\det\boldsymbol{G} \tag{28}$$

therefore:

$$\boldsymbol{F} = \sum_{k=1}^{\infty}\frac{1}{k\,!}\,[\operatorname{adj}\boldsymbol{G}]\,\boldsymbol{G}\boldsymbol{G}^{k-1}(t-t_0)^k = \det\boldsymbol{G}\sum_{k=1}^{\infty}\frac{1}{k\,!}\,\boldsymbol{G}^{k-1}(t-t_0)^k \tag{29}$$

It results so that because $\det \boldsymbol{G}\big|_{b_i} = 0$, therefore it is necessary (at the same determinant in the nominative) to differentiate the numerator and the nominative $r$-tuplely, where $r$ is a multiplicity of the eigenvalue adequate to the exponent of $i$-th component of the control vector. Because in such case, for $\lambda_i = b_i$, it $\det\,[\boldsymbol{A}-\boldsymbol{I}b_i]$ is equal to the characteristic polynomial of the matrix $\boldsymbol{A}$:

$$\det\,[\boldsymbol{A}-\boldsymbol{I}\lambda] = w(\lambda) = (-1)^n\sum_{i=1}^{n} a_{n-i}\lambda^i \tag{30}$$

this $r$-th derivative of the polynomial $w(\lambda)$ in the point $\lambda_i = b_i$ can record as:

$$\frac{d^r}{d\lambda^r}w(\lambda)\bigg|_{\lambda=b_i} = (-1)^n\sum_{k=0}^{n-r}\frac{(r+k)!}{k!}a_{n-r-k}\lambda^k \tag{31}$$

Differentiation of the numerator $r$-tuplely on the basis of formula of the multiple differentiation of the product of two functions gives the following result:

$$\mathrm{e}^{b_i t_0}(-1)^n\sum_{k=0}^{n-r}\frac{r!}{(r-k)!k!}t_0^{r-k}\frac{d^k}{d\lambda^k}\boldsymbol{G}(b_i) \tag{32}$$

Finally, after execution of operation we receive a formula on $i$-th component of state vector:

$$\boldsymbol{x}_w^i(t) = \frac{\mathrm{e}^{A(t-t_0)}\,\mathrm{e}^{b_i t_0}}{\left.\frac{d^r}{d\lambda^r}w(\lambda)\right|_{\lambda=b_i}} \sum_{k=0}^{r} a_i \binom{r}{k} t_0^{r-k} \left.\frac{d^k}{d\lambda^k}\mathrm{adj}[\boldsymbol{A}-\boldsymbol{I}\lambda]\right|_{\lambda=b_i} \boldsymbol{B}_i -$$

$$-\frac{e^{b_i t}}{\left.\frac{d^r}{d\lambda^r}w(\lambda)\right|_{\lambda=b_i}} \sum_{k=0}^{r} a_i \binom{r}{k} t^{r-k} \left.\frac{d^k}{d\lambda^k}\mathrm{adj}[\boldsymbol{A}-\boldsymbol{I}\lambda]\right|_{\lambda=b_i} \boldsymbol{B}_i \tag{33}$$

## 2.2. Sinusoidal control with different Circular frequencies

The sinusoidal control vector can be presented in the form:

$$\boldsymbol{u}(t) = \begin{bmatrix} U_{1\,m}\sin(\omega_1 t \pm \psi_1) \\ \vdots \\ U_{i\,m}\sin(\omega_i t \pm \psi_i) \\ \vdots \\ U_{p\,m}\sin(\omega_p t \pm \psi_p) \end{bmatrix} = \boldsymbol{U}_m\,\boldsymbol{sin}(\omega t + \psi) \tag{34}$$

where: $i = 1,\dots,p$, $\boldsymbol{U}_m = \mathrm{diag}[U_{1\,m},\dots,U_{i\,m},\dots,U_{p\,m}]$ and:

$$[\boldsymbol{sin}(\omega t \pm \psi)]^{\mathrm{T}} = [\sin(\omega_1 t \pm \psi_1),\dots,\sin(\omega_i t \pm \psi_i),\dots,\sin(\omega_p t \pm \psi_p)]^{\mathrm{T}}$$

To designate the general integral of solution of the state equation with the control vector (34) we can use the dependence (7), where:

$$u_i(t) = U_{i\,m}\sin(\omega_i t \pm \psi_i) \tag{35}$$

On this base and with the use (10) we can record the expression:

$$\mathrm{e}^{At}\int_{t_0}^{t}\mathrm{e}^{-A\tau}\boldsymbol{B}\,\boldsymbol{u}(\tau)\,d\tau = \mathrm{e}^{At}\sum_{i=1}^{p}\left(\int_{t_0}^{t}\mathrm{e}^{-A\tau}\,u_i(\tau)\,d\tau\right)\boldsymbol{B}_i \tag{36}$$

At first we designate the integral from the expression (36) in a form:

$$\int \mathrm{e}^{-A\tau}u_i(\tau)\,d\tau = \int \mathrm{e}^{-A\tau}U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\,d\tau \tag{37}$$

Integrating twice by parts the right side (37), and then multiplying both sides of the obtained result by $\omega_i^{-2}$ and subtracting from each other both integrals in this expression we receive:

$$[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]\int \mathrm{e}^{-A\tau}U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\,d\tau =$$

$$= -\,\omega_i\,U_{i\,m}\cos(\omega_i\tau \pm \psi_i)\,\mathrm{e}^{-A\tau} - U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\,\boldsymbol{A}\,\mathrm{e}^{-A\tau}$$

Finally:

$$\int \mathrm{e}^{-A\tau}U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\,d\tau =$$

$$= -\,\omega_i U_{i\,m}\cos(\omega_i\tau \pm \psi_i)[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\,\mathrm{e}^{-\boldsymbol{A}\tau} -$$

$$-\,U_{i\,m}\sin(\omega_i\tau \pm \psi_i)[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{A}\,\mathrm{e}^{-\boldsymbol{A}\tau} \tag{38}$$

Taking into account (38) in (37) we obtain for $i$-th component:

$$\left(\mathrm{e}^{\boldsymbol{A}t}\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}u_i(\tau)\,d\tau\right)\boldsymbol{B}_i =$$

$$= -\,\omega_i U_{i\,m}\cos(\omega_i\tau \pm \psi_i)\;\mathrm{e}^{\boldsymbol{A}t}[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{B}_i -$$

$$-\,U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\;\mathrm{e}^{\boldsymbol{A}t}[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\boldsymbol{A}\,\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{B}_i +$$

$$+\,\omega_i U_{i\,m}\cos(\omega_i\tau \pm \psi_i)\;\mathrm{e}^{\boldsymbol{A}t}[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\mathrm{e}^{\boldsymbol{A}t_0}\boldsymbol{B}_i +$$

$$+\,U_{i\,m}\sin(\omega_i\tau \pm \psi_i)\;\mathrm{e}^{-\boldsymbol{A}t}[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\boldsymbol{A}\,\mathrm{e}^{\boldsymbol{A}t_0}\boldsymbol{B}_i$$

We can easily show the commutativity of the product of the matrix:

$$t = t_0$$

now then:

$$\left(\mathrm{e}^{\boldsymbol{A}t}\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}u_i(\tau)\,d\tau\right)\boldsymbol{B}_i =$$

$$= \mathrm{e}^{\boldsymbol{A}(t-t_0)}[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}[\boldsymbol{A}\boldsymbol{B}_i\,U_{i\,m}\sin(\omega_i\,t_0 \pm \psi_i)\; + \boldsymbol{B}_i\;\omega_i\,U_{i\,m}\cos(\omega_i\,t_0 \pm \psi_i)] -$$

$$-[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}[\boldsymbol{A}\boldsymbol{B}_i\,U_{i\,m}\sin(\omega_i\,t \pm \psi_i)\; + \boldsymbol{B}_i\;\omega_i\,U_{i\,m}\cos(\omega_i\,t \pm \psi_i)]$$

Finally:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\sum_{i=1}^{p}\boldsymbol{V}_{\mathrm{SP}}^{i} \;-\sum_{i=1}^{p}[\boldsymbol{V}_{\mathrm{SS}}^{i}\sin(\omega_i\,t_0 \pm \psi_i) + \boldsymbol{V}_{\mathrm{SC}}^{i}\cos(\omega_i\,t_0 \pm \psi_i)] \tag{39}$$

where:

$$\boldsymbol{V}_{\mathrm{SP}}^{i} = \boldsymbol{V}_{\mathrm{SS}}^{i}\sin(\omega_i\,t_0 \pm \psi_i) + \boldsymbol{V}_{\mathrm{SC}}^{i}\cos(\omega_i\,t_0 \pm \psi_i)$$

$$\boldsymbol{V}_{\mathrm{SS}}^{i} = U_{i\,m}\,[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{A}\boldsymbol{B}_i$$

$$\boldsymbol{V}_{\mathrm{SC}}^{i} = \omega_i\,U_{i\,m}\,[\boldsymbol{I}\,\omega_i^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{B}_i$$

## 2.3. Sinusoidal multiple control

The multiple sinusoidal control vector can be presented in the form:

$$\boldsymbol{u}(t)=\begin{bmatrix}\sum\limits_{j=0}^{M_1}U_{1\,m}^{j}\ \sin(\omega_1^j t\pm\psi_1^j)\\ \dots\\ \sum\limits_{j=0}^{M_i}U_{i\,m}^{j}\ \sin(\omega_i^j t\pm\psi_i^j)\\ \dots\\ \sum\limits_{j=0}^{M_p}U_{p\,m}^{j}\ \sin(\omega_p^j t\pm\psi_p^j)\end{bmatrix}=\sum_{j=0}^{M}\boldsymbol{u}^j(t)=\sum_{j=0}^{M}\boldsymbol{U}^j\,\boldsymbol{sin}(\omega^j t\pm\psi^j) \tag{40}$$

where $M=\max\{M_1,\dots,M_i,\dots,M_p\}$, and $\boldsymbol{U}^j=\operatorname{diag}[U_{1\,m}^j,\dots,U_{i\,m}^j,\dots,U_{p\,m}^j]$, $[\boldsymbol{sin}(\omega^j t\pm\psi^j)]^{\mathrm{T}}=[\sin(\omega_1^j t\pm\psi_1^j),\dots,\sin(\omega_p^j t\pm\psi_p^j)]^{\mathrm{T}}$, $i=1,\dots,p$, $j=1,\dots,M$, $\boldsymbol{B}=[\boldsymbol{B}_1|\dots|\boldsymbol{B}_i|\dots|\boldsymbol{B}_p]$.

On the basis (7) for purpose of designating the general integral of solution at the control vector (40) let us record $j$-th component of this vector in the form:

$$\boldsymbol{u}^j(t)=\sum_{i=1}^{p}u_i^j(t)\,\boldsymbol{n}_i \tag{41}$$

whereat:

$$u_i^j(t)=U_{i\,m}^j\sin(\omega_i^j t\pm\psi_i^j) \tag{42}$$

After the substitution (40), (41) and (42) to the generalized formula we receive:

$$\mathrm{e}^{\boldsymbol{A}t}\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\,\boldsymbol{u}(\tau)\,d\tau=\mathrm{e}^{\boldsymbol{A}t}\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\left(\sum_{j=0}^{M}\boldsymbol{u}^j(\tau)\right)d\tau=$$

$$=\mathrm{e}^{\boldsymbol{A}t}\sum_{j=0}^{M}\left(\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\,\boldsymbol{u}^j(\tau)\,d\tau\right)=\mathrm{e}^{\boldsymbol{A}t}\sum_{j=0}^{M}\left(\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\sum_{i=1}^{p}u_i^j(\tau)\,\boldsymbol{n}_i\,d\tau\right)=$$

$$=\mathrm{e}^{\boldsymbol{A}t}\sum_{j=0}^{M}\left(\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}\left(\sum_{i=1}^{p}u_i^j(\tau)\,\boldsymbol{B}_i\right)d\tau\right)=\mathrm{e}^{\boldsymbol{A}t}\sum_{i=1}^{p}\left(\sum_{j=0}^{M}\left(\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}\tau}u_i^j(\tau)\,d\tau\right)\boldsymbol{B}_i\right) \tag{43}$$

To calculate the integral from the expression (44) we can use the relation (42), therefore:

$$\int\mathrm{e}^{-\boldsymbol{A}\tau}U_{i\,m}^j\sin(\omega_i^j\tau\pm\psi_i^j)\,d\tau=$$

$$\boldsymbol{b}$$

$$-\,U_{i\,m}^j\sin(\omega_i^j\tau\pm\psi_i^j)[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\boldsymbol{A}\,\mathrm{e}^{-\boldsymbol{A}\tau} \tag{44}$$

Putting this expression to (43) we receive:

$$\mathrm{e}^{At}\sum_{i=1}^{p}\left(\sum_{j=0}^{M}\left(\int_{t_0}^{t}\mathrm{e}^{-A\tau}u_i^j(\tau)\,d\tau\right)\boldsymbol{B}_i\right)=$$

$$=\mathrm{e}^{At}\sum_{i=1}^{p}\sum_{j=0}^{M}\Big(-\,\omega_i^j U_{i\,m}^j\sin(\omega_i^j t\pm\psi_i^j)\mathrm{e}^{At}\,[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\boldsymbol{A}\,\mathrm{e}^{-At}\boldsymbol{B}_i-$$

***a***

$$+\;U_{i\,m}^j\sin(\omega_i^j t\pm\psi_i^j)\,\mathrm{e}^{At}[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\boldsymbol{A}\,\mathrm{e}^{-At_0}\boldsymbol{B}_i+$$

$$+\;\omega_i^j U_{i\,m}^j\cos(\omega_i^j \tau\pm\psi_i^j)\,\mathrm{e}^{At}[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\,\mathrm{e}^{-At_0}\boldsymbol{B}_i\Big)$$

Finally:

$$\boldsymbol{x}_w(t)=\mathrm{e}^{\boldsymbol{A}(t-t_0)}\sum_{i=1}^{p}\sum_{j=0}^{M}\boldsymbol{V}_{i\,SP}^j-\sum_{i=1}^{p}\sum_{j=0}^{M}[\boldsymbol{V}_{i\,SS}^j\sin(\omega_i^j t\pm\psi_i^j)+\boldsymbol{V}_{i\,SC}^j\cos(\omega_i^j t\pm\psi_i^j)] \quad (45)$$

where:

$$\boldsymbol{V}_{i\,SP}^j=\boldsymbol{V}_{i\,SS}^j\sin(\omega_i^j t_0\pm\psi_i^j)+\boldsymbol{V}_{i\,SC}^j\cos(\omega_i^j t_0\pm\psi_i^j)$$

$$\boldsymbol{V}_{i\,SS}^j=U_{i\,m}^j\,[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\boldsymbol{A}\boldsymbol{B}_i$$

$$\boldsymbol{V}_{i\,SC}^j=\omega_i^j U_{i\,m}^j\,[\boldsymbol{I}\,(\omega_i^j)^2+\boldsymbol{A}^2]^{-1}\boldsymbol{B}_i$$

## 3. OTHER ALGEBRAIC FORMS FOR DETERMINED CONTROL VECTORS

In literature we can find also other algebraic forms for some, typical for application control vectors. They appear dispersed in literature therefore it seems useful to collect them for possible applications in one elaboration.

- **Control vector with impulse components** [1, 8]
  For the control vector in the form:

$$\boldsymbol{u}(t)=\boldsymbol{w}\;\delta(t) \quad (46)$$

where $\boldsymbol{w}$ is $p$-dimensional vector ($\boldsymbol{w}\in\Re^p$) with components representing values (fields) of $p$ impulse functions applied in the moment $t=t_0=0$, the forced solution has a form:

$$\boldsymbol{x}_w(t)=\;\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{B}\,\boldsymbol{w} \quad (47)$$

- **Control vector with constant components** [1, 8]
  For the control vector in ta form:

$$\boldsymbol{u}(t)=\boldsymbol{a} \quad (48)$$

where $\boldsymbol{a}$ is $p$-with the dimensional vector ($\boldsymbol{a}\in\Re^p$) with constant components, applied in moment $t = t_0$, the forced solution has a form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)} \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{a} - \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{a} \tag{49}$$

- **Control vector with linear components** [1, 8]
  For a control vector in the form:

$$\boldsymbol{u}(t) = \boldsymbol{a}\, t \tag{50}$$

where $\boldsymbol{a}$ is $p$-with the dimensional vector ($\boldsymbol{a} \in \Re^p$) with linear components, applied in the moment $t = t_0$, the forced solution is in form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)} [\boldsymbol{A}^{-2} \boldsymbol{B}\, \boldsymbol{a} + \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{a}\, t_0] - \boldsymbol{A}^{-2} \boldsymbol{B}\, \boldsymbol{a} - \boldsymbol{A}^{-1} \boldsymbol{B}\, \boldsymbol{a}\, t \tag{51}$$

- **Control vector with multinomial components** [6]
  For control vector in the form:

$$\boldsymbol{u}(t) = \begin{bmatrix} \sum_{j=0}^{N} a_1^j\, t^j & \cdots & \sum_{j=0}^{N} a_i^j\, t^j & \cdots & \sum_{j=0}^{N} a_p^j\, t^j \end{bmatrix}^{\mathrm{T}} = \sum_{j=0}^{N} \boldsymbol{a}^j t^j \tag{52}$$

where $\boldsymbol{a}^j$ is $p$-dimensional vector ($\boldsymbol{a}^j \in \Re^p$) with linear components, applied in the moment $t = t_0$, forced solution has a form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)} \sum_{j=0}^{N} \sum_{k=j}^{N} \boldsymbol{V}_{WP}^{j\,k} - \sum_{j=0}^{N} \sum_{k=j}^{N} \boldsymbol{V}_{WW}^{j\,k} \tag{53}$$

where:

$$\boldsymbol{V}_{WP}^{j\,k} = \frac{k!}{j!} \boldsymbol{A}^{-(k-j+1)}\, \boldsymbol{B}\, \boldsymbol{a}^k t_0^j$$

$$\boldsymbol{V}_{WW}^{j\,k} = \frac{k!}{j!} \boldsymbol{A}^{-(k-j+1)}\, \boldsymbol{B}\, \boldsymbol{a}^k t^j$$

- **Control vector with sinusoidal components with constant circular frequencies** [6]
  For control vector in the form:

$$\boldsymbol{u}(t) = \begin{bmatrix} U_{1\,m} \sin(\omega t \pm \psi_1) \\ \cdots \\ U_{i\,m} \sin(\omega t \pm \psi_i) \\ \cdots \\ U_{p\,m} \sin(\omega t \pm \psi_p) \end{bmatrix} = \boldsymbol{U}\, \boldsymbol{sin}(\omega t + \boldsymbol{\psi}) \tag{54}$$

where $i = 1, \ldots, p$ and:

$$\boldsymbol{U} = \mathrm{diag}[U_{1\,m}, \ldots, U_{i\,m}, \ldots, U_{p\,m}] \tag{55}$$

$$[\boldsymbol{sin}(\omega t \pm \boldsymbol{\psi})]^{\mathrm{T}} = [\sin(\omega t \pm \psi_1),\ldots,\sin(\omega t \pm \psi_i),\ldots,\sin(\omega t \pm \psi_p)]^{\mathrm{T}} \tag{56}$$

applied in the moment $t = t_0$, forced solution has a form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\,\boldsymbol{V}_{\mathrm{SP}} - \boldsymbol{M}_{\mathrm{SS}}\boldsymbol{sin}(\omega t + \boldsymbol{\psi}) - \boldsymbol{M}_{\mathrm{SC}}\boldsymbol{cos}(\omega t + \boldsymbol{\psi}) \tag{57}$$

where:

$$[\boldsymbol{cos}(\omega t \pm \boldsymbol{\psi})]^{\mathrm{T}} = [\cos(\omega t \pm \psi_1),\ldots,\cos(\omega t \pm \psi_i),\ldots,\cos(\omega t \pm \psi_p)]^{\mathrm{T}}$$

$$\boldsymbol{V}_{\mathrm{SP}} = \boldsymbol{M}_{\mathrm{SS}}\,\boldsymbol{sin}(\omega t_0 + \boldsymbol{\psi}) + \boldsymbol{M}_{\mathrm{SC}}\,\boldsymbol{cos}(\omega t_0 + \boldsymbol{\psi})$$

$$\boldsymbol{M}_{\mathrm{SS}} = [\boldsymbol{I}\,\omega^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{A}\boldsymbol{B}\,\boldsymbol{U}$$

$$\boldsymbol{M}_{\mathrm{SC}} = \omega\,[\boldsymbol{I}\,\omega^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{B}\,\boldsymbol{U}$$

- **Control vector with harmonics components** [6]

  For a control vector in the form:

$$\boldsymbol{u}(t) = \begin{bmatrix} \sum\limits_{h=0}^{\infty} U_{1\,m}^{h}\ \sin(h\omega^1 t \pm \psi_1^h) \\ \cdots \\ \sum\limits_{h=0}^{\infty} U_{i\,m}^{h}\ \sin(h\omega^1 t \pm \psi_i^h) \\ \cdots \\ \sum\limits_{h=0}^{\infty} U_{p\,m}^{h}\ \sin(h\omega^1 t \pm \psi_p^h) \end{bmatrix} = \sum_{h=0}^{\infty} \boldsymbol{u}^h(t) = \sum_{h=0}^{\infty} \boldsymbol{U}^h\ \boldsymbol{sin}(h\omega^1 t \pm \boldsymbol{\psi}^h) \tag{58}$$

where: $i = 1,\ldots,p$, $\omega^1$ – the circular frequency of basic of harmonic and:

$$\boldsymbol{U}^h = \mathrm{diag}[U_{1\,m}^{h},\ldots,U_{i\,m}^{h},\ldots,U_{p\,m}^{h}] \tag{59}$$

$$[\boldsymbol{sin}(h\omega^1 t \pm \boldsymbol{\psi}^h)]^{\mathrm{T}} = [\sin(h\omega^1 t \pm \psi_1^h),\ldots,\sin(h\omega^1 t \pm \psi_i^h),\ldots,\sin(h\omega^1 t \pm \psi_p^h)] \tag{60}$$

applied in the moment $t = t_0$, the forced solution is in form:

$$\boldsymbol{x}_w(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)} \sum_{h=0}^{\infty} \boldsymbol{V}_{\mathrm{SP}}^{h} - \sum_{h=0}^{\infty} [\boldsymbol{M}_{\mathrm{SS}}^{h}\boldsymbol{sin}(h\omega^1 t_0 \pm \boldsymbol{\psi}^h) + \boldsymbol{M}_{\mathrm{SC}}^{h}\boldsymbol{cos}(h\omega^1 t_0 \pm \boldsymbol{\psi}^h)] \tag{61}$$

where:

$$\boldsymbol{V}_{\mathrm{SP}}^{h} = \boldsymbol{M}_{\mathrm{SS}}^{h}\,\boldsymbol{sin}(h\omega^1 t_0 \pm \boldsymbol{\psi}^h) + \boldsymbol{M}_{\mathrm{SC}}^{h}\,\boldsymbol{cos}(h\omega^1 t_0 \pm \boldsymbol{\psi}^h)$$

$$\boldsymbol{M}_{\mathrm{SS}}^{h} = [\boldsymbol{I}\,(h\omega^1)^2 + \boldsymbol{A}^2]^{-1}\boldsymbol{A}\boldsymbol{B}\,\boldsymbol{U}^h$$

$$\boldsymbol{M}_{\mathrm{SC}}^{h} = h\omega^1\,[\boldsymbol{I}\,(h\omega^1)^2 + \boldsymbol{A}^2]^{-1}\,\boldsymbol{B}\,\boldsymbol{U}^h$$

## 4. CONCLUSION

In the work were presented forced components of analytical solutions of the state equation in a normal form for three kinds of control vector: exponential, sinusoidal with components having different amplitudes, circular frequencies and phases, and also for a sinusoidal multiple control for which each component of the control vector has sinusoidal components with different amplitudes, circular frequencies and phases. These solutions enable full qualitative and quantitative analysis how the individual values influence properties and shape of the obtained solutions. The work presents also other known from the literature [1, 4, 6, 8] algebraic forms of solutions of state equations. They can be useful in circuits theory, control theories and other fields in which mathematical models of systems are described by the use of linear state equation in a normal form.

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