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LIE SYMMETRY, CONVERGENCE ANALYSIS, EXPLICIT SOLUTIONS, AND CONSERVATION LAWS FOR THE TIME-FRACTIONAL MODIFIED BENJAMIN-BONA-MAHONY EQUATION

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Abstract. Lie symmetry analysis is considered as one of the most powerful techniques that has been used for analyzing and extracting various types of solutions to partial differential equations. Conservation laws reflect important aspects of the behavior and properties of physical systems. This paper focuses on the investigation of the $(1+1)$-dimensional time-fractional modified Benjamin-Bona-Mahony equation (mBBM) incorporating Riemann-Liouville derivatives (RLD). Through the application of Lie symmetry analysis, the study explores similarity reductions and transforms the problem into a nonlinear ordinary differential equation with fractional order. A power series solution is obtained using the Erdelyi-Kober fractional operator, and the convergence of the solutions is analyzed. Furthermore, novel conservation laws for the time-fractional mBBM equation are established. The findings of the current work contribute to a deeper understanding of the dynamics of this fractional evolution equation and provide valuable insights into its behavior.

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1. Introduction

Nowadays, fractional differential equations represent various physical applications and phenomena in nature and sciences [1, 2]. There are many schemes being developed to find numerical and analytical solutions to fractional problems, such as the collocation methods, finite-difference methods, and reproducing kernel approaches [3–5], different forms of fractional power series [6, 7], homotopy perturbation technique and its updates [8], combined Laplace transform and fractional power series [9, 10], and many others.

For decades, Lie symmetry analysis [11–14] has been considered as one of the most powerful techniques that has been used for analyzing and extracting various types of solutions to partial differential equations. Lie-symmetry analysis guarantees the existence of analytic solutions or convert partial differential equations to ordinary differential equations that can be solved. Moreover, symmetry methods can identify special transformations that map a given differential equation to a simpler one with known solutions. It provides a powerful framework for studying symmetries, conservation laws, and the dynamics of various physical, chemistry, and engineering phenomena.

Conservation laws are rooted in the fundamental principles of physics and mathematics. They reflect important aspects of the behavior and properties of physical systems, such as the conservation of energy, momentum, mass, charge, and angular momentum. These principles provide a foundation for understanding the underlying dynamics and governing equations of various phenomena.

Recently, there has been interest in exploring conservation laws to fractional physical models via its new Lie-symmetry by considering different new linear combination to prove the existence and uniqueness of solutions for fractional partial differential [15].

In this research article, our focus is on exploring novel solutions to the nonlinear time-fractional modified Benjamin-Bona-Mahony (mBBM) equation using Lie symmetry and conservation laws. The equation is considered in the context of Riemann--Liouville derivatives (RLD). By formulating the fractional mBBM equation in vector equation form, we aim to uncover new solutions that exhibit intriguing properties and shed light on the dynamics of this nonlinear fractional equation, which is given as

$$D_t^{\omega} h + \lambda_1 \, h^m \, \frac{\partial h}{\partial x} - \lambda_2 \, \frac{\partial^3 h}{\partial x^2 \partial t} = 0, \tag{1}$$

where λ_1, λ_2 are real numbers and represent the nonlinearity and dispersion parameters, respectively. D_t^{ω} is the RLD of order $\omega > 0$ which is given by [16]

$$D_t^{\omega} h(x,t) = \begin{cases} \partial_{t^n} h(x,t), & \omega = n, \\ \dfrac{1}{\Gamma(n-\omega)} \partial_{t^n} \displaystyle\int_0^t (t-s)^{n-\omega-1} h(x,s) ds, & 0 \leq n-1 < \omega < n, \end{cases} \tag{2}$$

The time-fractional modified Benjamin-Bona-Mahony equation [17–19] is a simulation of the Korteweg-de Vries (KdV) and describes the motion of shallow waves propagation with equal width to all wave amplitudes. It has been generalized into a time-fractional mode, which is a generalization of the classical time-derivative.

2. The Lie point symmetry representation

This section is devoted to presenting some concepts which are related to the Lie symmetry analysis. Let us start with the subordinate FPDE which is given by

$$D_t^{\omega} h - H(x,t,h,h_x,h_{xx},...) = 0, \omega > 0, \tag{3}$$

where $h(x,t)$ is the variable function that depends on t and x, and H is a relation among other terms of the FPDE.

It is well known that the one-parameter (β) Lie group is given by the transformation

$$t^* = t + \beta.\tau_1(x,t,h) + O(\beta^2), \qquad x^* = x + \beta.\tau_2(x,t,h) + O(\beta^2),$$

$$h^* = h + \beta.\urcorner_1(x,t,h) + O(\beta^2), \qquad D_t^{\omega} h^* = D_t^{\omega} h + \beta.\urcorner_{\omega}^0(x,t,h) + O(\beta^2),$$

$$\frac{\partial h^*}{\partial x} = h_x + \beta.\urcorner^x(x,t,h) + O(\beta^2), \ \frac{\partial^3 h^*}{\partial x^2 \partial t} = h_{xxt} + \beta.\urcorner^{xxt}(x,t,h) + O(\beta^2), \tag{4}$$

where $\urcorner$ and τ_1, τ_2 are the infinitesimals of the transformations for independent and dependent variables, simultaneously, and $\beta \ll 1$ is Lie group parameter. Hence, the explicit expressions of $\urcorner^x, \urcorner^t, \urcorner^{xt}, \urcorner^{xx}$, and $\urcorner^{xxt}$ are given as

$$\begin{aligned}
\urcorner^x &= D_x(\urcorner) - h_x D_x(\tau_2) - h_t D_x(\tau_1), \qquad \urcorner^t = D_t(\urcorner) - h_x D_t(\tau_2) - h_t D_t(\tau_1),\\
\urcorner^{xt} &= D_x(\urcorner^t) - h_{xt} D_x(\tau_2) - h_{tt} D_x(\tau_1), \quad \urcorner^{xx} = D_x(\urcorner^x) - h_{xx} D_x(\tau_2) - h_{xt} D_x(\tau_1),\\
\urcorner^{xxt} &= D_x(\urcorner^{xt}) - h_{xxx} D_x(\tau_2) - h_{xxt} D_x(\tau_1),
\end{aligned} \tag{5}$$

where D_x is the total derivative that is defined by $D_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h}$.

The corresponding Lie algebra of symmetries is formed by a collection of the following vector fields:

$$X = \tau_2(x,t,h)\frac{\partial}{\partial x} + \tau_1(x,t,h)\frac{\partial}{\partial t} + \urcorner(x,t,h)\frac{\partial}{\partial h}. \tag{6}$$

On the infinitesimal invariance criterion, it is easy to obtain that

$$Pr^{(\omega,3)} X[\nabla] = 0 \text{ when } \nabla = 0, \tag{7}$$

where $\nabla = D_t^{\omega} h - H(x,t,h,h_x,h_{xx},\ldots)$. The prolongation operator $Pr^{(\omega,3)}X$ of the Lie group takes the form

$$Pr^{(\omega,3)}X = X + \daleth_{\omega}^{0} \frac{\partial}{\partial_t^{\omega} h} + \daleth^{x} \frac{\partial}{\partial h_x} + \daleth^{xxt} \frac{\partial}{\partial h_{xxt}}. \tag{8}$$

Thus, when $t = 0$, the invariance condition leads to $\tau_1(x,t,h) = 0$, which gives that the ω^{th} extended infinitesimal associated with RLD can be performed as follows:

$$\daleth_{\omega}^{0} = D_t^{\omega}(\daleth) + \tau_2 D_t^{\omega}(h_x) - D_t^{\omega}(\tau_2 h_x) + D_t^{\omega}(D_x(\tau_1)h) - D_t^{\omega+1}(\tau_1 h) + \tau_1 D_t^{\omega+1}(h), \tag{9}$$

where D_t^{ω} is the total fractional derivative operator. Hence, the generalized Leibnitz rule gives that

$$D_t^{\omega}(g(t)f(t)) = \sum_{n=0}^{\infty} \binom{\omega}{n} D_t^n g(t) D_t^{\omega-n} f(t) : \binom{\omega}{n} = \frac{(-1)^{n-1}\omega\Gamma(n-\omega)}{\Gamma(n+1)\Gamma(1-\omega)}.$$

This leads, by employing the Leibnitz rule, to

$$\daleth_{\omega}^{0} = D_t^{\omega}(\daleth) - \omega D_t(\tau_1) D_t^{\omega}(h) - \sum_{n=1}^{\infty} \binom{\omega}{n} D_t^n(\tau_2) D_t^{\omega-n}(h_x) - \sum_{n=1}^{\infty} \binom{\omega}{n+1} D_t^n(\tau_1) D_t^{\omega-n}(h).$$

Notice that the chain rule for differentiating $(\phi \circ J)(t) = \phi(J(t))$ is given by

$$\frac{d^n \phi(J(t))}{dt^n \phi} = \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{\omega}{n+1} \frac{1}{k!} [-J(t)]^r \frac{d^n}{dt^n} \left[(J(t))^{k-r} \right] \frac{d^n \phi(J)}{dJ^k}.$$

Hence, we deduce that

$$D_t^{\omega}(\daleth) = \frac{\partial^{\omega} h}{\partial t^{\omega}} + \daleth_h \frac{\partial^{\omega} h}{\partial t^{\omega}} - h \frac{\partial^{\omega} \daleth_h}{\partial t^{\omega}} + \sum_{n=0}^{\infty} \binom{\omega}{n} \frac{\partial^{\omega} \daleth_h}{\partial t^{\omega}} D_t^{\omega-n}(\tau_2) D_t^{\omega-n}(h) + \mu,$$

where μ is given by

$$\mu = \sum_{n=1}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\omega}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\omega}}{\Gamma(n+1-\omega)} (-h)^r \frac{\partial^m}{\partial t^m} (h^{k-r}) \frac{\partial^{n-m+k} \daleth}{\partial x^{n-m} \partial h^k}.$$

Therefore, the ω^{th} extended infinitesimal of (9) reduces to the formula

$$\begin{aligned} \daleth_{\omega}^{0} = & \frac{\partial^{\omega} \daleth}{\partial t^{\omega}} + (\daleth_h - \omega D_t(\tau_1)) \frac{\partial^{\omega} h}{\partial t^{\omega}} - h \frac{\partial^{\omega} \daleth_h}{\partial t^{\omega}} + \mu \\ & + \sum_{n=0}^{\infty} \left[\binom{\omega}{n} \frac{\partial^{\omega} \daleth_h}{\partial t^{\omega}} - \binom{\omega}{n+1} D_t^{n+1}(\tau_1) \right] D_t^{\omega-n}(h) - \sum_{n=1}^{\infty} \binom{\omega}{n} D_t^n(\tau_2) D_t^{\omega-n}(h_x). \end{aligned} \tag{10}$$

We point out that a function $h = \theta(x,t)$ is called an invariant surface if

$$X\theta = 0 \Leftrightarrow \left[\tau_2(x,t,h)\frac{\partial}{\partial x} + \tau_1(x,t,h)\frac{\partial}{\partial t} + \daleth(x,t,h)\frac{\partial}{\partial h}\right]\theta = 0.$$

Theorem 1. *[20] A function $h = \theta(x,t)$ is an invariant solution to (3) if and only if it is an invariant surface that satisfies (4).*

3. The LS and reductions

In this section, we establish the reduction equations by deriving the characteristic formulas of vector fields. Precisely, we reduce the time-fractional RLD given in (2) to a FDE that is supported by Erdelyi-Kober FDO, and then solve it. In fact, we assume that the Equation (5) is invariant under one parameter conversion of (6). Hence, the following subordinate conversed equation

$$D_t^{\omega} h^* + \lambda_1 \, (h^*)^m \, \frac{\partial h^*}{\partial x} - \lambda_2 \, \frac{\partial^3 h^*}{\partial x^2 \partial t} = 0 \tag{11}$$

is obtained. Now substitute (4) in (11) to get the subordinate symmetry determining equation

$$\daleth_{\omega}^{0} + \lambda_1 (h^m \, \daleth^{x} + h^{m-1} \, \daleth h_x) - \lambda_2 \daleth^{xxt} = 0. \tag{12}$$

Hence, by (10) and (11), we get

$$\begin{aligned}
0 = {} & \frac{\partial^{\omega}\daleth}{\partial t^{\omega}} + (\daleth_h - \omega D_t(\tau_1))\frac{\partial^{\omega} h}{\partial t^{\omega}} - h\frac{\partial^{\omega}\daleth_h}{\partial t^{\omega}} + \mu - \sum_{n=1}^{\infty}\binom{\omega}{n} D_t^{n}(\tau_2) D_t^{\omega-n}(h_x) \\
& + \sum_{n=0}^{\infty}\left[\binom{\omega}{n}\frac{\partial^{\omega}\daleth_h}{\partial t^{\omega}} - \binom{\omega}{n+1} D_t^{n+1}(\tau_1)\right] D_t^{\omega-n}(h) + \lambda_1 h^m(\daleth_x + h_x(\daleth_h - \tau_{2_x}) \\
& - (h_x)^2\tau_{2_h} - h_t\tau_{1_x} - h_t h_x \tau_{1_h}) + \lambda_1 h^{m-1}\daleth h_x - \lambda_2(\daleth_{xxt} + (2\daleth_{xth} - \tau_{2_{txx}})h_x \\
& + (\daleth_{th} - 2\tau_{2_{tx}})h_{xx} - (\daleth_{thh} - 2\tau_{2_{txh}})(h_x)^2 + (\daleth_h - \tau_{2_x} - \tau_{1_x} - \tau_{1_t})h_{txx} \\
& + (2\daleth_{hx} - \tau_{2_{xx}} - \tau_{1_{tx}})h_{tx} + (2\daleth_{hh} - 4\tau_{2_{hx}} - 2\tau_{1_{th}})h_x h_{tx} + (\daleth_{hxx} - \tau_{1_{txx}})h_t \\
& + (\daleth_{hh} - 2\tau_{2_{hx}} - \tau_{th})h_{xx}h_t + (\daleth_{hhh} - \tau_{2_{hhx}} - \tau_{1_{thh}})(h_x)^2 h_t - 3\tau_{2_{hh}} h_{xx} h_x \\
& - 3\tau_{2_{hh}}(h_x)^2 h_{tx} - 3\tau_{2_{hhh}}(h_x)^3 h_t - \tau_{1_h}(h_{tx})^2 - 2\tau_{1_h} h_t h_{txx} - 4\tau_{1_{hx}} h_x h_{tx} \\
& - (\tau_{1_{hhh}} + 2\tau_{1_{hhx}})(h_t)^2 h_x - \tau_{1_{hh}}(h_t)^2 h_{xx} - \tau_{1_x} h_{xtt} - 2\tau_{1_{hx}} h_x h_{tt} - \tau_{1_h} h_x h_{ttx} \\
& - \tau_{2_h} h_{xxx} h_x - \tau_{1_{xx}} h_{tt} - h_{xxt}(\tau_{1_x} + h_x\tau_{1_h}) - h_{xxx}(\tau_{2_x} + h_x\tau_{2_h}) - (2\tau_{2_h} - \tau_{1_h})h_x h_{txx} \\
& + (2\daleth_{hhx} - \tau_{2_{hxx}} - 2\tau_{1_{txh}})h_x h_t - 4\tau_{1_{hh}} h_x h_t h_{tx} - \tau_{1_{hxx}}(h_t)^2 - \tau_{1_h} h_{xx} h_{tt} - \tau_{1_{hh}}(h_x)^2 h_{tt}).
\end{aligned} \tag{13}$$

Set each power of derivatives to zero, and then solve it, we get the following subordinate infinitesimals:

$$\tau_2 = (\omega-1)mxc_1 + c_2, \quad \tau_1 = 2mtc_1$$
$$\text{and } \urcorner = -(\omega+1)hc_1,$$

where c_1, c_2 are free constants. Notice that the Lie algebra of infinitesimal symmetries of (1) is given by

$$X_1 = \frac{\partial}{\partial x} \text{ and } X_2 = (\omega-1)mx\frac{\partial}{\partial x} + 2mt\frac{\partial}{\partial t} - (\omega+1)h\frac{\partial}{\partial h}.$$

The reduction equations related to our work are the following:

Case 1: Express the characteristic equation for X_1 symbolically as $\frac{dx}{1} = \frac{dt}{0} = \frac{dh}{0}$, then we obtain that $t = z$ and $h = y(z)$, where $y(z)$ is any function that satisfies $D_t^{\omega} y(z) = 0$. Hence, the group invariant solution associated with X_1 is of the form $h_1 = k_1 t^{\omega-1}$ with a free variable k_1.

Case 2: Express the characteristic equation for X_2 symbolically as $\frac{dx}{(\omega-1)mx} = \frac{dt}{2mt} = \frac{dh}{-(\omega+1)h}$, we get $z_1 = x\,t^{\frac{1-\omega}{2}}$ and $z_2 = t^{\frac{-(\omega+1)}{2m}}$. Therefore, by the symmetry of X_2, we deduce that the group invariant solution as

$$z = x\,t^{\frac{1-\omega}{2}} \text{ and } h = t^{\frac{-(\omega+1)}{2m}} Q(z). \tag{14}$$

The function $Q(z)$ is to be determined later through the power series expansion, and thus is to be used to identify the solution $h = h(x,t)$.

Theorem 2. *The transformation given in (14) reduces the governing equation into the following subordinate nonlinear FODE:*

$$\left(P_{\frac{2}{\omega-1}}^{1+\frac{-\omega(2m+1)-1}{2m},\omega} Q\right)(z) + \lambda_1 Q(z)^m Q'(z) + \lambda_2 \frac{(\omega+1)+2m(1-\omega)}{2m} Q''(z)$$
$$-\lambda_2 Q'''(z) = 0,$$

where $P_{\beta}^{\zeta,\alpha}$ is the Erdélyi-Kober FDO given by

$$\left(P_{\beta}^{\zeta,\alpha} f\right)(z) = \prod_{j=0}^{n-1}(\zeta - j - \frac{1}{\beta} z\frac{d}{dz})(k_{\beta}^{\zeta+\alpha,n-\alpha} f)(z), \ n = \begin{cases} \alpha, & \alpha \in \mathbb{N}, \\ 1+[\alpha], & \alpha \notin \mathbb{N}, \end{cases}$$

$$(k_{\beta}^{\zeta,\alpha} f)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \left(u^{-(\zeta+\alpha)} f(zu^{\frac{1}{\beta}})\right)(u-1)^{\alpha-1} du, & \alpha > 0 \\ f(z), & \alpha = 0. \end{cases}$$

PROOF Let $\omega \in (n-1,n)$ with $n=1,2,3,\ldots$. Then, by the similarity transformation of (11), the RLD turns into

$$\frac{\partial^{\omega} h}{\partial t^{\omega}} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\omega)}\int_0^t s^{\frac{-(\omega+1)}{2m}} Q(xs^{\frac{-(\omega+1)}{2m}})(t-s)^{n-\omega-1}ds\right] \tag{15}$$

Let $r=\frac{t}{s}$, then, we have

$$\begin{aligned}\frac{\partial^{\omega} h}{\partial t^{\omega}} &= \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\omega)}\int_0^t [s^{n-\omega-1}\left(\frac{t}{s}-1\right)^{n-\omega-1} s^{\frac{-(\omega+1)}{2m}} Q(x\frac{t^{\frac{\omega+1}{2}}}{s^{\frac{\omega+1}{2}}}t^{\frac{1-\omega}{2}})ds\right]\\ &= \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\omega+\frac{-(\omega+1)}{2m}}}{\Gamma(n-\omega)}\int_1^{\infty}[r^{-\left(n-\omega+1+\frac{-(\omega+1)}{2m}\right)}(r-1)^{n-\omega-1}Q\left(zr^{\frac{\omega-1}{2}}\right)dr\right]\\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right].\end{aligned}$$

Take $\varphi \in C^1(0,\infty)$ and $z=xt^{\frac{1-\omega}{2}}$, we get that

$$t\frac{\partial}{\partial t}\varphi(z) = tx\left(\frac{1-\omega}{2}\right)t^{\frac{1-\omega}{2}-1}\varphi'(z) = \frac{1-\omega}{2}z\frac{\partial}{\partial z}\varphi(z).$$

Hence, the following subordinate is generated:

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(n-\omega+\frac{-(\omega+1)}{2m}+\frac{1-\omega}{2}z\frac{\partial}{\partial z}\right)\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right].\end{aligned}$$

Repeating the same argument $n-1$ times, one acquires

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right]\\ &= t^{-\omega+\frac{-(\omega+1)}{2m}}\prod_{k=1}^{n}\left[\left(1-\omega+\frac{-(\omega+1)}{2}+k+\frac{1-\omega}{2}z\frac{\partial}{\partial z}\right)\left(k_{\frac{2}{\omega-1}}^{1+\frac{-(\omega+1)}{2m},n-\omega}Q\right)(z)\right],\end{aligned}$$

which can be simplified, by Erdélyi-Kober FDO, into

$$\frac{\partial^{\omega} h}{\partial t^{\omega}} = t^{n-\omega+\frac{-(\omega+1)}{2m}}\left(P_{\frac{2}{\omega-1}}^{1+\frac{-\omega(2m+1)-1}{2m},\omega}Q\right)(z) = t^{\frac{-\omega(2m+1)-1}{2m}}\left(P_{\frac{2}{\omega-1}}^{1+\frac{-\omega(2m+1)-1}{2m},\omega}Q\right)(z).$$

By this, the proof is complete. ■

4. Explicit power series analysis

In this section, we discuss the explicit solution of the nonlinear FDEs by employing the power series technique [21]. It is clear that once we obtain the explicit solution of FDE, then we can get the power series solutions to FPDE (1). Consider the subordinate expansion $Q(z) = \sum_{n=0}^{\infty} a_n z^n$. Then by simple calculations and comparing the coefficients we conclude that the explicit solution to (14) is represented in the form:

$$Q(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \sum_{n=1}^{\infty} a_{n+3} z^{n+3}, \tag{16}$$

where a_3 and a_{n+3} for $n \geq 1$ are found as

$$a_3 = \frac{1}{6\lambda_2}\left(\frac{\Gamma\left(2+\frac{-\omega(2m+1)-1}{2m}\right)}{\Gamma\left(2-\frac{\omega+1}{2m}\right)} a_0 + \lambda_1 (a_0)^m a_1 + 2\lambda_2 \frac{(\omega+1)+2m(1-\omega)}{2m} a_2\right),$$

$$\begin{aligned} a_{n+3} = {} & \frac{1}{(n+1)(n+2)(n+3)\lambda_2}\left(\frac{\Gamma\left(2+\frac{-\omega(2m+1)-1}{2m}-\frac{n(\omega-1)}{2}\right)}{\Gamma\left(2-\frac{\omega+1}{2m}-\frac{n(\omega-1)}{2}\right)} a_n\right) \\ & + \lambda_1 \sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}} (n-i_1+1) a_{i_m} a_{i_{m-1}-i_m} a_{i_{m-2}-i_{m-1}} \cdots a_{i_1-i_2} a_{n-i_1+1} \\ & + \lambda_2 (n+1)(n+2)\frac{(\omega+1)+2m(\omega-1)}{2m} a_{n+2}. \end{aligned} \tag{17}$$

Therefore, for a free selection of a_0, a_1, a_2, the explicit power series solution to (1) is

$$\begin{aligned} h(x,t) = {} & a_0 t^{\frac{-(\omega+1)}{2m}} + a_1 x\, t^{\frac{m(1-\omega)-(\omega+1)}{2m}} + a_2 x^2\, t^{\frac{2m(1-\omega)-(\omega+1)}{2m}} \\ & + \left(\frac{\Gamma\left(\frac{-\omega(2m+1)-1}{2m}+2\right)}{\Gamma\left(2-\frac{\omega+1}{2m}\right)} a_0 + \lambda_1 (a_0)^m a_1 + 2\lambda_2 \frac{2m(1-\omega)+(\omega+1)}{2m} a_2\right) \\ & \times \frac{x^3\, t^{\frac{3m(1-\omega)-(\omega+1)}{2m}}}{6\lambda_2} + \sum_{n=1}^{\infty} a_{n+3} x^{n+3}\, t^{\frac{m(n+3)(1-\omega)-(\omega+1)}{2m}}. \end{aligned} \tag{18}$$

5. The convergent analysis

The radius of convergence determines the range of values for which the series converges and provides accurate approximations. Understanding the convergence behavior and radius of convergence is essential for determining the applicability and limitations of power series methods. In this section, we study the convergence of the explicit power series solution that we obtained in the previous section. In fact, we have

$$\begin{aligned}|a_{n+3}| \leq [\frac{1}{|\lambda_2|}&\left(\frac{\Gamma\left(2+\frac{-\omega(2m+1)-1}{2m}-\frac{n(\omega-1)}{2}\right)}{\Gamma\left(2-\frac{\omega+1}{2m}-\frac{n(\omega-1)}{2}\right)}|a_n|\right.\\&+|\lambda_1|\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\dots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}}|a_{i_m}|\left|a_{i_{m-1}-i_m}\right|\left|a_{i_{m-2}-i_{m-1}}\right|\dots|a_{i_1-i_2}|\left|a_{i_{n-i_1+1}}\right|\\&\left.+|\lambda_2|\left|\frac{(\omega+1)+2m(\omega-1)}{2m}\right||a_{n+2}|\right).\\&\leq K\left(|a_n|+|a_{n+2}|+\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\dots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}}|a_{i_m}|\left|a_{i_{m-1}-i_m}\right|\dots|a_{i_1-i_2}|\left|a_{i_{n-i_1+1}}\right|\right),\end{aligned} \tag{19}$$

where $K=\max\{\frac{1}{|\lambda_2|},\frac{|\lambda_1|}{|\lambda_2|},\left|\frac{(\omega+1)+2m(\omega-1)}{2m}\right|\}$.

Set the power series $S(z)=\sum_{n=0}^{\infty}m_n z^n$ with $m_i=|a_i|$ for all $i\in\{0,1,2,\dots\}$. Then, we have

$$m_{n+3}\leq K[m_n+m_{n+2}+\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\dots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}}|m_{i_m}|\left|m_{i_{m-1}-i_m}\right|\dots|m_{i_1-i_2}|\left|m_{i_{n-i_1+1}}\right|].$$

Hence, we get that $|a_n|\leq m_n$ and $S(z)$ is the majorant series to $Q(z)$. Therefore,

$$\begin{aligned}S(z)=m_0+m_1z+m_2z^2+m_3z^3+K\sum_{n=0}^{\infty}m_nz^{n+3}+K\sum_{n=0}^{\infty}m_{n+2}z^{n+3}\\+K\sum_{n=0}^{\infty}\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\dots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}}m_{i_m}m_{i_{m-1}-i_m}\dots m_{i_1-i_2}m_{i_{n-i_1+1}}z^{n+3}.\end{aligned} \tag{20}$$

This means that the series $S(z)$ possesses a positive radius of convergence, which allows us to assume that the implicit functional equation with respect to z is represented in the form:

$$\begin{aligned}R(z,S)=S(z)-m_0-m_1z-m_2z^2-m_3z^3-K\sum_{n=0}^{\infty}m_nz^{n+3}-K\sum_{n=0}^{\infty}m_{n+2}z^{n+3}\\-K\sum_{n=0}^{\infty}\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\dots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}}m_{i_m}m_{i_{m-1}-i_m}\dots m_{i_1-i_2}m_{i_{n-i_1+1}}z^{n+3}=0.\end{aligned}$$

Since $R(z,S)$ is analytic in a neighborhood $(0,m_0)$ with $R(0,m_0)=0$ and $\frac{\partial}{\partial S}R(0,m_0)\neq$ $\neq 0$, then, by the implicit function theorem [21], the convergence is satisfied.

6. Conservation laws

Conservation laws are very important parts of the discussing the FPDEs. Actually, they prove the existence and uniqueness of solutions, provide conserved quantities for all the solutions and also explain the linearization. Let us establish the conservation laws of the (1) regarding the formal Lagrangian and LPSs. Consider a vector $C = (C^x, C^t)$ that admits the subordinate conservation equation

$$D_x(C^x) + D_t(C^t) = 0,$$

where $C^x = C^x(x,t,h,...)$ and $C^t = C^t(x,t,h,...)$ are the conserved vectors for (1). According to the conservation theorem in [22], we get that the formal Lagrangian to (1) is given by

$$\mathscr{L} = w(x,t)\left[D_t^{\omega}h + \lambda_1 h^m \frac{\partial h}{\partial x} - \lambda_2 \frac{\partial^3 h}{\partial x^2 \partial t}\right],$$

where $w(x,t)$ is a smooth function. Depending on the definition of Lagrangian, we obtain the following action integral:

$$\int_0^t \int_\Omega L(x,t,h,w,D_t^{\omega},h_x,h_{xx},h_{xxt})dxdt.$$

We point out that the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta h} = \frac{\partial}{\partial h} + (D_t^{\omega})^* \frac{\partial}{\partial D_t^{\omega}h} + D_x \frac{\partial}{\partial h_x} + D_x D_x D_t \frac{\partial}{\partial h_{xxt}},$$

where $(D_t^{\omega})^*$ is the adjoint operator of D_t^{ω}, the Riemann-Liouville left-sided time-fractional derivative given by

$$_0D_t^{\omega} = D_t^n \left({}_0I_t^{n-\omega}\right)$$

$$_0I_t^{n-\omega} f(x,t) = \frac{1}{\Gamma(n-\omega)} \int_0^t (t-\tau)^{n-\omega-1} f(\tau,x)d\tau.$$

Now, consider the dependent variable $\vartheta = \vartheta(x,t)$ to get the subordinate result:

$$R^* + D_x(\tau_1)I + D_t(\tau_2)I = W\frac{\delta}{\delta h} + D_x(C^x) + D_t(C^t),$$

where I represents the identity operator, and R^* is given by

$$R^* = \tau_2 \frac{\partial}{\partial x} + \tau_1 \frac{\partial}{\partial t} + \daleth \frac{\partial}{\partial h} + \daleth_t^{\omega} \frac{\partial}{\partial D_t^{\omega}h} + \daleth^x \frac{\partial}{\partial h_x} + \daleth^{xxt} \frac{\partial}{\partial h_{xxt}}.$$

Moreover, W is the Lie characteristic function defined as:

$$W = \urcorner - \tau_2 h_x - \tau_1 h_t.$$

Apply the RLD to equation (1), the density component C^t of conservation low is given by:

$$C^t = \tau_1 \mathscr{L} + \sum_{k=0}^{n-1} (-1)^k D_t^{\omega-1-k}(W_s) D_t^k \frac{\partial \mathscr{L}}{\partial ({}_0D_t^{\omega} h)} - (-1)^n J(W_s, D_t^n \frac{\partial \mathscr{L}}{\partial ({}_0D_t^{\omega} h)}),$$

where J is defined by $J(f,g) = \dfrac{1}{\Gamma(n-\omega)} \int_0^t \int_t^T \dfrac{f(\tau,x) g(\mu,x)}{(\mu-\tau)^{\omega-1-n}} d\mu d\tau.$

The other component C^x is given by

$$C^x = \tau_2 \mathscr{L} + W_s \left(\frac{\partial \mathscr{L}}{\partial h_x} - D_x \frac{\partial \mathscr{L}}{\partial h_{xx}} + D_x D_t \frac{\partial \mathscr{L}}{\partial h_{xxt}} \right) + D_x(W_s) \left(\frac{\partial \mathscr{L}}{\partial h_{xx}} - D_t \frac{\partial \mathscr{L}}{\partial h_{xxt}} \right)$$
$$+ D_x D_t (W_s) \left(\frac{\partial \mathscr{L}}{\partial h_{xxt}} \right) + D_t(W_s) \left(\frac{\partial \mathscr{L}}{\partial h_{xt}} - D_x \frac{\partial \mathscr{L}}{\partial h_{xxt}} \right).$$

We indicate here that W_s represents different forms of the function W. Now, we generate the components of the conservation laws for equation (1).

Case 1: For the case $W_1 = -h_x$, we get that the t and x components of the conserved vectors are:

$$C^t = w D_t^{\omega-1}(-h_x) + J(-h_x, w_t), \quad C^x = w(D_t^{\omega} h) - \lambda_2 h_{xx} w_t - \lambda_2 h_{xt} w_x + \lambda_2 h_x w_{xt}.$$

Case 2: For the case $W_2 = -(\omega+1)h - (\omega-1)mxh_x - 2mth_t$, the t and x components of the conserved vectors are

$$C^t = 2mtw \left[D_t^{\omega} h + \lambda_1 h^m \frac{\partial h}{\partial x} - \lambda_2 \frac{\partial^3 h}{\partial x^2 \partial t} \right] - I^{1-\alpha}((\omega+1)h - (\omega-1)mxh_x - 2mth_t)$$
$$+ J(-(\omega+1)h - (\omega-1)mxh_x - 2mth_t, w_t).$$

$$C^x = (-(\omega+1)h - (\omega-1)mxh_x - 2mth_t)(\lambda_1 h^m w - \lambda_2 w_{xt}) + \lambda_2 D_x(W_2) w_t$$
$$+ \lambda_2 D_t(W_2) w_x - \lambda_2 D_x D_t (W_2) w.$$

7. Conclusions

In conclusion, this research has contributed to the understanding of the (1+1)-dimensional time-fractional modified Benjamin-Bona-Mahony equation equipped with Riemann-Liouville derivatives. By utilizing the Erdelyi-Kober fractional operator, we derived a power series solution and analyzed its convergence properties. Furthermore, we established novel conservation laws for the equation. These findings lay the groundwork for future studies that can explore higher-dimensional versions, validate the obtained solution through numerical simulations, and investigate additional conservation laws. Overall, this research enhances our understanding of the dynamics of the time-fractional modified Benjamin-Bona-Mahony equation and opens up possibilities for further exploration in this field.

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