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EXPANDING THE PROBLEMS OR FRUITFUL RESULTS OF SIMPLE QUESTIONS IN MATHEMATICS

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Abstract:

In this article, three famous, ancient geometric problems (angle trisection, squaring the circle, and the Delian problem) are described as expanding mathematical problems, i.e., a transition from the problems already solved to new problems of a higher level.

The author recalled centuries of work by mathematicians trying to solve the above mentioned problems of geometry.

It was emphasized that only the use of algebra in the nineteenth century made it possible to give (unfortunately negative) answer to the question about the feasibility of these geometry structures.

The article highlights the need for putting simple questions (e.g. in the category of "expanding problems"). Asking such questions may lead to new, not always easy, but often interesting problems.

Keywords:

expanding problems, angle trisection, squaring the circle, the Delian problem

INTRODUCTION

Modern mass education of mathematics, both at a basic and at academic level often seems to be reduced to teaching the students, how to solve several types of rather easy problems and this in accordance to a few particular, predefined schemes to boot. Meanwhile, the skill of proper ordering of the knowledge already gained (not only of mathematics!), the ability to sum up the results of the student's own work and, afterwards, an attempt at answering the question: "Can I go even further; could I push myself even more?" is a vital component of the educational process. This is not far from the issue

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announced in the title of the paper, which is expanding the problems. This issue has been illustrated basing on the three ancient geometric problems.

1. CLASSICAL PROBLEMS OF ANTIQUITY. EXPANDING THE PROBLEMS

Ancient Greece produced three basing geometric problems:

- angle trisection;
- squaring the circle;
- the Delian problem.

These issues were to have been solved using compass and straightedge (a sort of ruler with no equally spaced markings on it). Initially, the condition of using only these two tools seemed to be a mere impediment, but not an insurmountable obstacle in solving these problems. However, the two thousand years have revealed that the above-mentioned condition of using just the two tools cannot be fulfilled. These problems of ancient Greek mathematics could be looked on as the issues which arose out of a natural willingness to solve the issues which seemed only slightly more difficult than the problems which had already been solved. In the terminology of a famed mathematics teacher and the creator of the Cracow school of mathematics teaching, Zofia Anna Krygowska, one can say that we are dealing with "expansion" of a mathematical problem ([3], p. 101).

2. ANGLE TRISECTION

Trisection of any angle can be interpreted as an expansion of the problem of bisecting any angle into two congruent angles. Being able to bisect any angle into two congruent angles, naturally we start asking ourselves about the possibility of dissecting any angle into three congruent angles, that is the possibility of ***angle trisection***. As has turned out, the trisection of some angles (such as 90° 45°) was relatively easy but there were such angles which "were not eager" to undergo trisection using just the compass and straightedge. The 60° angle was the most "reluctant" angle. Impossibility of its trisection automatically yields a negative answer to the question of the possibility of constructing a 40° angle. Subsequently, this negative result leads to a conclusion that is impossible to construct a nonagon using a compass and straightedge. For ancient Greek mathematicians, those admirers of construction and harmony in geometry, this posed a vital problem.

Mathematicians in the Middle Ages noted that the issue of angle trisection is connected with solving a cubic equation. This was, therefore, not "expansion" of a problem, but rather its substitution with an equivalent problem in a branch of mathematics different than geometry. This is how the issue of angle trisection impacted the development of medieval algebra.

In modern age, some schoolmen came into the conclusion that angle trisection using the compass and straightedge only is impossible. René Descartes (Latinised: Renatus Cartesius) was one among those who, in 1637, expressed his doubt, as far as the possibility of trisecting any angle using compass and straightedge is concerned. The doubts, however, even the doubts of a great thinker like Descartes, cannot be substitute for a proof. And for such proof the world had to wait almost another 200 years, since only in the 19^{th} century, in 1837, did a French mathematician Pierre Laurent Wantzel (1814–

1848) announce a precise mathematical proof of the impossibility of trisecting any angle using compass and straightedge method. This was a great achievement for a man of 23.

3. THE DELIAN PROBLEM

The Delian problem relates to an ancient prophecy by an oracle in Delos. The Athenians were advised by the God Apollo to double his altar (which had the form of a cube) without changing its shape. The altar was a regular hexahedron of side length a. Therefore the altar which was to be built had to be a cube with volume of $2a^3$, which means that its side would equal $\sqrt[3]{2}$, Using the language of modern algebra, a number equal $\sqrt[3]{2}$, had to be constructed, which despite being an algebraic number (fulfils the equation $x^3 - 2 = 0$), was not a constructible number. This is so because the degree to which field can be expanded $Q(\sqrt[3]{2})$ in relation to field Q (field of rational numbers) equals 3.

Therefore, the necessary and sufficient condition of a complex number being constructible is not fulfilled. This condition is as follows ([1], p. 163): *complex number is constructible using compass and straightedge if and only if the extension field $Q(a)$ of the field Q has degree a power of 2.*

The Delian problem can be looked on as an expansion of the problem of how to construct a square (circle) with the field doubling a given square (circles). However, this issue is easily solved. The area of a square with side 1 equals 1; the area of a square with side $\sqrt{2}$ equals 2, which is twice as large as the area of a square with side 1. Similarly, the area of circle with radius $\sqrt{2}$ is twice as large as the area of the circle with radius 1. Thus it is enough to construct a line segment equal $\sqrt{2}$, which is generally considered to be easy, since the length of diagonal of a square with side 1 equals $\sqrt{2}$. It is only after 2000 years that the problem of cube duplication proved to be impossible to solve.

The nonexistence of a solution for the Delian problem was finally proven by the abovementioned French scholar, Pierre Laurent Wantzel, in 1837.

4. SQUARING THE CIRCLE. RECTIFICATION OF THE CIRCLE

Among the construction problems of antique Greece, one which posed greatest difficulty for the mathematicians and which puzzled them was the challenge of ***squaring the circle*** and the related rectification (straightening) of circle. Nowadays, the expression: “squaring a circle” is used as a metaphor for trying to do the impossible or attain the unattainable. In mathematical sense, it means constructing a square with the same area as a given circle and doing it only with compass and straightedge. The issue can also be studied as expansion of the problem, *how to construct a square with the same area as two given squares with compass and straightedge only.*

The solution to this problem is known to anyone who can geometrically interpret the Pythagorean theorem which states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. If, in the problem above, we should substitute the squares with a circle, and if such square was to be of the same area as the area of the said circle, we obtain a problem with which mathematicians tackled for several centuries.

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The issue similar to the issue of squaring the circle is ***rectifying a circle,*** that is constructing an ideal straight line equal in length to the circumference of the circle with compass and straightedge. Adam Adamandy Kochański – Polish Jesuit and court mathematician for king John III Sobieski – published the approximate construction of rectifying a circle in 1685, in a science magazine entitled "Acta Eruditorum". In the literature devoted to mathematics, Kochański's construction is considered to be one of the simplest and most elegant approximate solutions of rectifying a circle. Is this approximate construction THE solution to the problem? For the mere users of mathematics: yes, for mathematicians: most surely no.

Similarly as with the angle trisection and the Delian problem, the solution to both the rectification of a circle and squiring of circle problems was produced using algebraic methods. It is sufficient to note that squaring of a circle with radius $r = 1$ requires construction of a square with area equal $\pi r^2 = \pi \cdot 1^2 = \pi,$ which means that its side equals $\sqrt{\pi}$. Rectification of a circle with radius $r = 0{,}5$ requires construction of a line segment, the length of which equals the circumference of the circle, that is $2\pi r = 2\pi \cdot 0{,}5 = \pi$. Since the constructability of the number $\sqrt{\pi}$. results from the constructability of the number π., both these problems are connected with the problem of constructability of the number π. It is only in 1882 when a German mathematician Ferdinand Lindemann (1852-1939) proved that π. is a transcendental number, meaning it is not a root of any polynomial with rational coefficients. The conclusion from the π. transcendence is its lack of constructability and thus the impossibility of squaring the circle and rectification of the circle.

CONCLUSION

As we can see, posing simple questions, on the surface not much more complex than the problems which had already been solved, leads to new problems, often very complicated ones but which contribute to the development and growth of mathematics as a science. Studying mathematics involves discovery of subjectively new algorithms and theorems of this discipline. Therefore we should not only refuse to avoid asking simple question, but encourage the students to pose such questions, including those which are deemed to be "expansions" of the problems already solved.

A brilliant mathematician and a Stanford University professor of Hungarian descent, Geörgy (George) Pólya in his influential book [4] thus recommended that after a problem had been solved, we should not be satisfied with the mere solution but should strive (having this problem solved) to once again reflect on the solution of the problem in order to improve it and perhaps also to "expand" the problem already solved.

Expansion of the problems involves a reverse operation – reduction of a problem to the problems which have already been solved. Theorem of the reduction of order of linear homogeneous differential equation ([2], p. 48), which is encountered by the students studying differential equations, can serve as an example.

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