Application of Uninorms to Aggregate Uncertainty From Many Classifiers

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Abstract:

In this contribution we want to present the concept of uncertainty area of classifiers and an algorithm that uses uninorms to minimize the area of uncertainty in the prediction of new objects by complex classifiers.

Keywords: *aggregation function, decision making, classifier, uncertainty area*

1. Introduction

The main task of the classification constituting one of the important methods of data mining is the creation of models, called classifiers (also classifying algorithms or decision algorithms), describing dependencies between the given class (category) of objects and their characteristics. Discovered classification models are then used to classify new objects of the unknown class membership (see e.g., [27]). We will consider a problem of approximation of concepts (classes) based on a finite set of observations containing examples of positive and negative concepts. This finite set of observations may be represented using data tables. In this representation individual observations correspond to rows of a given data table and attributes to columns of a given data table. In this paper, we consider decision tables of the form $\mathbf{T} = (U, A, d)$ in Pawlak's sense (cf. [23]) for representation of data tables, where U is a set of objects (rows) in the data table, A is a set of attributes or columns in the data table, and d is a distinguished attribute from the set A called a decision attribute (in this paper, we consider problems for the case of a 2-class classification, e.g., for decision classes YES and NO or for decision classes 0 and 1, etc.).

The classifier assigns to the object a certain weight (classification coefficient) to classify the object. For a set range of the threshold parameter $t \in (0, 1)$, if the classification weight of the test object obtained from the classifier is greater than t, the object is classified into the main class (e.g., YES). However, if the weight is less than or equal to t, then the object is classified into a subordinate class (e.g., NO). However, for some neighborhood threshold t very small differences in the classification weight can lead to opposing decisions. In order to avoid the incorrect classification, we propose to introduce an uncertainty area, which if the classifier returns the classification weight from a certain neighborhood of threshold, will lead to abstain from the decision. When classifying objects, we can construct different classifiers. Often the decisions obtained differ for a some elements. Therefore, a conflict appears between the classifiers that operate on the basis of different sources or parameters, which must be resolved in order to finally classify the test object. For this purpose we suggest aggregation of values obtained by the individual classifiers using uninorms. As a result, we build a new compound classifier, which additionally reduces the measure of uncertainty area.

The paper is structured as follows. In Section 2, notions connected with aggregation operators are recalled. In Section 3, the motivation to consider new versions of classifier are provided as well as a description of them is given.

2. Agregations Operators

Firstly, we recall definition of an aggregation function. More details can be found in [5, 7, 8, 19, 28]

Definition 1 (cf. [6]). *A function $A : [0,1]^n \to [0,1]$, $n \in \mathbb{N}$, $n \geq 2$, which is increasing in each variable, i.e.*

$$(\forall_{1 \leq i \leq n} s_i \leq t_i) \Rightarrow A(s_1, \ldots, s_n) \leq A(t_1, \ldots, t_n), \quad (1)$$

for all $s_1, \ldots, s_n, t_1, \ldots, t_n \in [0,1]$ is called an aggregation function (aggregation operator) if $A(0, \ldots, 0) = 0$, $A(1, \ldots, 1) = 1$.

Definition 2 ([6]). *Let $n \geq 2$. $A : \mathbb{R}^n \to \mathbb{R}$ is a mean (average function) if it is increasing and idempotent, i.e.*

$$\forall_{s,t \in \mathbb{R}^n} (\forall_{1 \leq k \leq n} s_k \leq t_k) \Rightarrow A(s_1, \ldots, s_n) \leq A(t_1, \ldots, t_n),$$

and

$$\forall_{t \in \mathbb{R}} A(t, \ldots, t) = t.$$

Lemma 1. *For every mean A we have*

$$\forall t \in \mathbb{R}^n \min_{1 \leq k \leq n} t_k \leq A(t_1, \ldots, t_n) \leq \max_{1 \leq k \leq n} t_k. \quad (2)$$

From the above lemma we see that the mean can be restricted to any interval. Our domain of interest is the interval [0, 1]. In this case, the mean is the aggregation function.

Example 1. *Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and $t, w \in [0,1]^n$. We remind here two important examples of aggregation function: the quasi-arithmetic mean (cf. [1])*

$$A(t_1, \ldots, t_n) = \varphi^{-1}\left(\frac{1}{n} \sum_{k=1}^{n} \varphi(t_k)\right),$$

and the generalized weighted average (cf. [6])

$$A(t_1, \ldots, t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(t_k)\right).$$

Uninorms are a special kind of aggregation functions that generalise both t-norms and t-conorms (see [18, 21]). Uninorms are increasing, commutative and associative binary operators on the unit interval having a neutral element $e \in [0,1]$. They appear for the first time using the term uninorm in [29] (although the very related operators called Dombi's operators were already studied in [11]) with the idea of allowing certain kind of aggregation operators combining the maximum and the minimum, depending on an element $e \in (0,1)$. This idea was deeper studied in [18], where the structure of such operators was analysed and two first classes of uninorms were introduced: uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, and representable uninorms (extremely related with Dombi's operators introduced in [11]). We will assume the basic theory of t-norms and t-conorms. The definitions, notations and results on them can be found in [2,21]. We will just give in this section some basic facts about uninorms. More details can be found in [10, 14, 18, 20, 25].

Definition 3 ([29]). *Operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is commutative, associative, increasing and has a neutral element $e \in [0,1]$. A uninorm with neutral element $e = 1$ is called a triangular norm and a uninorm with neutral element $e = 0$ is called a triangular conorm.*

the general structure of the uninorm can be represented by the following theorems (cf. Figure 1).

Theorem 1 ([18]). *If a uninorm U has a neutral element $e \in (0,1)$, then there exist a triangular norm T and a triangular conorm S such that*

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \end{cases}$$

Lemma 2 (cf. [18]). *If U is increasing and has a neutral element $e \in (0,1)$, then*

$$\min \le U \le \max \text{ in } A(e) = [0,e)\times(e,1] \cup (e,1]\times[0,e).$$

Furthermore, if U is associative, then $U(0,1), U(1,0) \in \{0,1\}$.

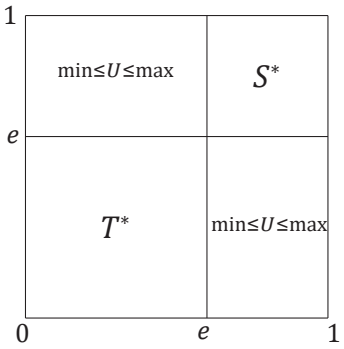

Fig. 1. The structure of uninorms

The most studied classes of uninorms are:

• Uninorms in $\mathcal{U}_{\min}$ (respectively $\mathcal{U}_{\max}$), those given by minimum (respectively maximum) in $A(e)$, that were characterized in [18].

• Representable uninorms, those that have additive generators. They were firstly introduced in [18] and then they were characterized as those uninorms that are continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$ in [24] and also as those uninorms that are strictly increasing and continuous in the open unit square in [17].

• Uninorms continuous in the open unit square $(0,1)^2$, that were characterized in [12,20] and that clearly includes the representables ones.

• Idempotent uninorms, those such that $U(x,x) = x$ for all $x \in [0,1]$. Their characterization was given in [25].

• Locally internal uninorms, those such that $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$. This class has been studied in [12, 13, 15] and a recent characterization of uninorms in this class having continuous underlying operators has been given in [14]. This class includes all idempotent uninorms.

• Uninorms with continuous underlying operators. This class is characterized via the ordinal sum construction of Clifford [22]. Again it is clear that this class includes all the previous ones except for the case of uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and for the case of locally internal uninorms.

In what follows we recall some results about the structure of several classes of uninorms.

Theorem 2. *([18]) Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in (0,1)$. Then, the sections $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ are continuous at each point except perhaps at e if and only if U is given by one of the following formulas:*

(a) If $U(1,0) = 0$, then

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x,y \in [0,e] \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } x,y \in [e,1] \\ \min(x,y) & \text{if } x,y \in A(e) \end{cases}$$

(b) If $U(0,1) = 1$, then the same structure holds, changing minimum by maximum in $A(e)$.

Example 2. *Let $e \in (0,1)$. The following operations are uninorms satisfying the conditions of Theorem 2.*

$$U_1(x,y) = \begin{cases} \min(x,y) & \text{if } x,y \le e, \\ \max(x,y) & otherwise \end{cases}$$

$$U_2(x,y) = \begin{cases} \max(x,y) & \text{if } x,y \ge e, \\ \min(x,y) & otherwise \end{cases}$$

The whole characterization of idempotent uninorms was definitively given in [25] as follows.

Theorem 3. *([25]) Consider $e \in (0,1)$. The following items are equivalent:*

(i) U is an idempotent uninorm with neutral element e.

(ii) There exists a decreasing function $g : [0,1] \to [0,1]$ with fixed point e, which is Id-symmetrical, such that

U is given as follows:

$$U(x,y)=\begin{cases}\min(x,y) & \text{if } y<g(x) \text{ or}\\ & (y=g(x) \text{ and } x<g^2(x)),\\ \max(x,y) & \text{if } y>g(x) \text{ or}\\ & (y=g(x) \text{ and } x>g^2(x)),\\ x \text{ or } y & \text{if } y=g(x) \text{ and } x=g^2(x).\end{cases}$$

being commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$.

Example 3. *The operations U_1 and U_2 from Example 2 are idempotent uninorms.*

As it turns out, there are no continuous uninorms with the neutral element $e \in (0, 1)$. Therefore, there is considered continuity on some subset.

Theorem 4 ([18]). *Let $U : [0, 1]^2 \to [0, 1]$ be a binary operation and $e \in]0, 1[$. The following statements are equivalent:*

(i) U is a uninorm with neutral element e that is strictly increasing on $]0, 1[^2$ and continuous on $[0, 1]^2 \setminus \{(0, 1), (1, 0)\}$.

(ii) There exists a strictly increasing bijection $u : [0, 1] \to [-\infty, +\infty]$ with $u(e) = 0$ such that for all $(x, y) \in [0, 1]^2$ it holds that $U(x, y) = u^{-1}(u(x) + u(y))$, where in case of a conjunctive uninorm U, we adopt the convention $(+\infty) + (-\infty) = -\infty$, while in case of a disjunctive uninorm, we adopt the convention $(+\infty) + (-\infty) = +\infty$.

If this representation holds, then the function u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.

Example 4. *Let $e \in (0, 1)$. The following operations are uninorms satisfying the conditions of above Theorem.*

$$U_3(x,y)=\begin{cases}0 & \text{if } x \text{ or } y=0,\\ \dfrac{\left(\frac{1}{e}-1\right)^2 xy}{\left(\frac{1}{e}-1\right)^2 xy+\left(\frac{1}{e}-1\right)(1-x)(1-y)} & \text{otherwise,}\end{cases}$$

$$U_4(x,y)=\begin{cases}1 & \text{if } x \text{ or } y=1,\\ \dfrac{\left(\frac{1}{e}-1\right)^2 xy}{\left(\frac{1}{e}-1\right)^2 xy+\left(\frac{1}{e}-1\right)(1-x)(1-y)} & \text{otherwise.}\end{cases}$$

3. Classifiers

During classification, the classifier assigns a certain classification weight to the object. For a set range of the threshold parameters $t \in (0, 1)$, the test objects are tested in such a way that if the classification weight of the test object obtained from the classifier is greater than t, the object is classified into the main class (e.g., YES). However, if the weight is less than or equal to t, then the object is classified into a subordinate class (e.g., NO). In this way, we obtain the decision value for the test object, which may be correct (consistent with the actual decision in the test table) or incorrect (we make a mistake in the classification).

To calculate the global classification quality of a given classifier with the fixed parameter t we use the accuracy of the classification which is the quotient of the number of correct classifications to the number of all classifications. Accuracy calculated for the test objects from the main class is called sensitivity, and the accuracy calculated for the test objects from a subordinate class we call specificity.

If the sensitivity is unsatisfactory, e.g., in medicine when trying to predict the occurrence of a disease of a patient, it may turn out that the sensitivity of the classification to the main class "sick" is too low, we can balance between sensitivity and specificity, i.e. increasing sensitivity at the expense of decreasing specificity. This approach leads to the concept of the ROC curve, where each point of the ROC curve corresponds to one setting of the classifier's performance (the parameter t).

ROC shows the dependence of sensitivity on error of the first type (one minus specificity) during calibration of the classifier (at various threshold settings).

For classifiers with this property (sensitivity and specificity regulation), the AUC parameter was used to assess their quality. AUC is the measure of the quality of a classifier which is the area under the ROC curve (cf. [16, 26]). The greater is the AUC value the better is the classifier.

3.1. Uncertainty Area

As it was mentioned above, in the method of sensitivity and specificity regulation, the sensitivity of the classifier can be increased at the expense of the decrease in specificity. This is not always an acceptable situation. It may happen that in the case of two decision classes main (eg. YES) and subordinate (eg. NO) instead of obtaining a higher sensitivity for the main class and reducing the specificity for the subordinate class, we would prefer that the classifier does not classify some cases by agreeing to reduce the coverage of the entire test data.

In other words, in the case of classifying test objects, the so-called area of uncertainty for which we abstain from the decision because we are not sure enough about it. Thanks to this, the classifier may make fewer mistakes while classifying, but from time to time, instead of the decision value, the classifier returns "I do not make decisions" or "I do not know".

Since the most errors of classification are made when the classification weight is close to the threshold parameter, we will refrain from the decision for this area. In this situation, the aforementioned ROC curve generation concept can be modified by introducing the uncertainty area. For this purpose, instead of simple threshold parameter t, we consider parameter ε such that $\varepsilon \in [0, \min(t, 1 - t)]$. For the set value of parameters t and ε, classification of the test objects is performed in such a way that if the classification weight of the test object obtained from the classifier is greater than $t + \varepsilon$, then the object is classified into the main class (eg YES). On the other hand, if the classification

weight is less than or equal to $t - \varepsilon$, then the object is classified into a subordinate class (e.g., NO).

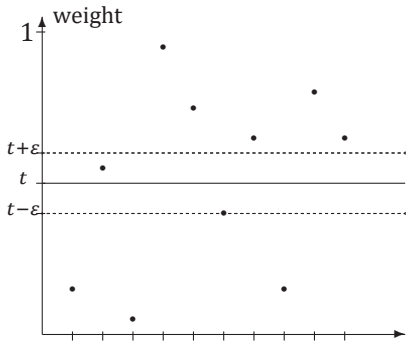

Fig. 2. The uncertainty area

In other cases, the object is classified into the so-called uncertainty area (see Figure 2). Similarly as before, for the calculation of the global classification quality of the given classifier with the parameters t and ε, we use classification accuracy for objects from the main class called sensitivity and accuracy calculated for the test objects from a subordinate class (specificity). In addition, for each experiment we obtain a third parameter, which we call a measure of the uncertainty of classification, which is a quotient of the number of test objects classified to the uncertainty area and the number of all test objects.

3.2. Two Versions of the Classifier

When classifying objects, we can construct different classifiers (based on different systems or based on different data sources, e.g., using several diagnostic devices - see [3, 4, 9]). Often the decisions obtained differ for a certain class of test elements. Therefore, a conflict appears between the classifiers that operate on the basis of different sources or parameters, which must be resolved in order to finally classify the test object. To get a final decision, we should create a new classifier that will take into account previous results. For this purpose we suggest aggregation of values obtained by the individual classifiers. As a result, we build a new compound classifier.

In this article, we suggest aggregating of the classification weights obtained by individual classifiers, and we propose two algorithms.

The first is when we use the means, denoted by the Algorithm M (Algorithm 1) and, in fact, we get a complex classifier.

Unfortunately, if we apply a quality assessment method that takes into account the uncertainty area, then it turns out that the measures of the uncertainty area for the M classifier is very high.

This is due to the Lemma 1, because for example, for two classifiers with classification weights p_1 and p_2 that classify an object to the uncertainty area (weights p_1, p_2 belong to the interval $[t - \varepsilon, t + \varepsilon]$) our classifier M will classify the object to the uncertainty area (the weight p of the aggregated M classification will belong to the same interval). In addition, if only one

Algorithm 1: Classification of a test object by the M classifier

Input:

1) training data set represented by decision table $\mathbf{T} = (U, A, d)$,
2) collection $C_1, ..., C_m$ of classifiers,
3) test object u,
4) aggregation M,
5) threshold parameters t and ε.

Output: The membership of the object u to the "main class" or to the "subordinate class" or "no decision"

```
1  begin
2      for i := 1 to m do
3          Compute a certain weight ("main class"
           membership probability) for the
           given test object u using the classifier
           C_i and assign it to p_i
4      end
5      Determine the final weight p for the object
       u by aggregating (with a use of the mean
       M e.g., arithmetic mean) the weights
       p_1,...,p_m.
6      if p > t + ε then
7          return u belongs to the "main class"
8      else
9          if p < t − ε then
10             return u belongs to the
               "subordinate class"
11         else
12             return we abstain from the decision
13         end
14     end
15 end
```

of the weights will belong to the interval $[t - \varepsilon, t + \varepsilon]$, the weight of the final classification may belong to that interval.

Thus, by creating a classifier in this way, we increase the measure of the uncertainty area, not necessarily significantly increasing the accuracy of classification of the new classifier M in relation to the accuracy of aggregated classifiers.

Therefore, in this paper, we propose also another method for aggregating classifiers based on the so-called a neutral element being a value from the interval $(0, 1)$. This method will be denoted by Algorithm U (Algorithm 2).

We assume here that the neutral element e of uninorm U will be equal to the threshold parameter t. Then, using Lemma 2 and Theorem 1, for example, for two classifiers with classification weights p_1 and p_2 belonging to the interval $[0, t]$ our classifier U assigns the classification weight of the object, which is less than or equal to the $\min(p_1, p_2)$. That is, if at least one of the weights p_1, p_2 is less than $t - \varepsilon$ then the object will be classified to the subordinate class. In other

Algorithm 2: Classification of a test object by the U classifier

Input:

1) training data set represented by decision table $\mathbf{T} = (U, A, d)$,
2) collection $C_1, ..., C_m$ of classifiers,
3) test object u,
4) uninorm U,
5) threshold parameters t and ε.

Output: The membership of the object u to the "main class" or to the "subordinate class" or "no decision"

```
begin
    for i := 1 to m do
        Compute a certain weight ("main class" membership probability) for the given test object u using the classifier C_i and assign it to p_i
    end
    Determine the final weight p for the object u by aggregating (with a use of the uninorm U e.g., representable uninorm ) the weights p_1,...,p_m.
    if p > t + ε then
        return u belongs to the "main class"
    else
        if p < t − ε then
            return u belongs to the "subordinate class"
        else
            return we abstain from the decision
        end
    end
end
```

cases, the object can be classified to the subordinate class or to the uncertainty area.

If the weights p_1 and p_2 belong to the interval $[t, 1]$, then the classifier U assigns the classification weight of the object, which is greater than or equal to the $\max(p_1, p_2)$. This means that if at least one of the weights p_1, p_2 is greater than $t + \varepsilon$, then this object will be assigned to the main class. In other cases, the object can be classified to the main class or to the uncertainty area.

In both cases the degree of membership of the object to the main class or subordinate class is increased. If the object is classified to the uncertainty area with weights p_1 and p_2, such that $p_1 < t < p_2$, then the classifier U will classify the object to the uncertainty area.

4. Conclusion

In this paper, we presented the concept of uncertainty area and the algorithm allowing the creation of a new classifier based on the known classifiers, which should significantly reduce the measure of uncertainty area. The next step will be to implement the algorithm and test the quality of the created classifier based on the actual data.

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