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ON SOME GENERAL SOLUTIONS OF TRANSIENT STOKES AND BRINKMAN EQUATIONS

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General solution representations for velocity and pressure fields describing transient flows at small Reynolds numbers (Stokes flows) and flows obeying Brinkman models are presented. The geometry dependent vector representations emerge from the incompressibility condition and are expressed in terms of just two scalar functions similar to the Papkovich-Neuber and Boussinesq-Galerkin solution type. We provide new formulae connecting our differential representations and other solutions describing unsteady Stokes flow including Lamb's (1932) general infinite series solution. The unified approach presented here further demonstrates an important link between oscillatory flows and flow through porous media using Brinkman models. It is shown that the solutions of boundary value problems in the latter can be obtained in a straightforward fashion, from the results of the former. This simple but surprising analogy is further explained using the properties (mathematical as well as physical) that are shared by the two different models. The construction of certain physical quantities is also illustrated for spherical and spheroidal inclusions. It is believed that the general solutions presented here will be useful in the computation of multi-particle interactions in transient and Brinkman flows and also in linear elasticity.

Keywords: transient Stokes flow, differential representation, Brinkman equations

1. Introduction

The linearized viscous flow at low-Reynolds numbers is described by a pair of partial differential equations connecting the velocity with the pressure field. The so called transient Stokes equations were solved many years ago for oscillatory motions of a sphere along a diameter by Stokes (1851). Subsequently, Lamb (1945) treated the problems of periodic motion in three-dimensions having special relations to spherical surfaces in a more general manner. In particular, Lamb (1945) presented a general solution of the oscillatory Stokes equations – in terms of three linearly independent scalar functions – suitable for spherical boundaries in the form of an infinite series by extending his idea for the steady case. In principle, all the problems of periodic motions involving a spherical boundary can be solved using Lamb's solution in terms of solid spherical harmonics. On the other hand, spherical geometry provides the most widely used framework for representing small particles and obstacles embedded within a viscous, incompressible fluid characterizing transient creeping flow. Other notable works on this topic include those due to Basset (1888), Mazur and Bedeaux (1974), Felderhof (1978), Lawrence and Weinbaum (1986), Yang and Leal (1991) amongst many others. A brief historical review by Pozrikidis (1994) contains further references on this subject.

There are many efficient methods in use to solve problems in low-Reynolds number flow theory such as numerical computation, stream function technique, analytical function methods and a differential representation technique. An important feature of the differential representation approach is that it leads to closed form analytic solutions in a fairly simple manner. The famous representations due to Boussinesq (1885), Papkovich-Neuber (Neuber, 1934) and Naghdi and Hsu (1961) have been used extensively to solve various steady and nonsteady problems in elasticity and fluid dynamics. In the interest of producing differential representations similar to Papkovich-Neuber and Boussinesq-Galerkin, a general solution in terms of two scalar functions A and B was proposed earlier in the case of steady-state creeping flow (Palaniappan *et al.*, 1992). For steady flows at small Reynolds numbers, the representation due to Palaniappan *et al.* (1992) yields a complete general solution (Padmavathi *et al.*, 1998). Such a representation emerges as a result of the incompressibility condition and therefore can serve as a complete general solution for unsteady creeping flow problems as well. Indeed the common incompressibility feature for steady and unsteady flows suggests that the representation given in Palaniappan *et al.* (1992), Palaniappan (2009) can yield a complete set of basis functions for transient flow problems as well. In the first part of this paper, we discuss the connection between the differential representation given in Palaniappan *et al.* (1992) and other solutions in the context of transient creeping flow. Specifically, we provide new formulae connecting the differential representation and other solutions describing unsteady viscous flow. In particular, we show that the Lamb's general solution follows from the differential representation by a suitable choice of the scalar functions. The connections to other representations are briefly discussed. Another differential representation suitable for bounded flows (Palaniappan, 2000, 2009) constrained by plane wall is also given. This general representation is shown to generate solution forms that are suitable for studying oscillatory motions of disks at low Reynolds numbers.

In the second part, we demonstrate the link between the oscillatory creeping flow and flow through porous media. The equations postulated by Brinkman (1947) in modeling of porous media have been found widely applicable for high porosity systems. The merits of these equations over Darcy's equations (modeling systems with low porosity) may be found in the work of Oomes *et al.* (1970), Neale *et al.* (1973), Masliah *et al.* (1987), and Higdon and Kojima (1981). Moreover, the validity of Brinkman's equations has been justified theoretically by Tam (1969), Saffman (1971), Lundgren (1972), Howells (1974). Furthermore, the problems involving multiparticle interactions in a porous medium (Kim and Russel, 1985) and porous spherical shells (Qin and Kaloni, 1993; Bhatt and Sacheti, 1994; Padmavathi and Amaranath, 1996) also use Brinkman's equations effectively. Some perspectives on convection problems indicating the merits and limitations of equations of porous media may be found in Nield and Bejan (1992). Although the Brinkman model is physically quite different from transient flow model, it has some interesting mathematical similarities as shown here.

2. Solutions of transient creeping flow

The linearized Navier Stokes equations describing the motion of a viscous, incompressible fluid are

$$\rho\frac{\partial\mathbf{u}}{\partial t} = \mu\nabla^2\mathbf{u} - \nabla p \qquad \nabla\cdot\mathbf{u} = 0 \tag{2.1}$$

where $\mathbf{u}$, p are the velocity and pressure fields for the fluid and μ, ρ are the fluid viscosity and fluid density, respectively. For oscillatory motions, $\mathbf{u}$ and p vary as $\mathrm{e}^{\alpha t}$ (α is assumed to be imaginary) that is, $\mathbf{u} = \mathbf{u}'\mathrm{e}^{\alpha t}$, $p = p'\mathrm{e}^{\alpha t}$ and therefore equation $(2.1)_1$ may be written as

$$\mu(\nabla^2 + h^2)\mathbf{u}' = \nabla p' \tag{2.2}$$

where $h^2 = -\alpha/\nu$, ν is the kinematic viscosity. Using $(2.1)_2$ in (2.2) we obtain

$$\nabla^2 p' = 0 \tag{2.3}$$

and now (2.2) reduces to

$$(\nabla^2 + h^2)\nabla^2\mathbf{u}' = 0 \tag{2.4}$$

Thus, the pressure in transient flow is harmonic as in the steady case but the Laplacian of the velocity vector $\mathbf{u}'$ satisfies the Helmholtz equation. It may be worthwhile to point out that in the steady-state creeping flow, the Laplacian of velocity vector is harmonic and, hence, it is biharmonic there. It is evident from (2.2) (by operating curl on both sides) that the vorticity vector satisfies the Helmholtz equation. Incorporating these features, Lamb (1945) presented a general solution of (2.2) and (2.3) as

$$\begin{aligned} \mathbf{u}' &= \sum_{n=-\infty}^{\infty} \Big[\frac{1}{h^2\mu}\nabla p_n + (n+1)\psi_{n-1}(hr)\nabla\phi_n \\ &\quad - n\psi_{n+1}(hr)h^2 r^{2n+1}\nabla\frac{\phi_n}{r^{2n+1}} + \psi_n(hr)\mathbf{r}\times\nabla\chi_n \\ p' &= p_n \end{aligned} \tag{2.5}$$

The solid harmonic functions ϕ_n, χ_n arise from the solution of the homogeneous equation and p_n is the particular solution of (2.2). The function ψ_n is related to the Bessel function of a fractional order that is finite at the origin. In the situation where the quantities of interest are finite at infinity, the function ψ_n is to be replaced by Ψ_n. Their relations to the Bessel functions are as follows

$$\zeta^n\psi_n(\zeta) = \sqrt{\frac{\pi}{2\zeta}}J_{n+\frac{1}{2}}(\zeta) \qquad \text{and} \qquad \zeta^n\Psi_n(\zeta) = (-)^n\sqrt{\frac{\pi}{2\zeta}}J_{-n-\frac{1}{2}}(\zeta)$$

It is worth mentioning here that equation (2.2) may be regarded as the Laplace transform of $(2.1)_1$ with the transform variable $s = h^2$. Then by taking the inverse Laplace transform of $(2.5)_1$, one could obtain a general solution of transient Stokes equations for arbitrary time-dependent flow.

2.1. A general representation for velocity and pressure fields

In the theory of isotropic elasticity and hydrodynamics, it is common to assume the solutions of the governing partial differential equations in terms of auxiliary scalar functions, often referred to as *differential representations*. For instance, in linear elasticity, the displacement vector may be represented in terms of scalar functions as in Boussinesq (1885), Neuber (1934), Naghdi and Hsu (1961) known as Boussinesq, Papkovich-Neuber and Naghdi-Hsu differential representations, respectively. This approach is also followed and applied to many problems in hydrodynamics including inviscid (Milne-Thomson, 1968) and viscous (Stokes, 1851; Palaniappan *et al.*, 1992; Dassios and Vafeas, 2004) flows. The crucial point in this technique is that the governing vector differential equations in each physical model reduce to solving scalar differential equations for the auxiliary functions. In many circumstances, the boundary conditions also turn out to be simpler to apply with the scalar functions technique. The differential representation approach is also applicable for time-dependent viscous flow models. Since the velocity vector ($\mathbf{u}$ and hence $\mathbf{u}'$) in transient flow is divergence-free, a suitable representation for this quantity is

$$\mathbf{u}' = \operatorname{curl}\operatorname{curl}(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B) \tag{2.6}$$

where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and A, B are the scalar functions. The representation (2.6) was originally proposed for steady-state incompressible flows (Palaniappan *et al.*, 1992) and has been shown to yield a complete general solution for creeping flows (Padmavathi *et al.*, 1998). This type of differential representation has also been employed in Feng *et al.* (1998) to investigate the general motion of a circular disk in a Brinkman medium (a problem that is mathematically equivalent to oscillatory Stokes flow discussed in Section 3 and 4 below).

Substitution of (2.6) into (2.2) yields the pressure

$$p' = \mu\frac{\partial}{\partial r}[r(\nabla^2 + h^2)A] \tag{2.7}$$

if

$$\nabla^2(\nabla^2 + h^2)A = 0 \qquad\qquad (\nabla^2 + h^2)B = 0 \tag{2.8}$$

Thus, vector equation $(2.1)_1$ for $\mathbf{u}$ reduces to solving scalar equations (2.8) for the functions A and B. The scalar function A can be decomposed into $A = A_1 + A_2$ where $\nabla^2 A_1 = 0$ and $(\nabla^2 + h^2)A_2 = 0$. It is interesting to note that the function A_1 belongs to the *kernel* of ∇^2 (the Laplace operator) and the functions A_2 and B belong to the *kernel* of $\nabla^2 + h^2$ (the Helmholtz operator). We now provide connection formulae by which we can transform any solution of oscillatory flow from Lamb's solution to differential representation form (2.6) and vice-versa. Following the steady case approach, we take

$$A = \sum_{-\infty}^{\infty}\Big[\frac{1}{h^2\mu}\frac{p_n}{n+1} + (2n+1)\psi_n(hr)\phi_n\Big] \qquad\qquad B = \sum_{-\infty}^{\infty}\psi_n(hr)\chi_n \tag{2.9}$$

Substitution of (2.9) into (2.6) and (2.7) yields Lamb's solution (2.5) after the use of some simple vector identities. Therefore, expressions (2.9) provide the formulae connecting differential representation (2.6) and Lamb's general solution for transient flows. Connections to other solution representations, including the cartesian tensor solution due to Felderhof (1978), may be established in a similar manner.

For flows with axial symmetry, the solution can be found using a single scalar function commonly known as the Stokes stream function. In (2.6), if we take (after expanding the curl operator)

$$\frac{\partial A}{\partial\theta} = \frac{\psi(r,\theta)}{r\sin\theta} \qquad\qquad B = 0 \tag{2.10}$$

((r,θ) refer to spherical polar coordinates) we then obtain the axisymmetric representation of the solution for oscillatory flow. In this case, the Stokes stream function $\psi(r,\theta)$ satisfies the fourth-order scalar equation

$$D^2(D^2 + h^2)\psi = 0 \qquad\qquad D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial\eta^2} \qquad\qquad \eta = \cos\theta$$

The stream function $\psi(r,\theta)$ belongs to the *kernel* of $D^2(D^2 + h^2)$ and can be decomposed into two functions of which one belongs to the *kernel* of D^2 and the other to the *kernel* of $(D^2 + h^2)$. Note that $\mathbf{u}'$ and p' are treated as time-independent quantities and so are the functions A and B. But it should be remembered that they are in fact related to the original velocity and presseure fields $\mathbf{u}$ and p through $\mathbf{u} = \mathbf{u}'\mathrm{e}^{\alpha t}$ and $p = p'\mathrm{e}^{\alpha t}$. For this reason, we retain the name *oscillatory flow* in the following examples, although, the time-dependence is not explicitly mentioned.

2.1.1. Examples

(i) Uniform oscillatory flow. For a uniform oscillatory flow along the x-axis, the scalar functions A and B in (2.6) are

$$A(r,\theta,\phi) = \frac{U}{2}r\sin\theta\cos\phi \qquad\qquad B = 0 \qquad\qquad U > 0 \tag{2.11}$$

The corresponding functions in Lamb's general solution (2.5) are obtained using connection formulae (2.9) as

$$\begin{aligned} & p_1 = Ur\sin\theta\cos\phi \qquad p_n = 0 \qquad \text{for} \quad n \geqslant 2 \\ & \phi_n = \chi_n = 0 \qquad \text{for all} \quad n \end{aligned} \tag{2.12}$$

For the oscillatory flow symmetrical about z-axis (axis of symmetry), (2.10) yields the stream function

$$\psi(r,\theta) = \frac{U}{2} r^2 \sin^2\theta, \tag{2.13}$$

which was used by Stokes (1851) to study the oscillation of a sphere along its diameter.

(ii) Oscillatory rotlet along z-axis. For the flow resulting from a rotlet located at the origin and whose axis along z-direction, one has

$$A = 0 \qquad B = \frac{\cos\theta}{r^2} \mathrm{e}^{-R}(R+1) \tag{2.14}$$

where $R = hr$. Comparison of (2.14) with (2.9) yields

$$\chi_1 = \frac{\cos\theta}{r^2} \qquad \psi_1 = \mathrm{e}^{-R}(R+1) \qquad p_n = \phi_n = 0 \tag{2.15}$$

which are the functions in Lamb's solution representing the rotlet flow. The function ψ_1 in (2.15) can also be expressed in terms of modified Bessel's functions.

In general, the transient flow fields are completely determined through the scalar functions A and B given in differential representation (2.6). This general representation yields a set of basis functions suitable for problems involving spherical boundaries. It may be possible to analyze the utility of (2.6) for other body shapes including spheroids and ellipsoids.

2.2. A general solution representation for a plane wall

For flows in a semi-infinite domain, which is constricted by a plane wall, a more appropriate complete general differential representation is (Palaniappan, 2000)

$$\begin{aligned} \mathbf{u} &= \operatorname{curl}\operatorname{curl}(\hat{\mathbf{e}}_z A) + \operatorname{curl}(\hat{\mathbf{e}}_z B) \\ p &= p_0 + \mu\frac{\partial}{\partial z}[(\nabla^2 + h^2)A] \end{aligned} \tag{2.16}$$

where $\nabla^2(\nabla^2 + h^2)A = 0$, $(\nabla^2 + h^2)B = 0$, and $\hat{\mathbf{e}}_z$ is the unit vector along z-direction. The image solutions for oscillating point singularities located in front of a plane wall (Pozrikidis, 1991) can be constructed using representation (2.16).

Differential representation (2.16) is also suitable for solving boundary value problems concerning the oscillatory motions of circular disks in a viscous fluid. To see this, we first write the velocity components and pressure in the cylindrical coordinates (ρ,θ,z) as

$$\begin{aligned} u_\rho &= \frac{\partial^2 A}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial B}{\partial\theta} \qquad u_\theta = \frac{1}{\rho}\frac{\partial^2 A}{\partial\theta\partial z} - \frac{\partial B}{\partial\rho} \\ u_z &= -\Big(\frac{\partial^2 A}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial\theta^2}\Big) \end{aligned} \tag{2.17}$$

with

$$p = p_0 + \mu\frac{\partial}{\partial z}(\nabla^2 + h^2)A$$

By taking

$$A(\rho,\theta,z)=\frac{\psi(\rho,z)}{\rho}\cos\theta \qquad\qquad B(\rho,\theta,z)=\frac{\chi(\rho,z)}{\rho}\sin\theta$$

we obtain

$$\begin{aligned}
&u_\rho=\Big[\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\Big)+\frac{\chi}{\rho^2}\Big]\cos\theta \qquad\qquad u_\theta=-\Big[\frac{1}{\rho^2}\frac{\partial\psi}{\partial z}+\frac{\partial}{\partial\rho}\Big(\frac{\chi}{\rho}\Big)\Big]\sin\theta\\
&u_z=-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\Big)\cos\theta
\end{aligned} \tag{2.18}$$

and

$$p=p_0+\mu\frac{\cos\theta}{\rho}\frac{\partial}{\partial z}(D^2+h^2)\psi$$

These are precisely the functional forms assumed by Zhang and Stone (1998) to solve edgewise translation of a disk. Likewise, the other particular functional forms that are zero and first order harmonics in θ used in Zhang and Stone (1998) can be deduced from (2.16). Clearly, the flow problems involving higher harmonics in θ do not belong to the class of solutions discussed in Zhang and Stone (1998) and require a more generic approach. Needless to say that such problems can be treated by the use of the general differential representation given in (2.16). Another general solution in cylindrical coordinates is given by Happel and Brenner (1983) in their monograph. Their solution is well suited for steady Stokes flow problems involving infinitely long circular cylinders. One can also find explicit relations connecting the differential representation given in (2.16) and Happel and Brenner's general solution.

It should be pointed out that differential representations (2.6) and $(2.16)_1$ emerge from the incompressibility condition and differ by the vectors $\mathbf{r}$ and $\mathbf{e}_z$. Note that for spherical boundaries $\mathbf{r}$ represents a normal vector to the sphere and for planar surfaces $\mathbf{e}_z$ is a vector normal to the xy-plane. It is evident that the general solution representation is *geometry dependent*. Indeed, the boundary conditions in terms of the scalar functions A and B become considerably simpler. This makes our solution representations more effective and may be preferred over other solution forms for solving boundary value problems in transient Stokes flows. Below, we discuss the solutions given in (2.6) and $(2.16)_1$ in the context of Brinkman's equations which model the flow through porous media.

3. Solutions of Brinkman model equations

Now the equations modeling the flow through a porous medium proposed by Brinkman are

$$\mu\nabla^2\mathbf{v}-\frac{\mu_1}{k}\mathbf{v}=\nabla p \qquad\qquad \nabla\cdot\mathbf{v}=0 \tag{3.1}$$

where $\mathbf{v}$, p and μ have the usual meaning, k is the permeability of the porous medium and μ_1 is the effective viscosity. Equation $(3.1)_1$ may conveniently be written as

$$\mu(\nabla^2-\lambda^2)\mathbf{v}=\nabla p \tag{3.2}$$

where $\lambda^2=\mu_1/(k\mu)$. Note that equations (2.2) and (3.2) have the same structure (except for the sign of λ^2 and h^2) and hence reveal the fact that the two different models are mathematically equivalent. It is straightforward to see (by doing the same operations as before) that the pressure is harmonic, and the Laplacian of velocity and vorticity satisfy Helmholtz's equation in the Brinkman medium as in the case of transient flow. Thus, the equations of porous media and oscillatory flows share some of the physical properties as well. The sign difference in (2.2) and (3.2) merely indicates the choice of the Bessel functions that are having complex and real arguments respectively (Watson, 1952). Hence, the functions J_n are to be replaced by I_n in the porous media model in order to obtain the general solution of Brinkman's equations. We therefore make the following observations:

- the velocity field given in (2.6) in terms of the scalar functions A and B is a suitable differential representation for Brinkman's equations.
- Lamb's general solution given in (2.5) provides an alternative representation of the solution for Brinkman's equations in terms of the scalar functions ϕ_n, p_n and χ_n (all time-independent in this case) with $h^2 = -\lambda^2$.

The above observations add to the list of mathematical similarities between the transient Stokes equations and Brinkman equations which have been noted earlier in the literature (Kim and Russel, 1985; Feng *et al.*, 1998; Pozrikidis, 1991). Similarities between the two models have also been observed for the motion of non-spherical objects (Feng *et al.*, 1998; Zhang and Stone, 1998). Therefore, it appears that with an appropriate interpretation, it is possible to obtain solution for one model from the other. Some specific cases illustrating the above observations are discussed in the next Section.

It may be noted that for planar boundaries in the Brinkman medium, the solution representation given in (2.16) provides a complete general solution. Indeed, this differential representation can be used to find wall images due to point singularities located in the vicinity of porous slabs. The different solution forms assumed in Feng *et al.* (1998) for studying motion of disks in the Brinkman medium can be deduced from (2.16) in the same way as explained in the previous Section. It follows that the scalar representation of solutions of the transient Stokes equations and Brinkman equations are exactly the same as in the case of the steady Stokes flow which follows from the incompressibility condition. It is therefore unnecessary to discuss such representations for these two models separately. Moreover, forms similar to (2.6) and $(2.16)_1$ can also be utilized in classical linear elasticity and elastodynamics as well.

4. Further illustration

We now turn our attention to justify the foregoing observations by considering some special situations. We restrict ourselves to a class of boundary value problems where the velocity vector vanishes at the surface of a body that is immersed in a flow (no-slip or stick boundary condition). Although the solution of the Brinkman equations can be obtained directly from Lamb's solution (Eqs. (2.5)), little care should be taken in extracting the physical quantities. Since unsteady flows exhibit acceleration, an additional term is always present for instance in the force or drag. It could be easily isolated from the force and the remaining part excluding the time factor gives the expression for the force acting on a particle submerged in a Brinkman medium. We illustrate these facts in the examples considered below. For convenience, we denote the quantities for time-dependent flows and porous media (Brinkman's model) with suffixed t and B, respectively and λ as a common parameter. But it should be understood that λ assumes its respective values in two different models.

4.1. Force on a sphere

Consider the Stokes problem of an oscillating spherical pendulum in a viscous fluid. The velocity and pressure fields for this problem are given elsewhere (Stokes, 1851; Lamb, 1945). The force acting on the blob of radius a oscillating about x-axis is

$$F_t = \Big[-6\pi\mu U a\Big(1+\lambda+\frac{\lambda^2}{3}\Big) + \frac{4}{3}\pi\mu U a\lambda^2\Big]\mathrm{e}^{\alpha t} \tag{4.1}$$

We note that the second term on the right hand side of (4.1) is purely due to inertia which is absent in the Brinkman model. Dropping the second term together with the time-dependent factor $\mathrm{e}^{\alpha t}$ in (4.1), one obtains

$$F_B = -6\pi\mu U a\Big(1+\lambda+\frac{\lambda^2}{3}\Big) \tag{4.2}$$

The above expression yields the drag on a test sphere in a Brinkman medium and agrees with the result given in Tam (1969). As said before, the suffixes denote the time-dependent flows (the explicit time-dependence is now clearly seen in (4.1)) and the Brinkman medium, respectively. We have used λ as a common parameter in both the cases and it takes different values in the respective phases.

4.2. Force on a spheroid

Now we consider a slightly oblate spheroid immersed in an oscillating flow. The expression for the force acting on the spheroid is (Lawrence and Weinbaum, 1986)

$$\begin{aligned} F_t = \Big\{&-6\pi\mu U a\Big[\Big(1+\lambda+\frac{\lambda^2}{3}\Big) + \frac{4}{5}\epsilon(1+2\lambda+\lambda^2) \\ &+ \frac{2}{175}\epsilon^2\Big(1+58\lambda+53\lambda^2+\frac{4\lambda^2}{3+3\lambda+\lambda^2}\Big)\Big] + \frac{4}{3}\pi\mu U a\lambda^2(1+\epsilon+\epsilon^2)\Big\}\mathrm{e}^{\alpha t} \end{aligned} \tag{4.3}$$

where ϵ is the departure from the spherical shape. For an oblate spheroid $\epsilon > 0$ and for a prolate spheroid $\epsilon < 0$. Dropping the last term and together with $\mathrm{e}^{\alpha t}$ as in the previous case, we obtain the new result

$$F_B = -6\pi\mu U a\Big[\Big(1+\lambda+\frac{\lambda^2}{3}\Big)+\frac{4}{5}\epsilon(1+2\lambda+\lambda^2)+\frac{2}{175}\epsilon^2\Big(1+58\lambda+53\lambda^2+\frac{4\lambda^2}{3+3\lambda+\lambda^2}\Big)\Big] \tag{4.4}$$

The above expression yields the hydrodynamic force acting on a spheroid submerged in a Brinkman medium. When the permeability is large i.e., when λ is small, we recover the Stokes resistance for a slightly oblate spheroid suspended in a steady flow (Happel and Brenner, 1983). For small permeability, equation (4.4) is dominated by λ^2 term. In this case, the $O(\lambda)$ correction to (4.4) is important because it describes the growth of the boundary layer at the body surface. It may also be noted that when $\epsilon = 0$, (4.4) reduces to that of a perfect sphere.

Finally, the sign difference in (2.4) and (3.2) (in front of h^2 and λ^2) should be interpreted with care in the two models. This difference may have significant impact on the physical quantities of interest. For instance, in the slow oscillatory flow of a viscous fluid past a sphere (Smith, 1995), the vorticity on the sphere $(r = 1)$ is given by

$$\omega_t = \Big[\frac{3}{2}(\sigma+1)\Big]\mathrm{e}^{\alpha t} \tag{4.5}$$

where $\sigma^2 = \mathrm{i}h$. For the slow flow through porous media, the vorticity on the sphere (Pop and Ingham, 1996) is

$$\omega_B = \Big[\frac{3}{2}(\sigma+1)\Big] \tag{4.6}$$

with $\sigma = \lambda$. The vorticity given in (4.5) vanishes for some values of σ (negative values of σ are admissible in this case) implying that flow separation is possible. However, the vorticity given in (4.6) never vanishes for $\sigma > 0$ (σ cannot take negative values here) and consequently there is no flow separation for this flow in the Brinkman medium.

5. Conclusion

General solutions for the transient Stokes flow and flow through porous media are discussed in this paper. Differential representation for transient flow is shown to be equivalent to Lamb's general solution and new formulae connecting the two solutions are given. An alternative representation for bounded flows constrained by a plane wall is also provided. The usefulness of this new solution representation for flows involving disks is briefly outlined. It is observed that the solution representations are geometry dependent. The analogy between the oscillatory flows and the flow through porous media is exploited to derive the solutions of the Brinkman equations. Some representative boundary value problems are considered to justify our observations concerning the two models. Our discussion indicates that the solutions of Brinkman models can be derived from the transient Stoke flow models and so the duplication may be avoided. Apart from genuine mathematical interest, the results provided here may found useful in practice where the general and generalized solutions (Shu and Chwang, 2001) are needed to understand the basic engineering aspects in fluid and elastic environments. Time-dependent multiparticle interactions and mobility of particles close to the Brinkman half-space (Damiano *et al.*, 2004) including electrophoretic motion of charged particles in porous media (Tsai *et al.*, 2011) are a few prospective topics for further research using our general solution representations.

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