

**UNIFORMLY CONTINUOUS COMPOSITION OPERATOR IN
THE SPACE OF FUNCTIONS OF TWO VARIABLES OF
BOUNDED Φ -VARIATION IN THE SENSE OF SCHRAMM**

*WADIE AZIZ, TOMAS EREÚ, NELSON MERENTES, JOSE L. SANCHEZ,
MAŁGORZATA WRÓBEL*

ABSTRACT

We prove in this paper that if the composition operator H , generated by a function $h : I_a^b \times C \rightarrow Y$, maps $\Phi_1 BV(I_a^b, C)$ into $\Phi_2 BV(I_a^b, Y)$ and is uniformly continuous, then the left-left regularization h^* of h is an affine function with respect to the third variable.

1. INTRODUCTION

Let I_a^b denote the rectangle $[a_1, b_1] \times [a_2, b_2]$. Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be real normed spaces and C be a convex cone in X . For a function $h : I_a^b \times C \rightarrow Y$, denote by $X^{I_a^b}$ the algebra of all functions $f : I_a^b \rightarrow X$ and by $H : X^{I_a^b} \rightarrow Y^{I_a^b}$ the Nemytskij operator generated by the function h defined by

$$(Hf)(t, s) = h(t, s, f(t, s)), \quad f \in X^{I_a^b}, (t, s) \in I_a^b.$$

Let $(\Phi BV(I_a^b, X), \|\cdot\|_\Phi)$ be a Banach space of functions $f \in X^{I_a^b}$ which have bounded Φ -variation in the sense of Schramm, where the norm $\|\cdot\|_\Phi$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [1, 16, 17].

Tomas Ereú, Nelson Merentes, Jose L. Sanchez — Universidad Central de Venezuela.

Wadie Aziz — Universidad de los Andes.

Małgorzata Wróbel — Jan Długosz University in Częstochowa.

Assume that H maps the set of functions $f \in \Phi BV(I_a^b, X)$ such that $f(I_a^b) \subset C$ into $\Phi BV(I_a^b, Y)$. In the present paper, we prove that, if H is uniformly continuous, then the left-left, right-right, left-right and right-left regularizations of its generator h with respect to first two variables are affine functions with respect to the third variable. This extends the main results of [2] and [14]. In some spaces the representation theorems for the Lipschitzian Nemytskij operators have been established before, see [3-8, 11].

2. PRELIMINARIES

In this section we recall some facts which will be in need in further considerations.

Denote by \mathbb{R} the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$. We say that a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a φ -function if φ is continuous on \mathbb{R}_+ , $\varphi(0) = 0$, φ is increasing on \mathbb{R}_+ and $\varphi(t) \rightarrow \infty$ when $t \rightarrow \infty$. Let us recall first the concept of the bounded φ -variation in the sense of Wiener ([12]). Namely, we say that a function $u : [a, b] \rightarrow \mathbb{R}$ has a φ -bounded variation in the Wiener sense with respect to a φ -function φ provided the quality $V_\varphi^W(u)$ defined by the formula

$$V_\varphi^W(u) = V_\varphi^W(u; [a, b]) = \sup_\pi \sum_{j=1}^n \varphi(|u(t_j) - u(t_{j-1})|)$$

is finite. Here the supremum is taken over all partitions π of the interval $[a, b]$.

Next, let $\Phi = \{\phi_n\}$ be a sequence of increasing convex functions, defined on the set of nonnegative real numbers and such that $\Phi_n(0) = 0$ and $\Phi_n(t) > 0$ for $t > 0$ and $n = 1, 2, \dots$. We say that Φ is Φ^* -sequence if $\phi_{n+1}(t) \leq \phi_n(t)$ for all n, t and Φ -sequences and in addition

$$\sum_{n=1}^{\infty} \phi_n(t) \quad \text{diverges for all } t > 0. \quad (1)$$

If Φ is either a Φ^* -sequence or a Φ -sequence, we say that a function u is of Φ -bounded variation in the Schramm sense if the Φ -sum $\sum \phi_n(|u(I_n)|)$ is finite for any non-overlapping collection $\{I_n\}$ of I ([10]). If $I_n = [a_n, b_n]$ is a subinterval of the interval I ($n = 1, 2, \dots$) we write $u(I_n) := u(b_n) - u(a_n)$.

We introduce the $\Phi = \{\phi_{n,m}\}$ two dimensional sequence of increasing convex functions, such that $\phi_{n,m}(0) = 0$ and $\phi_{n,m}(t) > 0$ for $t > 0$ and

$n, m = 1, 2, \dots$ We say that Φ is Φ -sequence [13, 14] if

$$\begin{aligned} \phi_{n',m'}(t) &\leq \phi_{n,m}(t) \text{ for each } n' \leq n, m' \leq m, t \in [0, \infty) \\ \text{and } \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}(t) &\text{ diverges for } t > 0. \end{aligned} \quad (2)$$

3. NOTATION, DEFINITIONS AND AUXILIARY FACTS

At the beginning assume that $a = (a_1, a_2), b = (b_1, b_2)$ are two fixed points in the plane \mathbb{R}^2 . Denote by I_a^b the rectangle generated by the points a and b , i.e., $I_a^b = [a_1, b_1] \times [a_2, b_2]$.

Next, let us assume that $\{I_n\}$ and $\{J_m\}$ are two sequences of closed subintervals of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. It means $I_n = [a_1^n, b_1^n]$, ($n = 1, 2, \dots$), $J_m = [a_2^m, b_2^m]$, ($m = 1, 2, \dots$).

Finally assume that $f : I \rightarrow \mathbb{R}$ is a given function and let $\Phi = \{\phi_{n,m}\}$ be a fixed double Φ sequence.

Fix $x_2 \in J_1 = [a_2, b_2]$ and consider the function $f(\cdot, x_2) : [a_1, b_1] \rightarrow \mathbb{R}$. The quantity V_{Φ, I_1}^S defined by the formula

$$\begin{aligned} V_{\Phi, I_1}^S(u) &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m} (|f(I_n, x_2)|) \\ &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m} (|f(b_n, x_2) - f(a_n, x_2)|) \\ &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m} (|f(b_n, x_2) - f(a_n, x_2)|), \end{aligned} \quad (3)$$

is said to be Φ -variation in the sense of Schramm of the function $f(\cdot, x_2)$. In the case when $V_{\Phi, I_1}^S(f) < \infty$ we will say that f has a bounded Φ -variation in the sense of Schramm with respect to the first variable (with fixed the second one). In the same way one can define the concept of the Φ -variation of the function $f(x_1, \cdot)$ in the Schramm sense. It is denoted by V_{Φ, J_1}^S . Obviously, if $V_{\Phi, J_1}^S(f) < \infty$ then one can say that f has bounded Φ -variation in the sense of Schramm with respect to the second variable (with fixed the first one).

Let us pay attention to the fact that the least upper bound in formula (3) is taken with respect to all sequences $\{I_n\}$ of subintervals of the interval I_1 . Analogously we understand the least upper bound in the definition of the

quantity V_{Φ, J_1}^S [13, 14]. Further, we provide the definition of the concept of two dimensional (or bi-dimensional) variation in the sense of Schramm.

Definition 1. *The quantity $V_{\Phi, I_a^b}^S(f)$ defined by the formula*

$$\begin{aligned} V_{\Phi, I_a^b}^S(f) &= \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} (|f(I_n, J_m)|) = \\ &= \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} (|f(b_n, J_m) - f(a_n, J_m)|) = \\ &= \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} (|f(a_n, c_m) + f(b_n, d_m) - f(a_n, d_m) - f(b_n, c_m)|), \end{aligned}$$

is said to be the bi-dimensional variation in the sense of Schramm of the function f where the least upper bound is considered on all collections of closed and bounded subintervals $\{I_n\}$, $\{J_m\}$ of intervals I_1 and J_1 respectively.

Finally, we introduce the definition of the main considered concept.

Definition 2. *We say that the quantity $TV_{\Phi}^S(f)$ defined by the formula*

$$TV_{\Phi}^S(f) = V_{\Phi, I_1}^S(f) + V_{\Phi, J_1}^S(f) + V_{\Phi, I_a^b}^S(f)$$

is the total Φ -variation of the function f in the sense of Schramm.

A function f is referred as a function with bounded total Φ -variation provided $TV_{\Phi}^S(f) < \infty$.

By $\Phi BV(I_a^b)$ we denote the set of all functions $f : I_a^b \rightarrow X$ which have bounded total Φ -variation in the sense of Schramm.

By P_{Φ} let us denote the functional defined on the set $\Phi BV(I_a^b)$ in the following way:

$$P_{\Phi}(f) = \inf \left\{ \epsilon > 0 : TV_{\Phi}^S \left(\frac{f}{\epsilon} \right) \leq 1 \right\}. \quad (4)$$

The main result in [13] asserts that the set $\Phi BV(I_a^b)$ forms a Banach algebra with the norm defined by the formula

$$\|f\|_{\Phi} = |f(a)| + P_{\Phi}(f). \quad (5)$$

Observation 1. If we take the Φ -sequence defined as follows

$$\Phi = \{\phi_{n,m} : \phi_{n,m}(t) = t^p; \quad 1 < p < \infty, n, m = 1, 2, \dots\}$$

then we can check that $P_{\Phi}(f) = (TV_{\Phi}^S(f))^{1/p}$.

Our next result is contained in the following Lemma.

Lemma 1. *Let $f \in \Phi BV(I_a^b; X)$ and $\Phi \in \Phi^*$. Then f has the following properties:*

- (1) *If $(t, s), (t', s') \in I_b^a$ then $|f(t, s) - f(t', s')| \leq 4\Phi_{n,m}^{-1}(\frac{1}{2}) P_\Phi(f)$.*
- (2) *If $P_\Phi(f) > 0$ then $TV_\Phi^S(f/P_\Phi(f)) \leq 1$.*
- (3) *Let $r > 0$. Then $TV_\Phi^S(f/r) \leq 1$ if and only if $P_\Phi(f) \leq r$.*

Observation 2. From part (1) of Lemma 1. we deduced that each function $f \in \Phi BV(I_a^b; X)$ is bounded. Moreover, the following estimation is satisfied

$$\|f\|_\infty = \sup \{|f(t, s)| : (t, s) \in I_a^b\} \leq |f(a)| + 4\Phi_{n,m}^{-1}(\frac{1}{2}) P_\Phi(f) \quad (6)$$

if $n, m = 1, 2, \dots$ where the symbol $\|f\|_\infty$ denotes the supremum norm, i.e.

$$\|f\|_\infty = \sup \{|f(t, s)| : (t, s) \in I_a^b\}$$

Let us fix arbitrary $f \in \Phi BV(I_a^b)$. Then the function $f^* : I_a^b \rightarrow X$ defined by formula

$$f^*(x_1, x_2) = \begin{cases} \lim_{(y_1, y_2) \rightarrow (x_1-0, x_2-0)} f(y_1, y_2), & (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2], \\ \lim_{(y_1, y_2) \rightarrow (x_1-0, a_2+0)} f(y_1, y_2), & x_1 \in (a_1, b_1] \text{ and } x_2 = a_2, \\ \lim_{(y_1, y_2) \rightarrow (a_1+0, x_2-0)} f(y_1, y_2), & x_1 = a_1 \text{ and } x_2 \in (a_2, b_2], \\ \lim_{(y_1, y_2) \rightarrow (a_1+0, a_2+0)} f(y_1, y_2), & x_1 = a_1 \text{ and } x_2 = a_2 \end{cases}$$

is called the left-left regularization of the function f . The existence of all one-sided limits used above was proved in [15].

Definition 3. *A function $f : I_a^b \rightarrow \mathbb{R}$ is said to be left-left continuous if*

$$\lim_{y_1 \rightarrow x_1-0, y_2 \rightarrow x_2-0} f(y_1, y_2) = f(x_1, x_2) \text{ for all } (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2].$$

By $\Phi BV^*(I_a^b)$ is denoted the subspace of $\Phi BV(I_a^b)$ consisting of those functions which are left-left continuous on $(a_1, b_1] \times (a_2, b_2]$ and by $\mathcal{L}(X, Y)$ the space defined by

$$\mathcal{L}(X, Y) := \{f : X \rightarrow Y : f \text{ is linear}\}$$

Lemma 2 ([14]). *If $f \in \Phi BV(I_a^b)$, then $f^* \in \Phi BV^*(I_a^b)$.*

In the sequel we are going to deal with the main result of this paper.

4. THE COMPOSITION OPERATOR

Our main result reads as follows:

Theorem 1. *Let $I_a^b \subset \mathbb{R}^2$ be a rectangle, $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ be a real Banach space, C be a convex cone in X . If the composition operator H generated by $h : I_a^b \times C \rightarrow Y$ transforms $\Phi_1 BV(I_a^b, C)$ into $\Phi_2 BV(I_a^b, Y)$ and is uniformly continuous, then there exist functions $A \in \mathcal{L}(X, Y)$ and $B \in \Phi_2 BV(I_a^b, Y)$ such that*

$$h^*(t, s, y) = A(t, s)y + B(t, s), \quad (t, s) \in I_a^b, \quad y \in C,$$

where h^* is the left-left regularization of h .

Proof. For every $y \in C$ the constant function $f(t, s) = y$ with $(t, s) \in I_a^b$ belongs to $\Phi_1 BV(I_a^b, C)$. Since H maps $\Phi_1 BV(I_a^b, C)$ into $\Phi_2 BV(I_a^b, Y)$, it follows that the function $(t, s) \mapsto h(t, s, y)$, $(t, s) \in I_a^b$, belongs to $\Phi_2 BV(I_a^b, Y)$. Now the completeness of $\Phi_2 BV(I_a^b, Y)$ implies the existence of the left-left regularization h^* of h .

By assumption H is uniformly continuous on $\Phi_1 BV(I_a^b, C)$. Let ω be the modulus of continuity of H that is

$$\omega(\rho) := \sup \left\{ \|H(f_1) - H(f_2)\|_{\Phi_2} : \|f_1 - f_2\|_{\Phi_1} \leq \rho; f_1, f_2 \in \Phi_1 BV(I_a^b, C) \right\}$$

for $\rho > 0$. Hence we get

$$\|H(f_1) - H(f_2)\|_{\Phi_2} \leq \omega(\|f_1 - f_2\|_{\Phi_1}), \quad \text{for } f_1, f_2 \in \Phi_1 BV(I_a^b, C). \quad (7)$$

From the definition of the norm $\|\cdot\|_{\Phi_2}$ we obtain

$$P_{\Phi_2}(H(f_1) - H(f_2)) \leq \|H(f_1) - H(f_2)\|_{\Phi_2}, \quad \text{for } f_1, f_2 \in \Phi_1 BV(I_a^b, C). \quad (8)$$

In view of (8), Definitions 1., 2. and Lemma 1.(3), if $\omega(\|f_1 - f_2\|_{\Phi_1}) > 0$, then

$$V_{\Phi, I_a^b}^S(f) \left(\frac{(H(f_1) - H(f_2))(\cdot, a_2)}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right) \leq TV_{\Phi_2}^S \left(\frac{Hf_1 - Hf_2}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right) \leq 1. \quad (9)$$

The definitions of the operator H and the functional $V_{\Phi, I_a^b}^S(f)$ imply that for any

$$a_1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n \leq b_1,$$

and

$$a_2 \leq \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \cdots < \bar{\alpha}_m < \bar{\beta}_m \leq b_2,$$

if $n, m \in \mathbb{N}$, the inequality

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^m \phi_{i,j} \left(\frac{|h(\alpha_i, \bar{\alpha}_j, f_1(\alpha_i, \bar{\alpha}_j)) - h(\alpha_i, \bar{\alpha}_j, f_2(\alpha_i, \bar{\alpha}_j)) - h(\alpha_i, \bar{\beta}_j, f_1(\alpha_i, \bar{\beta}_j)) + h(\alpha_i, \bar{\beta}_j, f_2(\alpha_i, \bar{\beta}_j))|}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right. \\ & \left. + \frac{-h(\beta_i, \bar{\alpha}_j, f_1(\beta_i, \bar{\alpha}_j)) + h(\beta_i, \bar{\alpha}_j, f_2(\beta_i, \bar{\alpha}_j)) + h(\beta_i, \bar{\beta}_j, f_1(\beta_i, \bar{\beta}_j)) - h(\beta_i, \bar{\beta}_j, f_2(\beta_i, \bar{\beta}_j))|}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right) \leq 1. \end{aligned} \quad (10)$$

holds.

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha, \beta} : \mathbb{R} \rightarrow [0, 1]$ by the following formula:

$$\eta_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha \\ \frac{t - \alpha}{\beta - \alpha} & \text{if } \alpha \leq t \leq \beta \\ 1 & \text{if } \beta \leq t. \end{cases} \quad (11)$$

First let us fix $t \in (a_1, b_1]$, $s \in (a_2, b_2]$ and $n, m \in \mathbb{N}$. For arbitrary sequences

$$a_1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n \leq t$$

$$a_2 \leq \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \dots < \bar{\alpha}_m < \bar{\beta}_m \leq s$$

and $y_1, y_2 \in C$, $y_1 \neq y_2$ the functions $f_1, f_2 : I \rightarrow X$ defined by

$$f_\ell(\tau, \gamma) := \frac{1}{2} \left[(\eta_{\alpha_i, \beta_i}(\tau) + \eta_{\bar{\alpha}_j, \bar{\beta}_j}(\gamma) - 1)(y_1 - y_2) + y_\ell + y_2 \right], \quad (12)$$

for every $(\tau, \gamma) \in I_a^b$, $\ell = 1, 2$; belong to the space $\Phi_1 BV(I_a^b, C)$. From this we infer that

$$f_1(\cdot, \cdot) - f_2(\cdot, \cdot) = \frac{y_1 - y_2}{2},$$

therefore

$$\|f_1 - f_2\|_{\Phi} = \left| \frac{y_1 - y_2}{2} \right|;$$

moreover

$$f_1(\alpha_i, \bar{\alpha}_j) = y_2; \quad f_2(\alpha_i, \bar{\alpha}_j) = \frac{-y_1 + 3y_2}{2};$$

$$f_1(\alpha_i, \bar{\beta}_j) = \frac{y_1 + y_2}{2}; \quad f_2(\alpha_i, \bar{\beta}_j) = y_2,$$

$$f_1(\beta_i, \bar{\alpha}_j) = y_2; \quad f_2(\beta_i, \bar{\alpha}_j) = \frac{-y_1 + 3y_2}{2};$$

$$f_1(\beta_i, \bar{\beta}_j) = x_1; \quad f_2(\beta_i, \bar{\beta}_j) = \frac{y_1 + y_2}{2}.$$

Applying (10), we get

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^m \phi_{i,j} \left(\frac{\left| h(\alpha_i, \bar{\alpha}_j, y_2) - h(\alpha_i, \bar{\alpha}_j, \frac{-y_1+3y_2}{2}) - h(\alpha_i, \bar{\beta}_j, \frac{y_1+y_2}{2}) + h(\alpha_i, \bar{\beta}_j, y_2) \right.}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right. \\ & \left. + \frac{-h(\beta_i, \bar{\alpha}_j, y_2) + h(\beta_i, \bar{\alpha}_j, \frac{-y_1+3y_2}{2}) + h(\beta_i, \bar{\beta}_j, x_1) - h(\beta_i, \bar{\beta}_j, \frac{y_1+y_2}{2}) \right|}{\omega(\|f_1 - f_2\|_{\Phi_1})} \left. \right) \leq 1. \end{aligned} \quad (13)$$

In view of continuity of $\phi_{i,j}$, and Lemma 2., the left-left continuity of h^* we infer that

$$\sum_{i=1}^n \sum_{j=1}^m \phi_{i,j}(x) \leq 1 \text{ for } n, m = 1, 2, \dots, \quad (14)$$

where

$$x = \frac{\left| h^*(t, s, y_1) - 2h^*\left(t, s, \frac{y_1+y_2}{2}\right) + h^*(t, s, y_2) \right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}.$$

Since $n, m \in \mathbb{N}$ are arbitrary, condition (14) implies inequality

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \phi_{i,j}(x) \leq 1.$$

In view of (2) we get $x = 0$, i.e.

$$h^*\left(t, s, \frac{y_1 + y_2}{2}\right) = \frac{h^*(t, s, y_1) + h^*(t, s, y_2)}{2} \quad (15)$$

for all $(t, s) \in (a_1, b_1] \times (a_2, b_2]$ and $y_1, y_2 \in C$.

For $t \in (a_1, b_1]$ and $s = b_2$ let

$$a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n < t$$

and

$$a_2 < \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \dots < \bar{\alpha}_m < \bar{\beta}_m < b_2.$$

Proceeding as above we get (13).

If $\alpha_1 \uparrow t$ and $\beta_m \downarrow s$ in (13), then we get (15).

The cases when $t = a_1$ and $s \in (a_2, b_2]$ or $t = a_1$ and $s = a_2$ can be treated similarly. Consequently

$$h^*\left(t, s, \frac{y_1 + y_2}{2}\right) = \frac{h^*(t, s, y_1) + h^*(t, s, y_2)}{2}$$

is valid for all $(t, s) \in I_a^b$ and all $y_1, y_2 \in C$.

Therefore, the function $h^*(t, s, \cdot)$ satisfies the Jensen functional equation in C for $(t, s) \in I_a^b$. Modifying the standard argument (Kuczma [9]), we conclude that for each $(t, s) \in I_a^b$ there exist additive functions $A(t, s) : C \rightarrow \mathcal{L}(X, Y)$ and $B(t, s) \in Y$ such that

$$h^*(\cdot, y) = A(\cdot)y + B(\cdot), \quad y \in C. \quad (16)$$

The uniform continuity of the operator $H : \Phi_1 BV(I_a^b, C) \rightarrow \Phi_2 BV(I_a^b, Y)$ implies the continuity of the additive function $A(t, s)$.

Consequently $A(t, s) \in \mathcal{L}(X, Y)$.

Finally, notice that $A(t, s)(0) = \{0\}$ for every $(t, s) \in I_a^b$. Therefore, putting $y = 0$ in (16), we get

$$h^*(t, s, 0) = B(t, s), \quad (t, s) \in I_a^b,$$

which implies $B \in \Phi_2 BV(I_a^b, Y)$. \square

Observation 3. A similar theorem to Theorem 1. is valid for the right-right, right-left and left-right regularizations of $h(\cdot, y)$, $y \in C$.

REFERENCES

- [1] H. Nakano, *Modulated Semi-Ordered Spaces*, Tokyo, 1950.
- [2] *Uniformly continuous composition operators in the space of functions of two variables of bounded φ -variation in the sense of Wiener*, Comment. Math., **50(1)**, (2010), 41-48.
- [3] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operator*, Cambridge University Press, New York, 1990.
- [4] J. Matkowski, A. Matkowska and N. Merentes, *Remark on Globally Lipschitzian Composition Operators*, Demonstratio. Math. **427(1)**, (1995), 1, 171-175.
- [5] J. Matkowski, *Lipschitzian composition operators in some function spaces*, Nonlinear Anal. **3** (1997), 719-726.
- [6] J. Matkowski and N. Merentes, *Characterization of Globally Lipschitzian Composition Operators in the Sobolev Space $W_p^n[a, b]$* , Zeszyty Nauk. Politech. Łódzkiej, Mat. **24** (1993), 90-99.
- [7] J. Matkowski, *On Nemytskij Operators*, Math. Japonica **33(1)**, (1988), 81-86.
- [8] J. Matkowski, J. Miś, *On a Characterization of Lipschitzian Operators of Substitution in the Space $BV(a, b)$* , Math. Nachr. **117** (1984), 155-159.
- [9] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Editors and Silesian University, Warszawa-Kraków-Katowice, 1985.
- [10] M. Schramm, *Funtions of ϕ -bounded Variation and Riemann-Stieltjes integration*, Transaction Amer. Math. Soc. **267** (1985), 49-63.

- [11] N. Merentes, S. Rivas, *On Characterization of the Lipschitzian Composition Operators between Spaces of Functions of Bounded p -variation*, Czechoslovak Mathematical Journal **45(120)**, (1995), 627-637.
- [12] N. Wiener, *The quadratic variation of function and its Fourier coefficients*, Massachusetts J. Math. **3** (1924), 72-94.
- [13] T. Ereú, N. Merentes, J. L. Sánchez, *Some remarks on the algebra of functions of two variables with bounded total Φ -variation in Schramm sense*, Comment. Math **50(1)**, (2010), 23-33.
- [14] T. Ereú, N. Merentes, B. Rzepka, J. L. Sánchez, *On composition in the algebra of functions of two variables with bounded total Φ -variation in Schramm sense*, J. Math. Appl. **33** (2010), 01-15.
- [15] W. Aziz, N. Merentes, J. L. Sánchez, *Regularized Functions on the Plane and Nemytskij Operator*, Preprint.
- [16] W. A. Luxemburg, *Banach Function Spaces*, Ph.D. Thesis, Technische Hogeschool te Delft, Netherlands, 1955.
- [17] W. Orlicz, *A note on modular spaces. I*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **9** (1961), 157-162.

Wadie Aziz,
UNIVERSIDAD DE LOS ANDES,
DEPARTAMENTO DE FÍSICA Y MATEMÁTICAS,
TRUJILLO-VENEZUELA
E-mail address: wadie@ula.ve

Tomas Ereú, Nelson Merentes, Jose L. Sanchez
UNIVERSIDAD CENTRAL DE VENEZUELA
ESCUELA DE MATEMÁTICAS,
CARACAS, VENEZUELA
E-mail address: tomasereu@gmail.com
E-mail address: nmer@ciens.ucv.ve

Małgorzata Wróbel
JAN DŁUGOSZ UNIVERSITY,
INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE,
42-200 CZĘSTOCHOWA, AL. ARMII KRAJOWEJ 13/15, POLAND
E-mail address: m.wrobel@ajd.czest.pl