AN EXAMPLE OF RELIABILITY ANALYSIS FOR STOCHASTIC NETWORK CONNECTION

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ABSTRACT

It is an introductory study which may serve as a basis for future development of a more general approach to reliability analysis of stochastic network connection. Since the coordinates of connection points are generally correlated variables, a method of reliability analysis should be that for systems with correlated observations. An approach proposed earlier by the author of the present paper, destined for the stage of network design, is used for this purpose. A notion of contact elements is introduced, i.e. the elements being geometrical quantities (e.g. distances, angles) defined on connection points. In addition to the use of given covariance matrices, a method of creating special covariance matrices for testing the connection structure is presented, the matrices being built for assumed accuracies of the contact elements.
A numerical example is provided showing the reliability analysis for connecting a new levelling network to existing benchmarks of given elevations and their covariance matrix. The effects on reliability indices of using different covariance matrices are studied.

INTRODUCTION

There are two main approaches to internal reliability analysis of systems with correlated observa-tions, i.e. the testing-based approach (Schaffrin 1997, Knight, Rizos and Wang 2010) and the disturbance/response approach (Prószyński 2010). In this paper the second approach is under consideration, since main emphasis is put on a design stage of network connection.
After derivation of the corresponding reliability matrix, a strategy for operating with covariance matrix for coordinates of connecting points is presented. This enables one to generate covariance matrices useful for the purposes of reliability analysis of network connection schemes.

1. DERIVATION OF RELIABILITY MATRIX FOR STOCHASTIC NETWORK CONNECTION

Stochastic network connection – is understood here as connecting a new network to a group of given points with the known covariance matrix of their coordinates. The task is illustrated schematically in Figure 1,

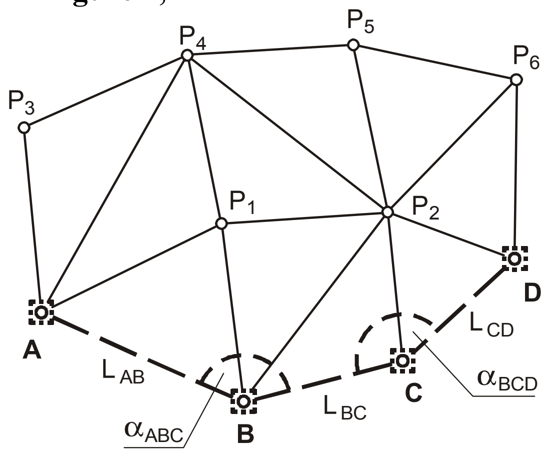

Fig. 1. The stochastic network connection.

where

A,B, C, D - connecting points; 1, 2,…,6 - new points; L, α - contact elements

We shall recall a well-known adjustment model for this type of connection (e.g. Prószyński 1990), which can be characterized more generally as a model for a network with the given a priori knowledge on positions and accuracies of some of its points

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}^{\mathrm{obs}} - \mathbf{y}^{\mathrm{o}} \\ \mathbf{X}_1^{\mathrm{obs}} - \mathbf{X}_1^{\mathrm{o}} \end{bmatrix} + \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} \quad ; \quad \begin{bmatrix} \mathbf{C}_{\mathbf{y}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathbf{X},1} \end{bmatrix} \tag{1}$$

or in a compact form

$$\mathbf{A}_* \mathbf{x}_* = \mathbf{y}_* + \mathbf{v}_* \quad ; \quad \mathbf{C}_* \tag{2}$$

where :

$\mathbf{x}_1(u_1 \times 1),\ \mathbf{x}_2(u_2 \times 1)$ **- the vectors of coordinate increments for the connecting points**
and the new points respectively;

$\mathbf{y}^{\mathrm{obs}}(n \times 1),\ \mathbf{X}_1^{\mathrm{obs}}(u_1 \times 1)$ **- the vectors of new observations and pseudo-observations res-pecttively (** $n > u_2$ **);** $\mathbf{y}^{\mathbf{o}},\ \mathbf{X}_1^{\mathrm{o}}$ **being vectors of approximate values;**

$\mathbf{v}(n_1 \times 1),\ \mathbf{w}(u_1 \times 1)$ **- the vectors of unknown true errors (taken with negative sign) in the observations and pseudo-observations respectively;** $\mathrm{E}(\mathbf{v}) = \mathbf{0},\ \ \mathrm{E}(\mathbf{w}) = \mathbf{0}$**;**

$\mathbf{X}_1^{\mathrm{obs}}(u_1 \times 1),\ \mathbf{X}_1^{\mathrm{o}}(u_1 \times 1)$ **- the vectors of given and approximate coordinates of connecting points; with** $\mathbf{X}_1^{\mathrm{o}} = \mathbf{X}_1^{\mathrm{obs}}$ **we have** $\mathbf{X}_1^{\mathrm{obs}} - \mathbf{X}_1^{\mathrm{o}} = \mathbf{0}$

$\mathbf{C}_{\mathbf{y}}(u_2 \times u_2), \mathbf{C}_{\mathbf{X},1}(u_1 \times u_1)$ **- covariance matrices for observations and pseudo-observa-tions (each pos. def.)**

$\mathbf{A}_*[(n + u_1) \times (u_1 + u_2)]$ **- the coefficient matrix (of full rank).**

The reliability matrix R for the model (2) will be (Schaffrin 1998)

$$\mathbf{H} = \mathbf{I} - \mathbf{A}_*(\mathbf{A}_*^{\mathbf{T}}\mathbf{C}_*^{-1}\mathbf{A}_*)^{-1}\mathbf{A}_*^{\mathbf{T}}\mathbf{C}_*^{-1} \tag{3}$$

Following the approach as in (Prószyński 2010) we shall modify the system (1) without affecting its least squares (LS) solution, by multiplying both sides by $\boldsymbol{\sigma}^{-1}$, where $\boldsymbol{\sigma} = (\operatorname{diag}\mathbf{C}_*)^{1/2}$, and transfor-ming the covariance matrix $\mathbf{C}_*$ accordingly, so we obtain

$$\mathbf{A}_{*,s}\mathbf{x} = \mathbf{y}_{*,s} - \mathbf{v}_{*,s}; \qquad \mathbf{C}_{*,s} \tag{4}$$

where:

$\mathbf{A}_{*,s} = \boldsymbol{\sigma}^{-1}\mathbf{A}_*$, $\mathbf{e}_{*,s} = \boldsymbol{\sigma}^{-1}\mathbf{e}_*$, $\mathbf{y}_{*,s} = \boldsymbol{\sigma}^{-1}\mathbf{y}_*$, $\mathbf{C}_{*,s} = \boldsymbol{\sigma}^{-1}\mathbf{C}_*\boldsymbol{\sigma}^{-1}$, $\mathbf{C}_{*,s}$ **is a correlation matrix, having the form**

$$\mathbf{C}_{*,s} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\rho}_{X,1} \end{bmatrix}$$

The reliability matrix H for the modified system is given by

$$\mathbf{H} = \mathbf{I} - \mathbf{A}_{*,s}(\mathbf{A}_{*,s}^{\mathbf{T}}\mathbf{C}_{*,s}^{-1}\mathbf{A}_{*,s})^{-1}\mathbf{A}_{*,s}^{\mathbf{T}}\mathbf{C}_{*,s}^{-1} \tag{5}$$

The matrices H and R are related by the formula

$$\mathbf{H} = \boldsymbol{\sigma}^{-1}\mathbf{R}\boldsymbol{\sigma}\,. \tag{6}$$

2. STRATEGY FOR OPERATING WITH COVARIANCE MATRIX FOR COORDINATES OF CONNECTING POINTS

For reliability analysis of stochastic connection we may operate either with the given or with specially constructed covariance matrices for coordinates of connecting points ($\mathbf{C}_{X,1}$). The former is an ordinary case, when we know the covariance matrix, either in a full or reduced form (e.g. standard error ellipses in 2D networks). The letter case seems to be specially suitable for research purposes or for design of complex setting out networks planned to be developed in several phases, i.e. by connecting the new network to a subset of points of the previously established network.

With a focus on the letter case we shall show how to construct covariance matrices for coordina-tes of connecting points, the matrices being meaningful for reliability analysis. The method is based on a statement, that the actual influence of the connecting points upon internal accuracy of a new network is exerted through geometrical elements (i.e. distances, angles) defined on these points. We shall be calling them "contact elements" So, we may form an auxiliary sub-network, by choosing the contact elements and assuming standard deviations of their masuement. The sub-network can be either with or without redundant observations.

Further steps can be the following:

a) form a coefficient matrix of the auxiliary sub-network. Let its standardized form be denoted by $\mathbf{B}_*$
b) find the covariance matrix for the coordinates of connecting points, using a local "free-type" reference system;

$$\mathbf{C}_{X1}\{\text{local}\} = (\mathbf{B}_*^{\mathrm{T}}\mathbf{B}_*)_{\mathbf{S_o}}^{-} \equiv (\mathbf{B}_*^{\mathrm{T}}\mathbf{B}_*)^{+} \tag{7}$$

where: $\mathrm{S_o}$ - the coefficient matrix In „free-type" conditions, i.e. such that $\mathbf{B}_*\mathbf{S}_{\mathrm{o}}^{\mathrm{T}} = \mathbf{0}$;
$(\circ)^{+}$ - the pseudo-inverse (Rao and Mitra 1971).

The two inverses in (7), although of identical values, are the result of different computation procedures.
c) transform the covariance matrix $\mathbf{C}_{X1}\{\text{local}\}$ to external reference system, using a method of ficticious external reference base (Prószyński 1990), i.e.

$$\mathbf{C}_{X1}\{\text{external}\} = \mathbf{C}_{X1}\{\text{local}\} + \mathbf{S}_{\mathrm{o}}^{\mathrm{T}}\mathbf{S}_{\mathrm{o}} \tag{8}$$

or substituting (7)

$$\mathbf{C}_{X1}\{\text{external}\} = (\mathbf{B}_*^{\mathrm{T}}\mathbf{B}_*)_{\mathbf{S_o}}^{-} + \mathbf{S}_{\mathrm{o}}^{\mathrm{T}}\mathbf{S}_{\mathrm{o}} = (\mathbf{B}_*^{\mathrm{T}}\mathbf{B}_* + \mathbf{S}_{\mathrm{o}}^{\mathrm{T}}\mathbf{S}_{\mathrm{o}})^{-1}$$

where $\mathbf{C}_{X1}\{\text{external}\}$ is a positive definite matrix.

We can easily prove that by using the transformation (8) we do not change the covariance matrix for the contact elements in a local reference system, denoted by $\mathbf{C}_{\mathrm{el}}\{\text{local}\}$. So, we obtain

$$\mathbf{C}_{\mathrm{el}}\{\text{external}\} = \mathbf{B}_* \cdot C_{X1}\{\text{external}\} \cdot \mathbf{B}_*^{\mathrm{T}} = \mathbf{B}_*(\mathbf{B}_*^{\mathrm{T}}\mathbf{B}_*)_{S_o}^{-}\mathbf{B}_*^{\mathrm{T}} + \mathbf{B}_*\mathbf{S}_{\mathrm{o}}^{\mathrm{T}}\mathbf{S}_{\mathrm{o}}\mathbf{B}_*^{\mathrm{T}}$$

and, due to $\mathbf{B}_*\mathbf{S}_{\mathrm{o}}^{\mathrm{T}} = \mathbf{0}$ (see (7)), finally $\mathbf{C}_{\mathrm{el}}\{\text{external}\} = \mathbf{C}_{\mathrm{el}}\{\text{local}\}$

3. INDICES OF INTERNAL RELIABILITY USED IN THE PAPER

Since for systems with correlated observations the matrix H is an oblique projector (see formula (5)), we shall be using a two-parameter reliability measure for the *i*-th observation as proposed in (Prószyński 2010)

$$\boldsymbol{h}_{(i)} = (\boldsymbol{h}_{ii},\ \boldsymbol{w}_{ii}) \tag{9}$$

where h_{ii} is the *i*–th diagonal element of H, and w_{ii} is the asymmetry index for the *i*-th row and the *i*-th column of H. The index h_{ii}, denoted also as $L_{i(i)}$, is called a local response of the model, i.e. the response in the *i*-th observation to a gross error in that observation.

It proved advantageous to use also a reliability measure being a pair of indices (h_{ii}, k_i), where $\mathrm{k_i}$ is a ratio of the squared quasi-global response $Q_{(i)}$ to a squared local response $L_{i(i)}$ of the model to a gross error in the *i*-th observation, i.e.

$$k_i = \frac{Q_{(i)}^2}{L_{i(i)}^2} = \frac{h_{ii} - h_{ii}^2 - w_{ii}}{h_{ii}^2} = \frac{1}{h_{ii}} - \frac{w_{ii}}{h_{ii}^2} - 1 \qquad (\text{for } h_{ii} \neq 0) \tag{10}$$

where quasi-global response $Q_{(i)}$ is a global response not covering the local response.

In the numerical example that will follow, the results of reliability analysis for stochastic network connection will be shown in a tabular and a graphic form for both pairs of indices, i.e. (h_{ii}, w_{ii}) and (h_{ii}, k_i).

4. NUMERICAL EXAMPLE

A numerical example is provided showing the reliability analysis for connecting a new levelling network to existing benchmarks of given elevations and their covariance matrix. The network and the chosen contact elements are shown in Fig. 2.

For specifying the options of the covariance matrix for elevations, used in the test, we will introduce the notation:

$\sigma_{\Delta H}$ **- standard deviation of difference in elevation between the existing benchmarks (used as connecting points),**

σ_h **- standard deviation of the new measurement (equal for all new leveling lines).**

The following options of covariance matrix will be used:

1) **matrix taken from practical task ($\sigma_{\Delta H} < \sigma_h$)**
2) **matrix constructed with the use of contact elements ($\sigma_{\Delta H} << \sigma_h$)**
3) **matrix constructed with the use of contact elements ($\sigma_{\Delta H} >> \sigma_h$)**

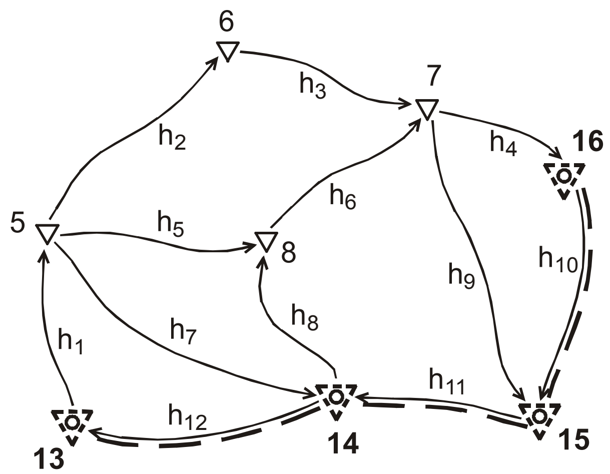

Fig. 2. Levelling network used in the test; 13 ÷16 are existing benchmarks, 13–14, 14-15, 15-16 are contact elements

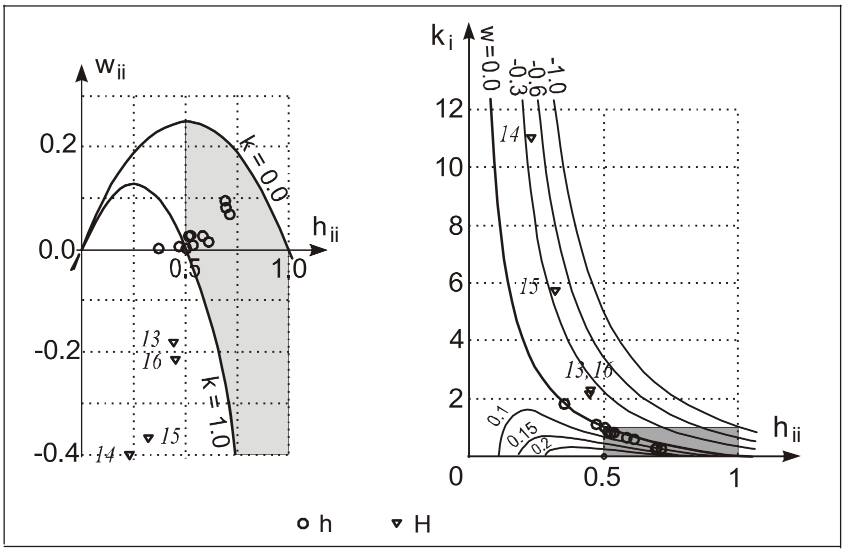

Fig. 3. Results of reliability analysis for Option 1 ($\sigma_{\Delta H} < \sigma_h$).

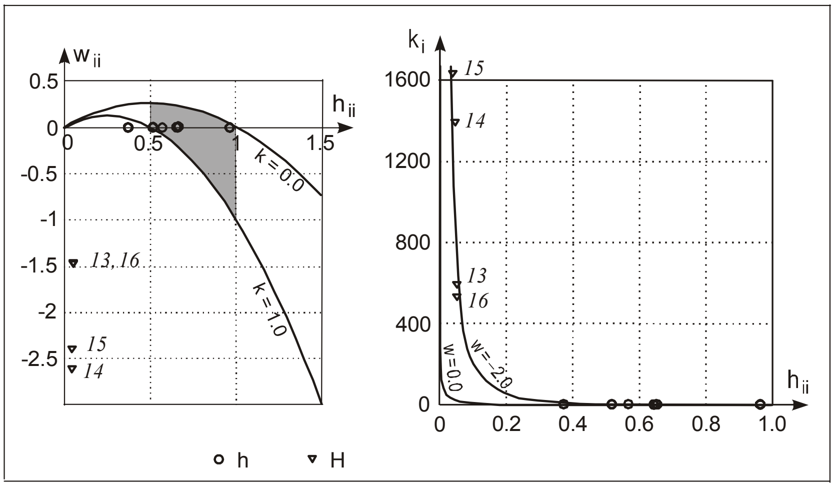

Fig. 4. Results of reliability analysis for Option 2 ($\sigma_{\Delta H} << \sigma_h$).

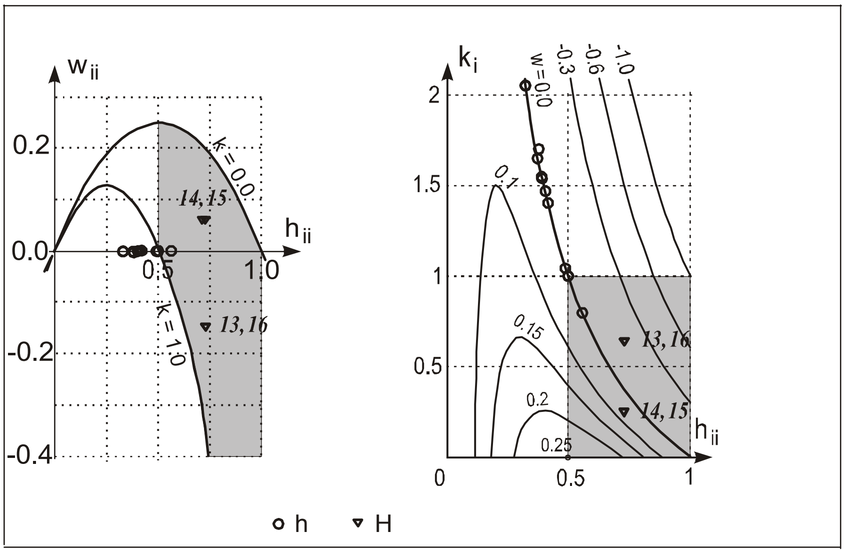

Fig. 5. Results of reliability analysis for Option 3 ($\sigma_{\Delta H} >> \sigma_h$).

We can see that with $\sigma_{\Delta H} < \sigma_h$ (Option 1) and to a greater extent with $\sigma_{\Delta H} << \sigma_h$ (Option 2), the new observations display higher reliability than the given elevations, since it is the elevations that control the new observations. Almost all the new observations satisfy the reliability criteria (see shaded area in Fig. 4). With $\sigma_{\Delta H} >> \sigma_h$ (Option 3) the situation is reverse, the elevations are con-trolled by the new observations. They all satisfy the reliability criteria (see shaded areas in Fig. 5).

5. CONCLUSIONS

We will formulate the conclusions extending them upon horizontal networks:
- the presented disturbance/response analysis of systems with correlated observations is a suitable tool for analyzing the internal reliability of stochastic network connection at the stage of its design;
- with the contact elements being of much smaller accuracy than that of new observations, the co-ordinates of connecting points will have more advantageous reliability indices, i.e. will be con-trolled by the new observations, and vice versa;
- by neglecting non-diagonal elements in the given full covariance matrix for connecting points and thus decreasing the accuracy of the contact elements, we loose the advantageous influence of connection upon the accuracy of the new network. What we gain is a better controllability of the coordinates of connecting points;
- there is a possibility of making a rough a priori reliability analysis – by comparing the intended accuracy of new observations with that of the contact elements;

The above mentioned conclusions can be easily interpreted using well known behaviour of obser-vation systems in the sphere of accuracy and reliability. The contact elements can be considered as additional network elements. Raising their accuracy results in decreasing their controllability by other network elements, and hence, decreasing their reliability. It is the elements of a new network that are then better controlled and gain higher reliability.

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