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Mathematical model of a leading line consisting of two pairs of beacons and its practical application

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Abstract

This paper presents a mathematical model of a leading line consisting of four beacons situated at the apexes of a rectangle. The ship handler compares angles between appropriate beacons to determine whether the vessel is in the middle of the fairway or not. This article assumes that the observer treats the two beacons on the front as a flat screen onto which rear beacons are projected. In this way he can estimate the quotients of sections created by appropriate beacons, from which he can find the lateral deviation from the waterway axis. This method of navigation is based on the Thales assertion. Described in detail, restrictions are the use of the model, precision assessment, and important in this respect, aspects of the waterway markings on Zalew Szczeciński.

Introduction

The system of the waterway marking on Zalew Szczeciński was created in the thirties of XX century and consists of four pairs of tower-beacons, which are resistant to ice masses. Pairs of the beacons are situated exactly on the great-circle at every 6750 meters and they create 11 Nm long waterway. Its width is 260 meters but only the 90 central meters were dredged and can be used by ships with maximum draft of 9.15 meters. In the epoch before the invention of GPS, when mariners were passing each other in this place, they used inaccurate sources of information to find the cross track error and distance to the dangerous depth contour. In good visibility and for vessels with deeper draft, navigators used to keep their vessels' portside close to the leading line formed by centers of pairs of beacons and buoys.

Increasingly deep draft vessel traffic, including container ships with tight schedules, necessitated that those vessels meet each other much more often in this narrow and dangerous passage, plagued by winds and cross currents. Before the use of GPS became common, terrestrial navigation solved this problem. Mariners would use leading lines formed by two pairs of beacons, named in this article rectangular leading lines.

Mathematical model of rectangular leading line

Two mathematical models based on two different coordinate systems can describe the rectangular leading line: radial and Cartesian. In both cases there is a rectangle marked A, A`, B and B`, with a beacon located at each apex (Figure 1). Additionally the principle has been adopted that points, sections and angles with measurements located below the X-axis and passing through the observer's position are marked by an appendix " ` " with respect to the corresponding points, sections and angles with its measurements situated above this axis. Therefore the long sides of the rectangle AB and A`B` are parallel to the X-axis, which is a center line of the leading line. The short sides, AA` and BB`, are called, according to the local terminology, "gates".

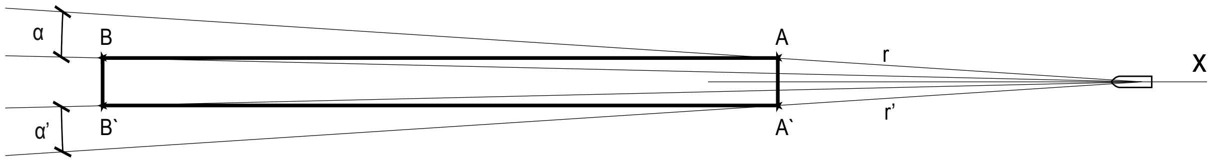

Figure 1. Mathematical model of a rectangular leading line in a radial coordinate system

Mathematical model of rectangular leading line in a radial coordinate system

A navigator observes angle α between beacons on one side of the waterway, angle α' between beacons on the other side, and measures by radar, distances r and r' to the beacons of the closest gate. Base on this data, it is possible to build a radial model, with the origin in the position of the observer and a reference direction as the direction of the X-axis.

Because the process of measuring angles with sextants, distances with radar and performing appropriate calculations takes too long, the navigator has a two-piece set of operands to find his position in relation to the center of the waterway i.e.: the angle between beacons on one side of the waterway is equal to the angle on the other side and contrary is not equal. The function value corresponding to such a domain also has a two-piece set, i.e.: the observer is in center of the waterway or contrary is not. The sensitivity and lateral deviation of this model are well known (Gucma, 2004).

Mathematical model of rectangular leading line in a Cartesian coordinate system

In the mathematical model based on a Cartesian coordinate system, the X-axis is a center line of the rectangular leading line, while the Y-axis coincides with section AA`, i.e. the gate situated closest to the observer (Figure 2).

In order to facilitate imaging and at the same time maintain veracity of deliberations, the longer sides of the discussed rectangle have been shortened (Figure 3). Points C and C` are created by intersecting the Y-axis with the lines connecting the observer with points B and B`, respectively.

Using Thales assertion the following equations can be formulated:

$$\frac{x_t}{R+x_t} = \frac{\frac{D}{2} - y_t - a}{\frac{D}{2} - y_t} \quad \text{and} \quad \frac{x_t}{R+x_t} = \frac{\frac{D}{2} + y_t - a'}{\frac{D}{2} + y_t}$$

and next:

$$\left(\frac{D}{2} - y_t - a\right)\left(\frac{D}{2} + y_t\right) = \left(\frac{D}{2} + y_t - a'\right)\left(\frac{D}{2} - y_t\right)$$

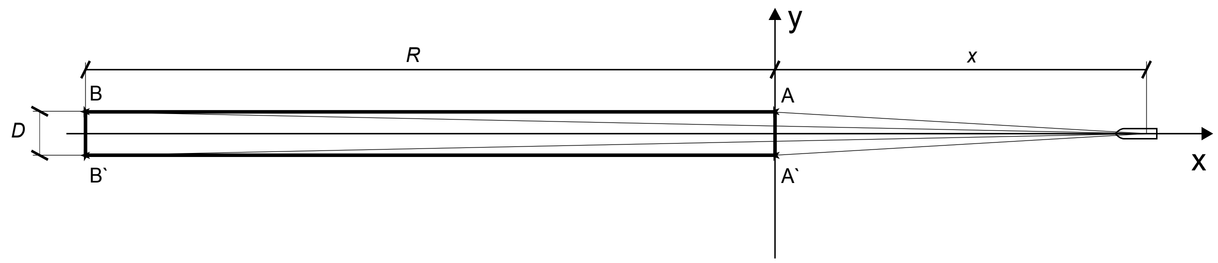

Figure 2. Mathematical model of a rectangular leading line in a Cartesian coordinate system

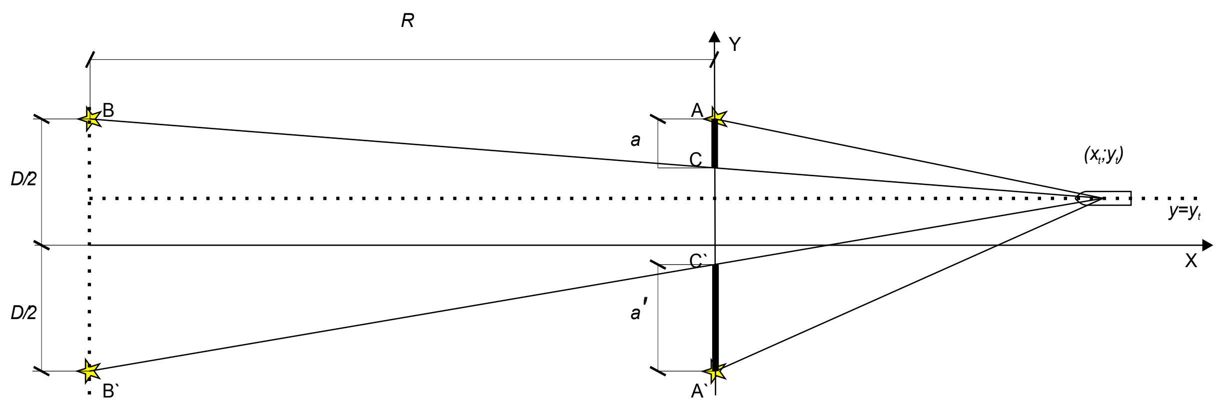

Figure 3. The use of Thales assertion in the description of a rectangular leading line in a Cartesian coordinate system

hence

$$\frac{\frac{D}{2}+y_t}{\frac{D}{2}-y_t}=\frac{a'}{a}=n_k \qquad (1)$$

where:

- x_t, y_t – the observer's coordinates in Cartesian system;
- R – distance between gates;
- D – distance between beacons within each gate;
- a – length of the section situated on the starboard side of the observer between beacon of the closer gate (point A) and projection of the beacon of further gate onto the Y-axis (point C);
- a' – length of the section situated on the port side of the observer between beacon of the closer gate (point A`) and projected of the beacon of further gate onto the Y-axis (point C`);
- n_k – perspective division factor, determining the quotient a' and a.

After transforming:

$$D=\frac{D}{2}-y_t+n_k\left(\frac{D}{2}-y_t\right)$$

and:

$$y_t=\frac{D}{2}-\frac{D}{n_k+1} \qquad (2)$$

Analyzing the above reveals the following relationships:

- n_k – the perspective division factor does not depend on x_t but only on y_t, i.e. on the lateral deviation from the waterway axis (Formula (1));
- for a given n_k the observer is situated on a vertical axis dividing D into $n_k + 1$ parts (Formula (2)).

The navigator treats the closest gate as a one-dimensional screen, on which beacons from the further gate have been thrown. In such a mathematical model, navigation is based on the estimation of the quotient of the length of sections of one side of the waterway and the other i.e. $n_k = a'/a$ (Figure 4).

Such a model is only approximate because in reality, the observer observes angles between beacons, as it was assumed in radial model.

Restrictions on the use of the Cartesian model can be formulated by analyzing Figure 5.

From Figure 5 the following formulas can be found:

$$\beta=\arctan\frac{\frac{D}{k+1}}{R+x_t} \quad \text{and} \quad \beta'=\arctan\frac{\frac{kD}{k+1}}{R+x_t}$$

$$y=\arctan\frac{\frac{D}{k+1}}{x_t} \quad \text{and} \quad y'=\arctan\frac{\frac{kD}{k+1}}{x_t}$$

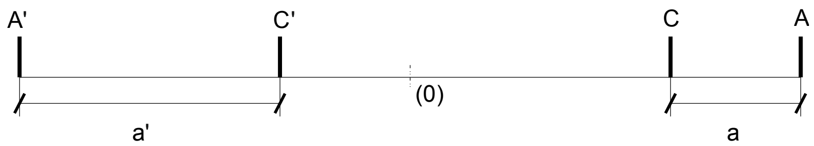

Figure 4. View of a rectangular leading line in a one-dimensional Cartesian coordination system

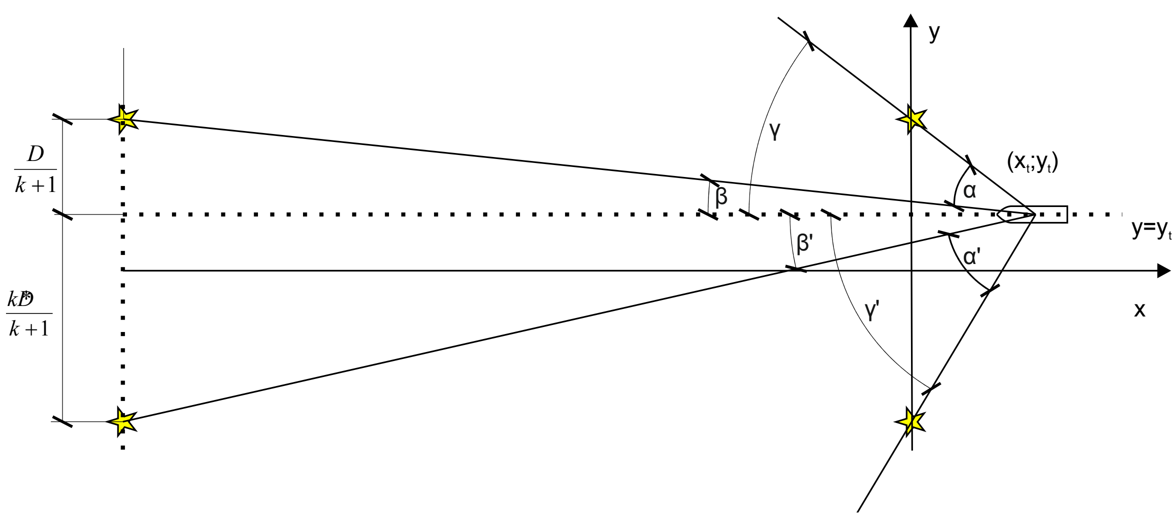

Figure 5. Rectangular leading line seen from short distance x_t

and next

$$\alpha = \arctan\frac{\frac{D}{k+1}}{x_t} - \arctan\frac{\frac{D}{k+1}}{R+x_t}$$

$$\alpha' = \arctan\frac{\frac{kD}{k+1}}{x_t} - \arctan\frac{\frac{kD}{k+1}}{R+x_t}$$

where:

β and γ – angles on starboard side of the observer between the direction of the waterway and a line connecting the observer and beacon of the respectively closer and further gate;

β' and γ' – angles on port side of the observer between the direction of the waterway and a line connecting the observer and beacon of the respectively closer and further gate;

k – factor of the length of the gate division, determining lateral deviation of the observer as a quotient of the distance of the beacon on the port side to the line $y = y_t$ and the beacon on starboard side to the same line $y = y_t$. For example for $k = 1$ the observer is on the center line of the leading line, for $k \to \infty$ the observer is on the line of the beacons on starboard side, for $k = 2$ the observer is on at 1/3 of width of rectangular leading line or in other words:

$$y_t = \frac{D}{2} - \frac{D}{k+1} = \frac{D}{2} - \frac{D}{3} = \frac{D}{6}.$$

Because the observer actually sees angles and not sections, factor $n_r = \alpha'/\alpha$ has been introduced and its relationships with navigator's position i.e. x_t, and k has been found using:

$$n_r = \frac{\alpha'}{\alpha} = \frac{\arctan\frac{\frac{kD}{k+1}}{x_t} - \arctan\frac{\frac{kD}{k+1}}{R+x_t}}{\arctan\frac{\frac{D}{k+1}}{x_t} - \arctan\frac{\frac{D}{k+1}}{R+x_t}} \quad (3)$$

In order to find an error committed by a navigator using the Cartesian model, in Formula (3) factor k has been replaced by n_k – the perspective division factor. This procedure helps answer the question: what real quotient of angles α' and α (No.) is seen by the observer when he compares the lengths of sections a' and a? The quotient of seen sections (perspective division factor n_k) has been calculated for various x_t using constant parameters $D = 260$ m, $R = 6750$ m (distances between beacons on Zalew Szczeciński waterway), and quotient seen angles $n_r = 2$ Substituting these data into the Formula (2), the constant absolute position error ΔY_t [m] can be found, which always moves the observer out of the leading line center.

Table 1. Comparing Cartesian and radial models of rectangular leading line for constant parameters $n_r = 2$, $D = 260$ m and $L = 6750$ m

$n_k = \frac{a}{a'}$	x_t	$\frac{D}{n_k+1}$	$n_r = \frac{\alpha'}{\alpha}$	% measurement error	$\frac{D}{n_r+1}$	Δy_t
3.270	**100**	60.89	2.000	38.83	86.66	25.77
2.369	**200**	77.17	2.000	15.58	86.67	9.497
2.171	**300**	81.99	2.000	7.89	86.67	4.68
2.099	**400**	83.9	2.000	4.70	86.66	2.759
2.064	**500**	84.86	2.000	3.11	86.67	1.817
2.054	**550**	85.13	2.000	2.61	86.65	1.52
2.050	**570**	**85.2**	**2.000**	**2.44**	**86.67**	**1.42**
2.045	**600**	85.39	2.000	2.22	86.68	1.29
2.034	**700**	85.7	2.000	1.66	86.66	0.965
2.026	**800**	85.92	2.000	1.29	86.67	0.751
2.021	**900**	86.06	2.000	1.04	86.67	0.602
2.017	**1000**	86.18	2.000	0.85	86.67	0.494
2.014	**1100**	86.26	2.000	0.71	86.68	0.414
2.012	**1200**	86.32	2.000	0.61	86.67	0.352
2.011	**1300**	86.35	2.000	0.53	86.65	0.304
2.009	**1400**	86.41	2.000	0.46	86.67	0.265
2.008	**1500**	86.44	2.000	0.40	86.67	0.234

Table 1 shows that for a distance of 570 meters from closer gate, the observed quotient of angles on the port and starboard sides is $n_r = 2$, which is different from $n_k = 2.05$ (2.44%), and also that the position error of the Cartesian model is 1.42 meters away from the leading line. From Table 1, it can be concluded that the Cartesian model may, without loss of positional correctness, replace the radial one when the distance of the observer to the closer gate is 500 meters or more.

Sensitivity and lateral deviation of the rectangular leading line in the Cartesian system

For the observer located at the center of the leading line, sensitivity is defined by safe lateral deviation at the level of 95% confidence, which is related to θ_z – traceability by human eye of the difference in angles (systematic uncertainty) and m_n – average error of the angles difference in the sector of its sensitivity (Gucma, 2004). The safe lateral deviation at the level of 95% confidence is:

$$Y_b = (\theta_z + m_n)x_t\left(1 + \frac{x_t}{R}\right) \quad (4)$$

where:

Y_b – lateral deviation;

θ_z – traceability by human eye of difference in angles;

m_n – average error of the angles difference.

The value $\theta_z + m_m$ of daytime observation is 4' ($1.16 \cdot 10^{-3}$ rad) (Kierzkowski, 1984) and 5' ($1.45 \cdot 10^{-3}$ rad) during nighttime. Table 2 shows the absolute value of this error in Zalew Szczeciński as a function of x_t.

Table 2. Lateral deviation of the observer in center of leading line in Zalew Szczeciński for $\theta_z + m_m = 4'$ as a function of x_t

D	R	$\theta_z + m_{n(95\%)}$	x_t	θ_b [m]
260	6750	0.00116	500	0.622963
260	6750	0.00116	1000	1.331852
260	6750	0.00116	1500	2.126667
260	6750	0.00116	2000	3.007407
260	6750	0.00116	2500	3.974074
260	6750	0.00116	3000	5.026667
260	6750	0.00116	3500	6.165185
260	6750	0.00116	4000	7.38963
260	6750	0.00116	4500	8.7
260	6750	0.00116	5000	10.0963
260	6750	0.00116	5500	11.57852
260	6750	0.00116	6000	13.14667
260	6750	0.00116	6500	14.80074
260	6750	0.00116	7000	16.54074

Practice and simple experiment, which consists of determining the change in perception of leading lines when the observer moves on the bridge, shows that the sensitivity of the rectangular leading line is much better and varies around 5 to 7 meters. However, the condition is that a lack of the visual acuity of closer and further gate is not blurring the picture of the entire leading line. For $n_k = 1$ and the naked eye this situation occurs when the distance x_t is greater than 8000 m. Determination of sensitivity and lateral deviation requires further studies, however bearing in mind the above-mentioned practical remarks, it is worth considering idea that sensitivity consists of a random uncertainty of the assessment of the perspective division factor n_k and systematic uncertainty associated with human eye detection of the boarder of construction edge background for 4 beacons (Szydłowski, 1981). This systematic uncertainty is given by

$$\theta_z = f(\theta, w, R, D)$$

$$Y_b = x_t \tan\theta_z + \left(\frac{D}{n_k + 1} - \frac{D}{(n_k + n_k m_k) + 1} \right) \quad (5)$$

where:

m_k – random uncertainty of the assessment of the factor of the perspective division n_k;

θ – angle of human eye detection of the border background construction edge;

w – factor determining conditions of look-out process.

The human eye detects border of construction edge background for θ angle less than 1', which is an angle formed by the 6-micron diameter of the macular and 17 mm focal length of eyeball. Additionally, contrast, brightness and color of the light shell need to be taken into account (Naskręcki, 2009). By Comparing the traceability by the human eye of a difference in angles (θ_z) in daytime for normal and rectangular leading lines, it can be concluded that in normal case, the rear beacon forms a background for the front beacon and in rectangular case, the observer assesses the quotient of sections between points situated on the much more contrasted background of horizon or shoreline. There is probably no difference between leading lines in the night and day. In connection with the above, the value of θ_z in rectangular leading line should be between 1' and 4'. Additionally, comparing the range of both types of leading lines, one can conclude that it is significantly influenced by the blurring of the picture caused by a lack of visual acuity. Due to the fact that angles between beacons in rectangular leading lines are many times greater than in normal ones, blurring of the picture is much more intensive in normal leading line than rectangular. Practice shows that for the above-mentioned conditions of Zalew Szczeciński, the range of a rectangular leading line is about two times longer than for a normal one.

The application of sea binoculars 7×50, increases detection angle, θ, *by* seven times. Further studies could assess for which position, x_t, an observer with and without binoculars can neglect the systematic uncertainty connected to θ_z, and the entire view of the closer gate can be watched in typical sea binoculars 7×50.

Without the systematic uncertainty, Formula (5) takes the form:

$$Y_b = \frac{D}{n_k + 1} - \frac{D}{n_k(1 + m_k) + 1} \quad (6)$$

For the waterway marking system on Zalew Szczeciński described earlier, $n_k = 1$ (vessel in the center of leading line) and the experimentally estimated sensitivity of a leading line, $Y_b = 5$–7 m, the random uncertainty of the perspective division factor (m_k) is between 8–11% which is comparable with the uncertainty of the readings of the optical measuring instruments, 10% (Szydłowski, 1981).

For $n_k = 2$, the random uncertainty of the perspective division factor (m_k) additionally depends on the observer's ability to conclude that the section/angle on one side is 2 times bigger than the section/angle on the other. This problem also requires further studies, but one can state that for the above-mentioned conditions of Zalew Szczeciński, $Y_{b(n=2)}$ is about 7 to 10 m, which, after placing this into the Formula (6), gives random uncertainty, m_k, of between 13.18% and 19.56%.

For other values n_k, for example 3 or 1.5, the observer should learn and exercise his ability to visually assess the quotient of section lengths.

The use of rectangular leading line on Zalew Szczeciński and in designing of waterway marking

The navigational markings on the main waterway on Zalew Szczeciński consist of 4 gates and 2 leading lines. The waterway axis runs through the middle, between the beacons of the gates and further in the axis of leading lines. The distance between gates is 6750 m and between beacons in each gate, 260 m. After taking into account certain restrictions, navigation can be based on the method described above. Furthermore, the width of the waterway is 90 m, which is 4 m more than 1/3 of the distance between beacons in gates. A Pilot/Capitan, giving way to another vessel, sticks to the starboard side of the waterway border, keeping the perspective division at 1 to 2. Such a leading line is very useful in terrestrial navigation in the range of about 4 Nm and using binocular even 6 Nm.

The greatest limitations derive from the fact that the beacons of gates on each side of the waterway and axis of the normal leading lines are located on great-circles. Therefore, beacons are moved away about 6 m from rhumb lines prolonging the longer side of the rectangular leading line. Similarly, the axis of the waterway runs exactly in the middle between beacons in each gate. The entire deviation between kilometer 11 and 42 of the waterway is about 40 m. Although the deviations of the rectangular leading lines from the waterway axis are within the limits of their sensitivity, inaccuracies are still visible. Figure 6 shows a schematic diagram of navigation in leading lines in the opposite directions.

As shown on Figure 6, there is left-handed traffic in same areas (on diagram shaded), but the scale of this phenomenon is of the order of the leading line sensitivity. On the other hand, lateral movement of the vessel, which is influenced by the wind, cross-currents, shape of waterway bed and the helmsmen, is of the same order of magnitude (PIANC, 2014). This phenomenon is, however, possible to observe. The method based on the Cartesian model is often the only terrestrial one that can be used when passing vessels on Zalew Szczeciński.

Fot. 1. Zalew Szczeciński, perspective division *k* = 1.8 – the observer in waterway limits on the east side of the fairway

When designing waterway beaconage consisting of rectangular leading lines, the following limitations should be also consider:

- Uneven background of both sides of the waterway, which is however, not relevant at night;
- Visibility not less than one and half distances between gates and two when the observer see gates only in front or only astern of him;
- Vessels sailing not in waterway axis may hinder look-out of the other navigators;
- Described model is most useful for maintaining the ship in safe sectors between lines parallel to the waterway axis and formed by dividing a short side of the rectangle 1:1 and 1:2 and also 1:2 and 1:3;
- Mismatched parameters *D* and *L* can hinder look-out, for example: too long of a distance be-

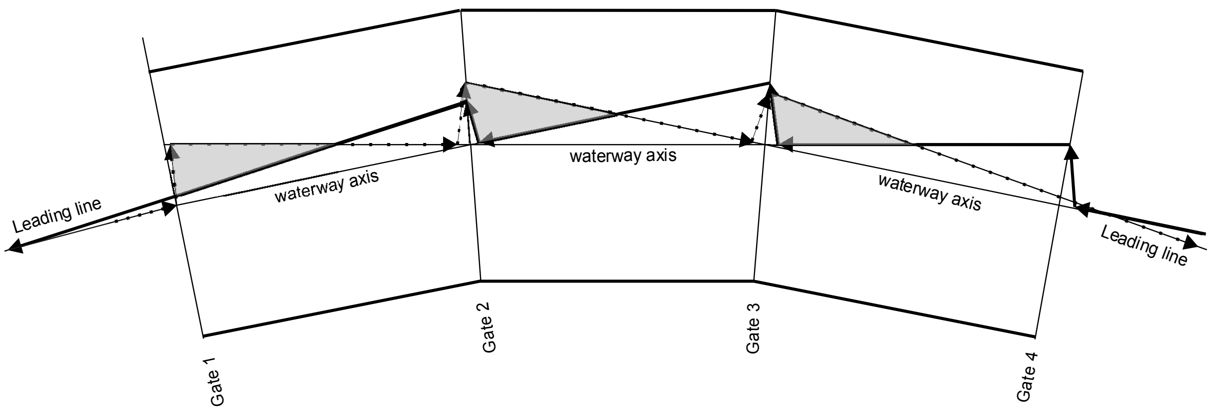

Figure 6. Schematic diagram of navigation in leading lines in the opposite directions

tween beacons in the gate can hamper the comparison of angles. Appropriate tests could be carried out on a full mission simulator.

Conclusions

This article presents leading lines consisting of four beacons situated at apexes of a rectangle, which can be used on long, rectilinear sections of the waterways. Navigation is based on comparing the perspectives created by beacons on both sides of the waterway. When the perspective of beacons on one side is two times bigger than perspective on the other side, the observer is in the one-third region of the leading line width. For the perspective division 1 to n the observer is in the $n + 1$ part of the leading line width. For the Zalew Szczeciński conditions i.e. rectangle with 6750 by 260 meters, the practical range of the leading line is about 300 to 8000 m. Lateral safe deviation in this range, specified by practice, for perspective division 1 and 2 is about 10 meters. Uncertainty in the measurements however, requires further studies in terms of angular resolution and perspective division quotient assessment. In practice, use of such a leading line consists in keeping the vessel in sectors of perspective divisions 1 to 2 and 2 to 3, giving certainty to the navigator that he maintains safe lateral distance from waterway axis.

References

1. GUCMA, S. (2004) *Nawigacja Pilotażowa.* Gdańsk: Fundacja Promocji Przemysłu Okrętowego i Gospodarki Morskiej.
2. KIERZKOWSKI, W. (1984) *Pomiary Morskie*. Gdynia: WSMW.
3. NASKRĘCKI, R. (2009) *Fizyka Procesu Widzenia*. Platforma Upowszechniania Nauki w Polsce.
4. SZYDŁOWSKI, H. (1981) *„Teoria Pomiarów"*. Warszawa: Państwowe Wydawnictwo Naukowe.
5. PIANC (2014) *Harbour Approach Channels Design Guidelines*. Bruxelles: The Word Association for Waterborne Transport Infrastructre.