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LAPLACE-CARSON INTEGRAL TRANSFORM FOR EXACT SOLUTIONS OF NON-INTEGER ORDER INITIAL VALUE PROBLEMS WITH CAPUTO OPERATOR

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Abstract. Finding the exact solution to dynamical systems in the field of mathematical modeling is extremely important and to achieve this goal, various integral transforms have been developed. In this research analysis, non-integer order ordinary differential equations are analytically solved via the Laplace-Carson integral transform technique, which is a technique that has not been previously employed to test the non-integer order differential systems. Firstly, it has proved that the Laplace-Carson transform for n-times repeated classical integrals can be computed by dividing the Laplace-Carson transform of the underlying function by n-th power of a real number p which later helped us to present a new result for getting the Laplace-Carson transform for δ-derivative of a function under the Caputo operator. Some initial value problems based upon Caputo type fractional operator have been precisely solved using the results obtained thereof.

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1. Introduction

A scientific field known by the name of fractional (non-integer order) calculus is considered to be as old as its counterpart called the traditional or classical calculus. The former field is mostly about finding derivatives and integrals of mathematical processes with an arbitrary order δ such that, in general, $\delta \in \mathbb{C}$ whereas the traditional calculus does not enjoy such a high degree of freedom, meaning that it allows us to take only integer order derivatives and integrals for the research problem under one's consideration. By degree of freedom we mean that one is allowed to take an infinite number of orders for differential and integral operators between any two successive integers thereby leading to have numerous choices for solutions of the problem in hand. This is also due to the non-local nature of fractional operators which has opened many doors for scientists and researchers to enable them to think outside of the box.

Most recently, exponential growth has been observed in research work carried out in the field of fractional calculus. Numerous publications and new research journals based upon mathematical problems being studied in this field are the major interests of scholars [1–3]. This is because of the ability of these non-local operators to more suitably capture the complicated and anomalous behavior of various physical and natural processes. Many such examples have been encountered in the past and in recent literature. For example, Abro in [4] has investigated the thermo-diffusion effects on unsteady-free convection flow, via fractional operator of Atangana-Baleanu for the governing mathematical equations in the presence of magnetic field. Qureshi and Atangana in [5] analyzed an epidemiological system for dengue fever while using fractional operators of Caputo, Caputo-Fabrizio and Atangana-Baleanu and proved with the assistance of experimental data that these non-local operators fit the data more accurately than the local operator from traditional calculus. In continuation, Qureshi et al. in [6] studied a blood ethanol concentration system in fractional settings and found a higher effectiveness of the fractional operators over the local ones. Qureshi and Yusuf in [7] designed three new versions of an epidemiological system for the chickenpox disease and proved the higher efficiency rates in statistical settings for the Caputo, Caputo-Fabrizio and Atangana-Baleanu operators. Various other mathematical models have been redefined in the framework of fractional calculus such as [8–15] and most of the references cited therein.

Mathematicians, engineers and physicists cannot deny the fact that finding exact solutions for fractional order dynamical systems is practically impossible. A similar sort of issue persists in the case of traditional dynamical systems, that is, the systems with integer-order derivatives and integral operators. However, when the system under investigation is simple (linear) then one can expect to have its exact or closed form solution. Growth and decay in population dynamics, RL-RC-RLC circuits in electronics, deflection of a beam in material science, and the mass-spring-damper system in mechanical engineering are the most suitable examples of linear models for which exact solutions are determinable.

In pursuance of getting an exact solution for the fractional order dynamical systems, integral transforms play a significant role. Perhaps the most popular and frequently used integral transform is the Laplace transform followed by others including the Fourier transform, Hankel transform, Sumudu transform, Natural transform, Mellin transform, Elzaki transform, Shehu transform, Aboodh transform, and the Mohand transform [16]. There is one more integral transform called the Laplace-Carson technique which, to the best of the authors' knowledge, has not been tested for getting the solutions of initial value problems defined by the Caputo operator. Thus, the present study is dedicated towards this goal.

2. Mathematical preliminaries

In order to perceive the main ambit of this research article, it is significant to know some basic information about non-integer order calculus and also some knowledge about the Laplace-Carson transform. In this regard, some relevant concepts are given below.

Definition 1 [17] The Laplace-Carson integral transform for a piecewise continuous function $g(t)$ with exponential order P is defined over the set of functions:

$$\mathbb{A} = \left\{ g(t) : \exists\, M, a_1, a_2 > 0,\ |g(t)| < M \exp\left(\frac{|t|}{a_i},\right), \text{if } t \in (-1)^i \times [0,\infty) \right\}, \tag{1}$$

via the given integral

$$LC[g(t)] = G(p) = p \int_0^{\infty} \exp(-pt) g(t) dt = pL\{g(t)\}, \tag{2}$$

where L shows the Laplace transform. □

Definition 2 [18] The non-integer order derivative known as the Caputo derivative in regards to function $g(t)$ with order $\delta > 0$ is justified and determined through the integral as given below

$$^{C}D_{0,t}^{\delta} g(t) = \frac{1}{\Gamma(m-\delta)} \int_0^t \frac{g^{(m)}(\mu)}{(t-\mu)^{\delta+1-m}} d\mu, \qquad m-1 < \delta \leq m \in \mathbb{N}. \tag{3}$$

Definition 3 [18] The fractional order integral called the Riemann-Liouville integral for function $g(t)$ having order $\delta > 0$ is defined by the following integral equation:

$$J_{0,t}^{\delta}[g(t)] = \frac{1}{\Gamma(\delta)} \int_0^t g(\mu)(t-\mu)^{\delta-1} d\mu, \quad t > 0, \tag{4}$$

where $\Gamma(\cdot)$ is known as the Euler gamma function. □

Theorem 1 *[17] Let $g^{(n)}(t)$ be the n-th order classical derivative of the function $g(t) \in \mathbb{A}$, there upon its Laplace-Carson transform can be shown in the following way:*

$$LC[g^{(n)}(t)] = p^n G(p) - \sum_{k=0}^{m-1} (p)^{m-k-1} g^{(k)}(0), \quad m \in \mathbb{N}. \tag{5}$$

Lemma 1 *[18] It is apparent that the non-integer order integral Riemann-Liouville and non-integer order Caputo derivative are similar to each other in accordance to the equation given below:*

$$^{C}D_{0,t}^{\delta} g(t) = J_{0,t}^{m-\delta}\left[D^m g(t)\right], \quad (m-1) < \delta < m, m \in \mathbb{N}. \tag{6}$$

3. Research methodology

This paper mainly aims to solve non-integer order Caputo type initial value problems that have fractional-order $\delta > 0$. Considering the similar point of view, we have derived the Laplace-Carson transform of classical integral of a function $g(t)$. After that, this property along with some fundamental concepts as presented in section 2 are used to prove a new theorem that will be used to get the exact solutions for the numerical examples under consideration.

Theorem 2 *If $G(s)$ is the Laplace-Carson integral transform of the function $g(t)$ then the Laplace-Carson integral transform for the classical integral $g(t)$ that can be described via the given equation:*

$$LC\left\{ \int_0^t g(u)du \right\} = \left(\frac{1}{p}\right)G(p). \tag{7}$$

PROOF Using the definition of the Laplace-Carson integral transform for $g(t)$, the following equation can be obtained

$$LC\left\{ \int_0^t g(u)du \right\} = p\int_0^\infty \exp(-pt)dt.\int_0^t g(u)du. \tag{8}$$

Using the product rule of integration, we get the equation as given below:

$$LC\left\{ \int_0^t g(u)du \right\} = p\left(\int_0^t g(u)du \int_0^\infty \exp\Big(-pt\Big)dt - \int_0^\infty \left\{ \frac{d}{dt}\int_0^t g(u)du \int_0^\infty \exp\Big(-pt\Big)dt \right\}dt \right),$$

$$LC\left\{ \int_0^t g(u)du \right\} = p\left(\int_0^t -g(u)du\Big(\frac{1}{p}\Big)\exp(-pt)\Big|_0^\infty + \Big(\frac{1}{p}\Big)\int_0^\infty \exp(-pt)g(t)dt \right),$$

$$LC\left\{ \int_0^t g(u)du \right\} = \left(\frac{1}{p}\right)G(p).$$

Correspondingly, $LC\left\{ \int_0^t \int_0^t g(u)du^2 \right\} = \left(\frac{1}{p}\right)^2 G(p)$. Continuing in this way, we obtain the general form as given below:

$$LC\left\{ \int_0^t \dots \int_0^t g(u)du^n \right\} = \left(\frac{1}{p}\right)^n G(p). \tag{9}$$

Next, we present a theorem which has been devised bearing in mind that it would be capable enough to get the exact solutions for fractional order ordinary differential equations under the Caputo differential operator having fractional order $\delta > 0$.

Theorem 3 *For instance, $G(p)$ is the Laplace-Carson integral transform of function $g(t)$ then the Laplace-Carson transform of fractional order derivative for $g(t)$ under the Caputo type operator having order $\delta > 0$ is suggested as given below:*

$$LC\{{}^{C}D_{0,t}^{\delta}g(t)\} = \Big(p\Big)^{\delta}G(p) - \sum_{k=0}^{m-1}\Big(p\Big)^{\delta-k}g^{(k)}(0). \tag{10}$$

PROOF Listing the Lemma 1, it can be written in the form of the given equation

$${}^{C}D_{0,t}^{\delta}g(t) = J_{0,t}^{m-\delta}\Big[D^{m}g(t)\Big]. \tag{11}$$

Applying the Laplace-Carson transform on both sides, the equation can be obtained as following:

$$LC\{{}^{C}D_{0,t}^{\delta}g(t)\} = LC\{J_{0,t}^{m-\delta}(D^{m}g(t))\}. \tag{12}$$

Forthwith, using the Laplace-Carson transform for the integral of a function as developed in the section 2, one can gain the following

$$\begin{aligned}
LC\{{}^{C}D_{0,t}^{\delta}g(t)\} &= \Big(\frac{1}{p}\Big)^{m-\delta}LC\{(D^{m}g(t))\},\\
LC\{{}^{C}D_{0,t}^{\delta}g(t)\} &= \Big(\frac{1}{p}\Big)^{m-\delta}\Big[\Big(p\Big)^{m}G(p) - \sum_{k=0}^{m-1}\Big(p\Big)^{m-k}g^{(k)}(0)\Big],\\
LC\{{}^{C}D_{0,t}^{\delta}g(t)\} &= \Big(p\Big)^{\delta}G(p) - \sum_{k=0}^{m-1}\Big(p\Big)^{\delta-k}g^{(k)}(0), \quad m \in \mathbb{N},\ \delta > 0.
\end{aligned} \tag{13}$$

In the above, it is wholly proved what it is needed for the Laplace-Carson integral transform of non integer order $\delta > 0$. With the help of this developed theorem we are able to solve the non-integer order linear initial value problems that are explained in the Caputo sense.

4. Results and discussion

Different kinds of fractional order initial value problems under the Caputo differential operator of order $\delta > 0$ can be solved using the Laplace-Carson integral transform technique. Considering the above derived theorem 3, we have solved some linear initial value problems under the Caputo differential operator. Few non-homogeneous initial value problems with fractional order $\delta = 1/2$ and $\delta = 3/2$ have been analytically solved with the Laplace-Carson integral transform technique. The obtained results in terms of basic mathematical functions agree welll with those obtained via integral transform techniques including the Shehu, Laplace, Mellin and Aboodh.

Consequently, the fractional order Caputo type continuous dynamical systems can be solved with the help of the Laplace-Carson integral transform technique. It is important to note that the technique of the Laplace-Carson integral transformation has so far been used for finding exact solutions of classical initial value problems. However, in this research article, this technique is used for the very first time to solve fractional order Caputo type initial value problems.

Example 1 Firstly, the given linear fractional order non-homogeneous initial value problem called the Bagley-Torvik equation is considered with $\delta = 3/2$:

$$D^2 g(t) + {}^C D_{0,t}^{\frac{3}{2}} g(t) + g(t) = t, \qquad g(0) = 0, g'(0) = 1. \tag{14}$$

Using the Laplace-Carson integral transform, the following equation is obtained:

$$\begin{aligned} &LC\{D^2 g(t)\} + LC\{{}^C D_{0,t}^{\frac{3}{2}} g(t)\} + LC\{g(t)\} = LC\{t\}, \\ &\Big(p\Big)^2 G(p) - p + \Big(p\Big)^{\frac{3}{2}} G(p) - \Big(p\Big)^{\frac{1}{2}} + G(p) = \Big(\frac{1}{p}\Big), \\ &G(p)\Big\{\Big(p\Big)^{\frac{3}{2}} + \Big(p\Big)^2 + 1\Big\} = \Big(\frac{1}{p}\Big) + p + \Big(p\Big)^{\frac{1}{2}}, \\ &G(p)\Big\{\Big(p\Big)^{\frac{3}{2}} + \Big(p\Big)^2 + 1\Big\} = \Big(\frac{1}{p}\Big)\Big\{\Big(p\Big)^{\frac{3}{2}} + \Big(p\Big)^2 + 1\Big\}, \\ &G(p) = \Big(\frac{1}{p}\Big). \end{aligned} \tag{15}$$

The exact solution is gained in the form of the equation given below. It is gained by applying the inverse Laplace-Carson transform.

$$g(t) = t. \tag{16}$$

Example 2 Consider the following linear fractional order in-homogeneous initial value problem with $\delta \in [1,2]$:

$$D^2 g(t) + {}^C D_{0,t}^{\delta} g(t) + g(t) = 1 + t, \qquad g(0) = g'(0) = 1. \tag{17}$$

By applying the Laplace-Carson integral transform, the following equation can be gained:

$$
\begin{aligned}
&LC\{D^2 g(t)\} + LC\{{}^C D_{0,t}^{\delta} g(t)\} + LC\{g(t)\} = LC\{1\} + LC\{t\},\\
&\Big\{\Big(p\Big)^2 G(p) - p^2 - p\Big\} + \Big\{\Big(p\Big)^{\delta} G(p) - p^{\delta} - p^{\delta-1}\Big\} + G(p) = 1 + \frac{1}{p},\\
&\Big\{\Big(p\Big)^2 G(p) + \Big(p\Big)^{\delta} G(p) + G(p)\Big\} = 1 + \frac{1}{p} + p^2 + p^{\delta} + p^{\delta-1},\\
&G(p)\Big\{\Big(p\Big)^2 + p^{\delta} + 1\Big\} = 1\Big(1 + \frac{1}{p}\Big) + p^2\Big(1 + \frac{1}{p}\Big) + p^{\delta}\Big(1 + \frac{1}{p}\Big),\\
&G(p)\Big\{\Big(p\Big)^2 + p^{\delta} + 1\Big\} = \Big(1 + \frac{1}{p}\Big)\Big(1 + p^2 + p^{\delta}\Big),\\
&G(p) = \Big(1 + \frac{1}{p}\Big).
\end{aligned}
$$

Similarly, applying the inverse Laplace-Carson transform, the absolute analytical solution can be obtained as following:

$$g(t) = 1 + t. \tag{18}$$

Example 3 Consider the below-mentioned linear fractional order non-homogeneous initial value problem with $\delta \in (0,1]$:

$${}^C D_{0,t}^{\delta} g(t) + g(t) = \frac{2t^{2-\delta}}{\Gamma(3-\delta)} + t^2, \qquad g(0) = 0. \tag{19}$$

Using the Laplace-Carson integral transform, the following equation can be obtained:

$$
\begin{aligned}
&LC\{{}^C D_{0,t}^{\delta} g(t)\} + LC\{g(t)\} = \frac{2}{\Gamma(3-\delta)} LC\{t^{2-\delta}\} + LC\{t^2\},\\
&\Big\{\Big(p\Big)^{\delta} G(p)\Big\} + G(p) = \frac{2}{\Gamma(3-\delta)} . \frac{\Gamma(2-\delta+1)}{p^{2-\delta}} + \Big(\frac{2}{p^2}\Big),\\
&G(p)\Big\{(p)^{\delta} + 1\Big\} = \frac{2}{p^{2-\delta}} + \Big(\frac{2}{p^2}\Big),\\
&G(p)\Big\{(p)^{\delta} + 1\Big\} = \frac{2}{p^2}\Big\{\frac{1}{p^{-\delta}} + 1\Big\},\\
&G(p)\Big\{(p)^{\delta} + 1\Big\} = \frac{2}{p^2}\Big\{(p)^{\delta} + 1\Big\},\\
&G(p) = \frac{2}{p^2}.
\end{aligned}
$$

By applying the inverse Laplace-Carson transform, the exact solution can be gained as:

$$g(t) = t^2. \tag{20}$$

Example 4 Consider the following linear fractional order non-homogeneous initial value problem with $\delta \in (0,1)$:

$$ {}^{C}D_{0,t}^{\delta} g(t) + g(t) = \frac{2t^{2-\delta}}{\Gamma(3-\delta)} - \frac{t^{1-\delta}}{\Gamma(2-\delta)} + t^2 - t, \quad g(0) = 0. \tag{21} $$

Using the Laplace-Carson integral transform, the following equation can be obtained:

$$ LC\{{}^{C}D_{0,t}^{\delta} g(t)\} + LC\{g(t)\} = \frac{2}{\Gamma(3-\delta)} LC\{t^{2-\delta}\} + \frac{1}{\Gamma(2-\delta)} LC\{t^{1-\delta}\} + LC\{t^2\} - LC\{t\}, $$

$$ \{p^{\delta} G(p)\} + \{G(p)\} = \frac{2}{\Gamma(3-\delta)} \cdot \frac{\Gamma(3-\delta)}{p^2 - \delta} - \frac{1}{\Gamma 2 - \delta} \frac{\Gamma 2 - \delta}{p^1 - \delta} + \frac{2}{p^2} + \frac{1}{p}, $$

$$ G(p)\Big\{p^{\delta} + 1\Big\} = \frac{2}{p^2 - \delta} - \frac{1}{p^1 - \delta} + \frac{2}{p^2} + \frac{1}{p}, $$

$$ G(p)\Big\{p^{\delta} + 1\Big\} = p^{\delta}\Big\{\frac{2}{p^2} - \frac{1}{p}\Big\} + 1\Big\{\frac{2}{p^2} - \frac{1}{p}\Big\}, $$

$$ G(p)\Big\{p^{\delta} + 1\Big\} = \Big(p^{\delta} + 1\Big)\Big\{\frac{2}{p^2} - \frac{1}{p}\Big\}, $$

$$ G(p) = \Big\{\frac{2}{p^2} - \frac{1}{p}\Big\}. $$

By implying the inverse Laplace-Carson transform, the exact solution can be gained as

$$ g(t) = t^2 - t. \tag{22} $$

Keeping in view the above results, it is hence verified that the Laplace-Carson integral transform technique can be used to solve any linear fractional order Caputo type ordinary differential equation with some initial condition(s).

5. Conclusion

In the current investigation of the Laplace-Carson integral transform technique, it has been proved that the technique can be used for linear non-integer order initial value problems. Furthermore, it can be stated that the technique is first time introduced in the current research study under the Caputo type operator. The main aim of this paper was to devise ways to use the Laplace-Carson integral transform technique for the Caputo type ordinary differential equations. In this regard, proof of a new theorem is also achieved by using the Caputo operator. The achieved theorem's quality is that it can help us to solve the fractional type of differential equations with the Caputo type operator. This can easily be compared with various transformation techniques available in the literature. In conjunction with the above-mentioned technique used to derive a new theorem, this research can pave the way to solving linear partial differential equations under the Caputo operator which is the research direction for our future work.

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