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A GENERALIZATION OF THE GRAPH LAPLACIAN WITH APPLICATION TO A DISTRIBUTED CONSENSUS ALGORITHM

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In order to describe the interconnection among agents with multi-dimensional states, we generalize the notion of a graph Laplacian by extending the adjacency weights (or weighted interconnection coefficients) from scalars to matrices. More precisely, we use positive definite matrices to denote full multi-dimensional interconnections, while using nonnegative definite matrices to denote partial multi-dimensional interconnections. We prove that the generalized graph Laplacian inherits the spectral properties of the graph Laplacian. As an application, we use the generalized graph Laplacian to establish a distributed consensus algorithm for agents described by multi-dimensional integrators.

Keywords: graph Laplacian, generalized graph Laplacian, adjacency weights, distributed consensus algorithm, cooperative control.

1. Introduction

There has been great interest in cooperative control of multi-agent systems, including collective behavior of flocks and swarms, sensor fusion, random networks, synchronization of coupled oscillators, formation control of multi robots, optimization-based cooperative control, etc. For more detailed information on this line, see the cornerstone paper by Vicsek *et al*. (1995), the survey papers of Olfati-Saber *et al*. (2007) and Bauer (2008), the book by Shamma (2008) and the references cited therein.

One significant control issue in cooperative control is the consensus problem, which means reaching an agreement regarding a certain quantity of interest that depends on the state of all agents. There are several important papers which have made great contribution to the consensus problem for self-organizing networked systems (Fax and Murray, 2004; Jadbabaie *et al*., 2003; Moreau, 2005; Ren and Beard, 2005; Cai and Ishii, 2012; Priolo *et al*., 2014). The approach of achieving consensus for general linear agents in the framework of matrix inequalities and stabilization is proposed by Zhai *et al*. (2009), and the extension to the consensus problem for networked nonholonomic systems is dealt with in another work of Zhai *et al*. (2010).

It is noted that the basic consensus problem requires that all agents' states converge to the same vector, and the well known existing method is to describe the agents' information flow (structure) as an interconnected graph and to use the graph Laplacian as a (negative) state feedback gain. The graph Laplacian is a matrix whose elements denote the adjacency weights (or weighted interconnection coefficients) among the agents. Such a scalar-weighted interconnection graph or the equivalent graph Laplacian is enough to describe the interconnection among one-dimensional agents or multi-dimensional agents whose states are connected to other agents uniformly. However, it cannot deal with the agents with multi-dimensional states where different state variables have different weights. For example, in the case of a family of moving vehicles, each agent's states are its position and velocity in general, and it may not be reasonable to describe the interconnection between the position and the velocity of any two vehicles by a single scalar. Based on this motivation, we generalize the graph Laplacian so as to describe the interconnection among different elements of the state.

Based on the above observation, we generalize the notion of a graph Laplacian by extending the adjacency weights from scalars to matrices. More precisely, we use positive definite matrices to denote full multi-dimensional interconnections, and nonnegative definite matrices to denote partial multi-dimensional interconnections (including the case of no interconnection, where zero matrices are used). We show that such a generalized graph Laplacian includes the graph Laplacian as a special case, and inherits the spectral properties of the graph Laplacian. Then, as an application example, we use the generalized graph Laplacian to establish a distributed consensus algorithm for agents described by multi-dimensional integrators.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries about the graph and the graph Laplacian, and state the Schur complement lemma and an inequality concerning the spectral property of the addition of two matrices. Section 3 establishes the generalization of the graph Laplacian, and states one important in spectral property of the generalized graph Laplacian. As an application example, in Section 4, present a new distributed consensus algorithm by using the concept of a generalized graph Laplacian, together with a numerical example confirming the effectiveness. Finally, Section 5 concludes the paper.

2. Preliminaries

2.1. Graph Laplacian. We first review the notion of a graph Laplacian in the literature. Usually, the interconnection of a family of agents is represented by using a directed graph (or a digraph) $G = (V, E)$ with the set of nodes $V = \{1, 2, \ldots, N\}$ (N is the number of agents) and edges $E \subset V \times V$. The edge $(j, i) \in E$ means that the information of the j-th agent is available for the i-th agent. If each pair of agents is bidirectional, i.e., $(j, i) \in E$ if and only if $(i, j) \in E$, then we omit the direction of the edges and use an undirected graph. As can be seen later, although most of the discussion can be extended to directed graphs, we focus our attention on the case of undirected graphs in this paper.

The set of neighbor agents of the i-th agent is defined as

$$N_i = \{j \in V \mid (j, i) \in E\}, \tag{1}$$

which is the index set of the agents from which the i-th agent can obtain necessary information. Then, the graph Laplacian of the agents' structure is defined as $L = [l_{ij}]_{N\times N}$, where

$$l_{ij} = \begin{cases} -a_{ij}, & j \in N_i, \\ \sum_{j\in N_i} a_{ij}, & j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

and a_{ij} is a positive scalar describing the adjacency weight, $|N_i|$ denotes the total adjacency weights of neighbor agents of the i-th agent (or the in-degree of the i-th agent). If, additionally, $a_{ij} = a_{ji}$ holds in the undirected graph, we say the interconnection (or the graph) is symmetric. Obviously, when a graph is symmetric, the graph Laplacian is a symmetric matrix.

Using the above definition, the graph Laplacian of the bidirectional graph in Fig. 1 is

$$L = \begin{bmatrix} a_{12}+a_{13} & -a_{12} & -a_{13} & 0 \\ -a_{21} & a_{21} & 0 & 0 \\ -a_{31} & 0 & a_{31}+a_{34} & -a_{34} \\ 0 & 0 & -a_{43} & a_{43} \end{bmatrix}, \tag{3}$$

and when the weights are the same between any pair of agents,

$$L = \begin{bmatrix} a_{12}+a_{13} & -a_{12} & -a_{13} & 0 \\ -a_{12} & a_{12} & 0 & 0 \\ -a_{13} & 0 & a_{13}+a_{34} & -a_{34} \\ 0 & 0 & -a_{34} & a_{34} \end{bmatrix}. \tag{4}$$

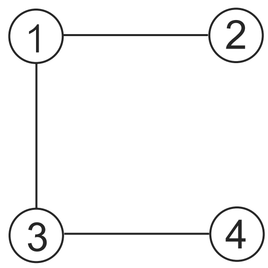

Fig. 1. Interconnection graph example.

From the definition (2) it is easy to see that all row-sums of L are zero, and thus L always has a zero eigenvalue and a corresponding eigenvector $\mathbf{1} = [1\ 1\ \cdots\ 1]^{\top}$. It is also known that the other eigenvalues of L have positive real parts when there is a spanning tree included in the graph. When the agents are bidirectional with the same weights (the graph is symmetric), the graph Laplacian L is a symmetric matrix, and thus it is nonnegative definite ($L \succeq 0$). For other spectral properties of graph Laplacians, see, for example, the work of Mohar (1991).

In the existing graph Laplacian method for achieving consensus, the entire controller takes the form of $u = -Lx$, where x and u are the collective (group) state and the collective (group) control input of multi-agents, respectively. The proof of achieving consensus with this controller is generally done by discussing the eigenvalues of the resultant closed-loop system matrix or using LaSalle's invariant principle (Khalil, 2002).

2.2. Notation and lemmas. Throughout this paper, the superscript "$\top$" represents the transpose of a matrix, and the superscript "-1" represents the inverse of a matrix. $W \succ 0$ ($W \prec 0$) means W is symmetric and positive (negative) definite, and $W_1 \succ W_2$ means $W_1 - W_2 \succ 0$.

$W \succeq 0$ ($W \preceq 0$) means W is symmetric and nonnegative (nonpositive) definite, and $W_1 \succeq W_2$ means $W_1 - W_2 \succeq 0$. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes its Euclidean norm. Denote by $\lambda_i(A)$ the i-th eigenvalue of the matrix A, and use $\lambda_M(A)$ (resp. $\lambda_m(A)$) to denote the maximum (minimum) eigenvalue of the symmetric matrix A. As used in most textbooks, $\mathrm{Re}(z)$ represents the real part of a complex scalar z.

The following two lemmas are necessary in the next section.

Lemma 1. (Schur complement lemma) (Gantmacher, 1959). *The following statements are equivalent:*

(i) $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succ 0$.

(ii) $A \succ 0$ *and* $C - B^\top A^{-1} B \succ 0$.

(iii) $C \succ 0$ *and* $A - BC^{-1}B^\top \succ 0$.

Lemma 2. (Gantmacher, 1959) *For any real matrices* $A, B \in \mathbb{R}^{n\times n}$,

$$\begin{aligned}\lambda_m\left(\frac{A^\top + A}{2}\right) + \lambda_m\left(\frac{B^\top + B}{2}\right) \\ \leq \mathrm{Re}\left(\lambda_i(A+B)\right) \\ \leq \lambda_M\left(\frac{A^\top + A}{2}\right) + \lambda_M\left(\frac{B^\top + B}{2}\right). \end{aligned} \tag{5}$$

When both A *and* B *are symmetric,*

$$\begin{aligned}\lambda_m(A) + \lambda_m(B) \leq \lambda_i(A+B) \\ \leq \lambda_M(A) + \lambda_M(B). \end{aligned} \tag{6}$$

3. Generalization of the graph Laplacian

3.1. Definition of the generalized graph Laplacian. As mentioned in the Introduction, the graph Laplacian defined in (2) basically deals with all states of every agent uniformly, which is not practical in real systems. In this paper, we propose to generalize the graph Laplacian so as to describe the interconnection among different elements of the state.

Suppose the dimension of all agents' dynamics is n and the entire interconnection graph is connected (there is no isolated agent). Then, the basic idea of our generalized graph Laplacian is to replace the adjacency weight a_{ij} with a matrix A_{ij}. In other words, if the i-th agent can obtain full state information from the j-th agent, we choose a positive definite matrix A_{ij} (depending on the real systems) to denote the bidirectional connection. If the i-th agent can only obtain information of a partial state from the j-th agent, we choose a nonnegative definite matrix A_{ij}, which includes the special case of setting $A_{ij} = 0$ when the i-th agent cannot obtain any state information from the j-th agent. The block-diagonal element is defined similarly as in (2). Therefore, the generalized graph Laplacian describing the agents' interconnection is defined as a matrix $L_G = [L_{ij}]_{N\times N}$ with the entire size $nN \times nN$, where

$$L_{ij} = \begin{cases} -A_{ij}, & j \neq i, \\ \displaystyle\sum_{j=1}^{N} A_{ij}, & j = i, \end{cases} \tag{7}$$

and all the matrices $A_{ij} \in \mathbb{R}^{n\times n}$ appearing in the above are appropriate positive (or nonnegative) definite matrices. When $A_{ij} = A_{ji}$, $\forall i \neq j$, we call the interconnection (or the graph) *symmetric*.

It is to be noted that the adjacency weight matrix A_{ij} is chosen nonnegative definite since it is a common sense of extending nonnegative scalars to nonnegative definite matrices which can represent the "nonnegative definiteness" in an entire sense. Concerning the individual elements of A_{ij}, we do not require that they should be nonnegative in this paper, although they are usually nonnegative in real applications.

Now, it is natural to define the neighbor agents set of the i-th agent as

$$N_i = \{j \in V \mid A_{ij} \succeq 0, A_{ij} \neq 0\}, \tag{8}$$

which actually consists of the full-connected set ($A_{ij} \succ 0$) and the partial-connected set ($A_{ij} \succeq 0$ but $A_{ij} \neq 0$). Then, the generalized graph Laplacian takes the form of

$$L_{ij} = \begin{cases} -A_{ij}, & j \in N_i, \\ \displaystyle\sum_{j\in N_i} A_{ij}, & j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

which is almost the same as (2).

Using the above definition for the agents in Fig. 1, when all the agents are multi-dimensional, we obtain the generalized graph Laplacian as

$$\begin{bmatrix} A_{12} + A_{13} & -A_{12} & -A_{13} & 0 \\ -A_{21} & A_{21} & 0 & 0 \\ -A_{31} & 0 & A_{31} + A_{34} & -A_{34} \\ 0 & 0 & -A_{43} & A_{43} \end{bmatrix}. \tag{10}$$

Remark 1. If we choose simply $A_{ij} = a_{ij} I_n$ with $a_{ij} > 0$, then the generalized graph Laplacian L_G substantially shrinks to the graph Laplacian in the existing literature.

Remark 2. As mentioned before, the adjacency weight matrix A_{ij} should be defined according to practical and physical requirements. In the case of moving vehicles on a two-dimensional plane without considering the velocities, the states are composed of each vehicle's x-axis position (the first element) and its y-axis position (the second element). Then

$$A_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

means the x-axis position is available,

$$A_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

means the y-axis position is available, and

$$A_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

or

$$A_{ij} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

mean that both the x and y-axis positions are available.

3.2. Spectral property. In this section, we prove that the generalized graph Laplacian L_G defined in (9) has the same spectral property as the graph Laplacian (Mohar, 1991). It is known that there is a great difference between directed graphs and undirected ones. Here, for simplicity, we focus on the case of symmetric graphs, i.e., $A_{ij} = A_{ji}, \forall i \neq j$. Moreover, as in the discussion for the graph Laplacian, we require that the interconnection graph be connected, which means there is a path connecting all the agents in the sense that the adjacency weight matrices on the path are positive definite.

Theorem 1. *When the interconnection graph is connected and symmetric, the generalized graph Laplacian L_G defined in (9) has n zero eigenvalues, and all the other eigenvalues are positive. Thus, L_G is nonnegative definite.*

Proof. The n zero eigenvalues of L_G can be confirmed by the following equation:

$$L_G \left[\ v^\top \quad v^\top \quad \cdots \quad v^\top \ \right]^\top = 0, \tag{11}$$

where v is an arbitrary vector in $\mathbb{R}^n$. ■

When $N = 2$, the generalized graph Laplacian is

$$L_G = \begin{bmatrix} A_{12} & -A_{12} \\ -A_{12} & A_{12} \end{bmatrix}, \tag{12}$$

where A_{12} is positive definite. Then, according to the similarity transformation

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}^{-1} L_G \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} 0 & -A_{12} \\ 0 & 2A_{12} \end{bmatrix}, \tag{13}$$

L_G has n zero eigenvalues, and the other eigenvalues are those of the matrix $2A_{12}$. Since A_{12} is positive definite, the eigenvalues of $2A_{12}$ are all positive.

When $N = 3$, the generalized graph Laplacian is

$$L_G = \begin{bmatrix} A_{12} + A_{13} & -A_{12} & -A_{13} \\ -A_{12} & A_{12} + A_{23} & -A_{23} \\ -A_{13} & -A_{23} & A_{13} + A_{23} \end{bmatrix}, \tag{14}$$

where all the matrices $A_{ij}, i \neq j$ are positive (or nonnegative) definite (and symmetric). Since L_G is symmetric and thus it has only real eigenvalues, the definiteness of L_G is equivalent to that of

$$\begin{aligned} \tilde{L}_G &= \begin{bmatrix} I & 0 & I \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}^\top L_G \begin{bmatrix} I & 0 & I \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A_{12} + A_{13} & -A_{12} & 0 \\ -A_{12} & A_{12} + A_{23} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{15}$$

It is obvious that $\tilde{L}_G$ has n zero eigenvalues, and the other eigenvalues are that of the matrix

$$\tilde{A} = \begin{bmatrix} A_{12} + A_{13} & -A_{12} \\ -A_{12} & A_{12} + A_{23} \end{bmatrix}. \tag{16}$$

Next, we prove that when the graph is connected, the matrix $\tilde{A}$ is positive definite.

Case 1: $A_{12} \succ 0$ and $A_{13} \succ 0$.
Since $A_{23} \succeq 0$, it is easy to obtain $A_{12} + A_{23} \succ 0$ and thus

$$\begin{aligned} & A_{12} + A_{13} - A_{12}(A_{12} + A_{23})^{-1} A_{12} \\ & \quad \succeq A_{12} + A_{13} - A_{12} A_{12}^{-1} A_{12} = A_{13} \succ 0 \,. \end{aligned} \tag{17}$$

Then, according to part (iii) of the Schur complement lemma, $\tilde{A}$ is positive definite.

Case 2: $A_{12} \succ 0$ and $A_{23} \succ 0$.
Since $A_{13} \succeq 0$, it is easy to obtain $A_{12} + A_{13} \succ 0$ and thus

$$\begin{aligned} & A_{12} + A_{23} - A_{12}(A_{12} + A_{13})^{-1} A_{12} \\ & \quad \succeq A_{12} + A_{23} - A_{12} A_{12}^{-1} A_{12} = A_{23} \succ 0 \,. \end{aligned} \tag{18}$$

Then, according to part (ii) of the Schur complement lemma, $\tilde{A}$ is positive definite.

Case 3: $A_{13} \succ 0$ and $A_{23} \succ 0$

Split the matrix $\tilde{A}$ into two parts $\tilde{A} = \tilde{A}_1 + \tilde{A}_2$, where

$$\tilde{A}_1 = \begin{bmatrix} A_{13} & 0 \\ 0 & A_{23} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_{12} & -A_{12} \\ -A_{12} & A_{12} \end{bmatrix}. \tag{19}$$

It is obtained from (13) that the eigenvalues of $\tilde{A}_2$ are nonnegative. Since $\tilde{A}_1 \succ 0$, we use Lemma 2 to obtain

$$\lambda_i(\tilde{A}) \geq \lambda_m(\tilde{A}_1) + \lambda_m(\tilde{A}_2) > 0, \tag{20}$$

and thus $\tilde{A}$ is positive definite.

To summarize, in all cases of $N = 3$, the matrix $\tilde{A}$ is positive definite, and all the eigenvalues of $\tilde{A}$ are positive. Therefore, the matrix $\tilde{L}_G$ has n zero eigenvalues, and the other eigenvalues are positive. Although the eigenvalues of L_G may be different from those of $\tilde{L}_G$, the definiteness properties of L_G and $\tilde{L}_G$ are equivalent, and thus L_G also has n zero eigenvalues, and all the other eigenvalues are positive.

The case of $N > 3$ can be proved similarly by induction, and therefore it is omitted here.

According to Theorem 1, when the interconnection is connected and symmetric, the generalized graph Laplacian L_G has n zero eigenvalues and $n(N-1)$ positive eigenvalues. In much the some way as in the literature, we call the smallest positive eigenvalue of L_G the *algebraic connectivity* of the interconnection, which determines the convergence rate of achieving consensus.

4. Application to a consensus algorithm

4.1. System and controller. As an application example, we consider N agents which are n dimensional integrators

$$\dot{X}_i(t) = U_i(t)\,, \tag{21}$$

where $X_i(t) \in \mathbb{R}^n$ is the state and $U_i(t) \in \mathbb{R}^n$ is the control input. We assume that the agents' interconnection is characterized by the generalized graph Laplacian L_G, and, for simplicity, assume that the interconnection is symmetric. The consensus problem is to design the control input, based on the information of its neighboring agents and itself, so that the states $X_i(t)$ converge to the same vector or, in other words,

$$\lim_{t\to\infty} \|X_i(t) - X_j(t)\| = 0\,. \tag{22}$$

As in the existing graph Laplacian consensus algorithm, we express the dynamics of all the agents in a compact form as

$$\dot{X}(t) = U(t)\,, \tag{23}$$

where $X(t) = [X_1^\top(t), \ldots, X_N^\top(t)]^\top$ is the collective state, and $U(t) = [U_1^\top(t), \ldots, U_N^\top(t)]^\top$ is the collective control input.

Now, we use the generalized graph Laplacian L_G to establish the control input as

$$U(t) = -L_G X(t) \tag{24}$$

or, equivalently,

$$U_i(t) = \sum_{j\in N_i} A_{ij}\left(X_j(t) - X_i(t)\right). \tag{25}$$

From (25) it is clear that the control input $U_i(t)$ feedbacks the states of its neighbor agents and itself, and thus has the desired distributed structure.

4.2. Consensus analysis. The closed-loop system composed of (23) and (24) is

$$\dot{X}(t) = -L_G X(t)\,, \tag{26}$$

for which we consider the Lyapunov-like function candidate $V(X(t)) = X^\top(t) L_G X(t)$. It is not difficult to get

$$V(X) = \frac{1}{2}\sum_{A_{ij}\neq 0} (X_i(t) - X_j(t))^\top A_{ij}(X_i(t) - X_j(t)), \tag{27}$$

and thus it is actually a quadratic disagreement function concerning consensus among the agents. In other words, the consensus is completely achieved ($X_1 = X_2 = \cdots = X_N$) if and only if $V(X) = 0$.

The time derivative of $V(X(t))$ along the trajectories of (26) is calculated as

$$\begin{aligned}\dot{V}(X(t)) &= \dot{X}^\top(t) L_G X(t) + X^\top(t) L_G \dot{X}(t) \\ &= -2(L_G X(t))^\top (L_G X(t))\,.\end{aligned} \tag{28}$$

If $\dot{V}(X(t)) = 0$, we obtain $L_G X(t) = 0$ and thus $V(X) = 0$, which is the consensus situation. Otherwise, it results in $\dot{V}(X(t)) < 0$, which means that the lower-bounded function $V(X(t))$ is decreasing. It is then concluded that

$$\lim_{t\to\infty} V(X(t)) = 0\,, \tag{29}$$

which leads to the consensus (22).

Furthermore, we obtain from (21) and (25) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} X_i(t) = \sum_{i=1}^{N}\sum_{j\in N_i} A_{ij}(X_j(t) - X_i(t)) = 0, \tag{30}$$

which implies that $\sum_{i=1}^{N} X_i(t)$ is an invariant quantity with the proposed consensus algorithm. Thus, all the agents' states converge to the same vector $\frac{1}{N}\sum_{i=1}^{N} X_i(0)$, which is the average of the initial states of all agents. As in the existing graph Laplacian method, it is said that the *average-consensus* is achieved.

We summarize the above discussion in the following theorem.

Theorem 2. *The average-consensus is achieved for the agents (21) with the distributed controller (algorithm) (25).*

From the above discussion it is observed that if there is a desirable consensus state X_f for the agents, we can choose appropriate initial states so that $\frac{1}{N}\sum_{i=1}^{N} X_i(0) = X_f$. If the initial states cannot be set arbitrarily, we can consider a virtual leader which has the same dynamics as the agents and the initial state can be adjusted as necessary. In addition to that, since the smallest positive eigenvalue of L_G dominates the convergence rate of achieving consensus, we can modify the distributed controller (25) as $U(t) = -\mu L_G X(t)$ with a large scalar $\mu > 0$, whenever it is desired.

Remark 3. Although only the average consensus has been discussed in the above, other group decision values can be easily achieved by choosing N positive scalars $d_1, \dots, d_N$ and modifying the consensus algorithm (control input) (25) as

$$\begin{aligned} U_i(t) &= d_i \sum_{j\in\mathcal{N}_i} A_{ij}(X_j(t) - X_i(t)), \quad i = 1, \dots, N \\ &\Longleftrightarrow U(t) = -\Gamma_d L_G X(t), \end{aligned} \tag{31}$$

where $\Gamma_d = \operatorname{diag}\{d_1, \dots, d_N\} \otimes I_n$. With this control input, the group decision value is

$$\frac{\sum_{i=1}^{N} d_i\, X_i(0)}{\sum_{i=1}^{N} d_i}. \tag{32}$$

Thus, one can choose appropriate scalars d_i so as to obtain another desired group decision value.

4.3. Numerical example.

Consider the case where four two-dimensional agents are fully interconnected as described in Fig. 1, with the adjacency weight matrices:

$$A_{12} = \begin{bmatrix} 10 & -4 \\ -4 & 5 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}, \quad A_{34} = \begin{bmatrix} 12 & 3 \\ 3 & 6 \end{bmatrix}. \tag{33}$$

Notice that all these matrices are positive definite. Moreover, we assume that there are two partial connections described by the following two nonnegative matrices

$$A_{23} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{24} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}. \tag{34}$$

Apply the distributed controller (25) for all the agents with the initial states

$$X_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad X_2(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad X_3(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \qquad X_4(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}. \tag{35}$$

Then, the differences between the states, $\|X_1(t) - X_2(t)\|$, $\|X_2(t) - X_3(t)\|$, $\|X_3(t) - X_4(t)\|$, and the value of the Lyapunov-like function $V(X)$ are depicted in Fig. 2. Clearly, consensus has been achieved among the four agents.

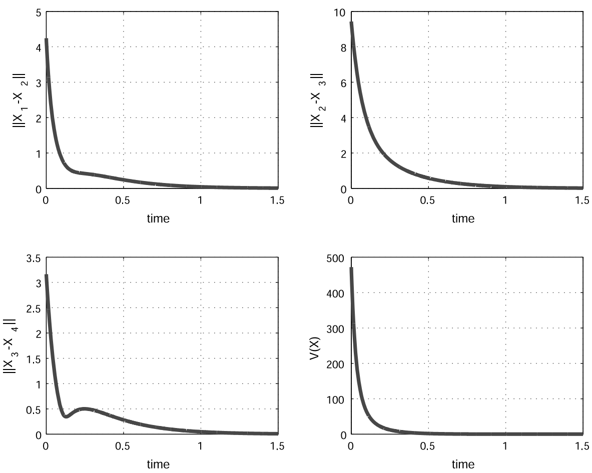

Fig. 2. Consensus achieved in the example.

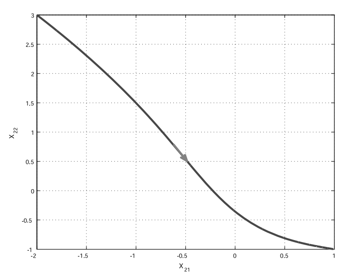

Fig. 3. Average consensus achieved (the second agent).

Furthermore, according to the discussion on the average-consensus, all the states should converge to the average of the initial states $(X_1(0) + X_2(0) + X_3(0) + X_4(0))/4 = [1 \ \ -1]^\top$. Figures 3 and 4 depict the state trajectories of the second and the fourth agent, respectively, as an example showing that $X_2(t) = [X_{21}(t) \quad X_{22}(t)]^\top$ and $X_4(t) = [X_{41}(t) \quad X_{42}(t)]^\top$ converge to the average vector.

Finally, for comparison, we show the simulation

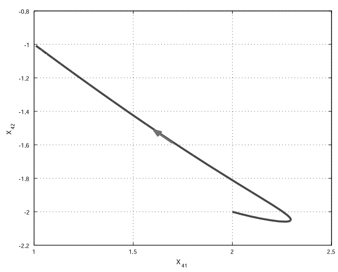

Fig. 4. Average consensus achieved (the fourth agent).

result of using the modified distributed controller

$$U(t) = -\mu L_G X(t)\,, \tag{36}$$

where $\mu = 10$ is used to obtain fast convergence. Figure 5 depicts the same information as described in Fig. 2. It is clear that although the trajectory curves are similar, the consensus has been achieved much more quickly with the controller (36).

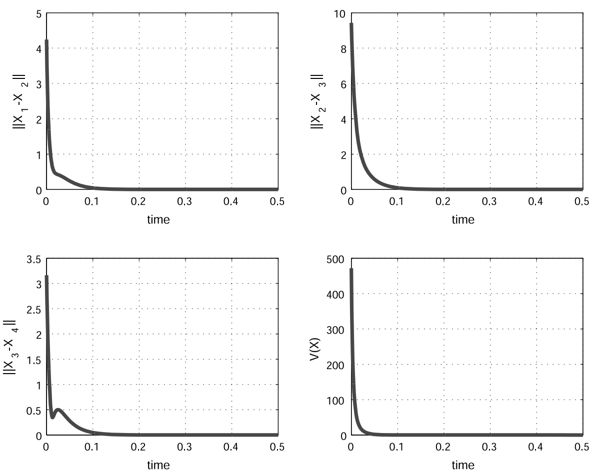

Fig. 5. Consensus achieved in the example ($\mu = 10$).

5. Concluding remarks

In this paper, we have generalized the notion of a graph Laplacian for networked agents by extending the adjacency weights from positive scalars to positive (or nonnegative) definite matrices. We have shown that the generalized graph Laplacian can describe the interconnection among agents with multi-dimensional states more practically, and it inherits the spectral properties of the graph Laplacian. Thus, most of the existing consensus algorithms can be applied in almost the same form. As an example, we have used the generalized graph Laplacian to establish a distributed consensus algorithm for agents described by multi-dimensional integrators, and have demonstrated the algorithm with a numerical example.

There are several open issues in our future research work. First, the discussion and the results in this paper are for symmetric network graphs, and the extension to non-symmetric graphs is desirable in real applications. Next, although the present generalization of the graph Laplacian is still valid in the case where time delays exist in the networked agents, we need some nontrivial modification when the interconnections among the agents are time varying. For an example, the connection between a pair of agents exists "entirely" but the connection associated with a certain state variable may be lost due to environmental changes.

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