Journal of Polish Safety and Reliability Association
Summer Safety and Reliability Seminars, Volume 4, Number 1, 2013

Soszyńska-Budny Joanna
Maritime University, Gdynia, Poland

Modelling safety of multistate systems with ageing components

Keywords

safety, risk, multistate system, ageing

Abstract

Basic notions of the ageing multistate systems safety analysis are introduced. The system components and the system multistate safety functions are defined. The mean values and variances of the multistate systems lifetimes in the safety state subsets and the mean values of their lifetimes in the particular safety states are defined. The multi-state system risk function and the moment of exceeding by the system the critical safety state are introduced. The exemplary safety structures of the multistate systems with ageing components are defined and their safety functions are determined. As a particular case, the safety functions of the considered multistate systems composed of components having exponential safety functions are determined. Applications of the proposed multistate system safety models to the evaluation and prediction of the safty characteristics of the exemplary series, parallel, "*m* out of *n*", consecutive "*m* out of *n*: F", series-parallel, parallel-series and "*m* out of *l*"-series systems are presented as well.

1. Introduction

Taking into account the importance of the safety and operating process effectiveness of real technical systems it seems reasonable to expand the two-state approach [8], [9] to multi-state approach [1]-[4], [6]-[21] in safety analysis. The assumption that the systems are composed of multi-state components with safety states degrading in time [11]-[13] gives the possibility for more precise analysis of their safety and operational processes' effectiveness. This assumption allows us to distinguish a system safety critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operation process effectiveness. Then, an important system safety characteristic is the time to the moment of exceeding the system safety critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state safety function that are basic characteristics of the multi-state system. The safety models of the considered here typical multistate system structures can be applied in the safety analysis of real complex technical systems. They may be successfully applied, for instance, to safety analysis, identification, prediction and optimization of the maritime transportation systems.

2. Safety analysis of multistate systems

In the multistate safety analysis to define the system with degrading components, we assume that:

- n is the number of the system components,
- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the safety state set $\{0,1,...,z\}$, $z \ge 1$,
- the safety states are ordered, the safety state 0 is the worst and the safety state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u,u+1,...,z\}$ while it was in the safety state z at the moment $t = 0$,
- the system states degrades with time t,
- $s_i(t)$ is a component E_i safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$,
- $s(t)$ is a system S safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$.

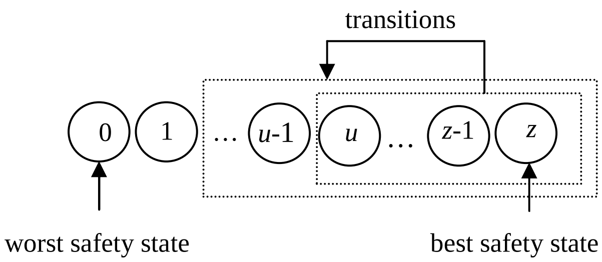

Figure 1. Illustration of a system and components safety states changing

The above assumptions mean that the safety states of the system with degrading components may be changed in time only from better to worse [11]-[13]. The way in which the components and the system safety states change is illustrated in *Figure 1.*

Definition 1. A vector

$$S_i(t,\cdot) = [S_i(t,0), S_i(t,1),...,S_i(t,z)],\ t \in <0,\infty), \qquad (1)$$

$$i = 1,2,...,n,$$

where

$$S_i(t,u) = P(s_i(t) \geq u \mid s_i(0) = z) = P(T_i(u) > t), \qquad (2)$$

$$t \in <0,\infty),\ u = 0,1,...,z,$$

is the probability that the component E_i is in the safety state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of a component E_i.

The safety functions $R_i(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (2) are called the coordinates of the component E_i, $i = 1,2,...,n$, multistate safety function $S_i(t,\cdot)$ given by (1). Thus, the relationship between the distribution function $F_i(t,u)$ of the component E_i, $i = 1,2,...,n$, lifetime $T_i(u)$ in the safety state subset $\{u, u+1,..., z\}$ and the coordinate $S_i(t,u)$ of its multistate safety function is given by

$$F_i(t,u) = P(T_i(u) \leq t) = 1 - P(T_i(u) > t) = 1 - S_i(t,u),$$

$$t \in <0,\infty),\ u = 0,1,...,z.$$

Under *Definition 1* and the agreements, we have the following property of the component multistate safety function coordinates

$$S_i(t,0) \geq S_i(t,1) \geq \ldots \geq S_i(t,z),\ t \in <0,\infty),$$

$$i = 1,2,...,n.$$

Further, if we denote by

$$p_i(t,u) = P(s_i(t) = u \mid s_i(0) = z),\ t \in <0,\infty),$$

$$u = 0,1,...,z,$$

the probability that the component E_i is in the safety state u at the moment t, while it was in the safety state z at the moment $t = 0$, then by (1)

$$S_i(t,0) = 1,\ S_i(t,z) = p_i(t,z),\ t \in <0,\infty), \qquad (3)$$

$$i = 1,2,...,n, (3)$$

and

$$p_i(t,u) = S_i(t,u) - S_i(t,u+1),\ u = 0,1,..., z-1, \qquad (4)$$

$$t \in <0,\infty),\ i = 1,2,...,n.$$

Moreover, if

$$S_i(t,u) = 1 \text{ for } t \leq 0,\ u = 1,2,...,z,\ i = 1,2,...,n,$$

then

$$\mu_i(u) = \int_0^\infty S_i(t,u)dt,\ u = 1,2,...,z,\quad i=1,2,...,n, \qquad (5)$$

is the mean lifetime of the component E_i in the safety state subset $\{u, u+1,..., z\}$,

$$\sigma_i(u) = \sqrt{n_i(u) - [\mu_i(u)]^2},\ u = 1,2,...,z, \qquad (6)$$

$$i = 1,2,...,n,$$

where

$$n_i(u) = 2\int_0^\infty t S_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \qquad (7)$$

is the standard deviation of the component E_i lifetime in the safety state subset $\{u, u+1,..., z\}$ and

$$\bar{\mu}_i(u) = \int_0^\infty p_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \qquad (8)$$

is the mean lifetime of the component E_i in the safety state u, in the case when the integrals defined by (5), (7) and (8) are convergent.

Next, according to (3), (4), (5) and (8), we have

$$\bar{\mu}_i(u) = \mu_i(u) - \mu_i(u+1),\ u = 0,1,..., z-1,$$

$$\bar{\mu}_i(z) = \mu_i(z), \ i = 1,2,...,n. \qquad (9)$$

Definition 2. A vector

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,0), \boldsymbol{S}(t,1),...,\boldsymbol{S}(t,z)], \ t \in <0,\infty), \qquad (10)$$

where

$$\boldsymbol{S}(t,u) = P(S(t) \geq u \mid S(0) = z) = P(T(u) > t), \qquad (11)$$

$$t \in <0,\infty), \ u = 0,1,...,z,$$

is the probability that the system is in the safety state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of this system.
The safety functions $\boldsymbol{S}(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (11) are called the coordinates of the system multistate safety function $\boldsymbol{S}(t,\cdot)$ given by (10). Consequently, the relationship between the distribution function $\boldsymbol{F}(t,u)$ of the system S lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ and the coordinate $\boldsymbol{S}(t,u)$ of its multistate safety function is given by

$$\boldsymbol{F}(t,u) = P(T(u) \leq t) = 1 - P(T(u) > t) = 1 - \boldsymbol{S}(t,u),$$

$$t \in <0,\infty), \ u = 0,1,...,z.$$

The exemplary graph of a four-state (z = 3) system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)], \ t \in <0,\infty),$$

is shown in *Figure 2*.

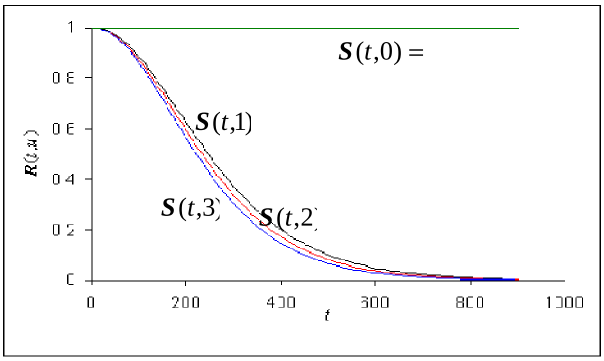

Figure 2. The graph of a four-state system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

Under *Definition 2*, we have

$$\boldsymbol{S}(t,0) \geq \boldsymbol{S}(t,1) \geq \ldots \geq \boldsymbol{S}(t,z), \ t \in <0,\infty),$$

and if

$$p(t,u) = P(S(t) = u \mid S(0) = z), \ t \in <0,\infty), \qquad (12)$$

$$u = 0,1,...,z,$$

is the probability that the system is in the safety state u at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment t = 0, then

$$\boldsymbol{S}(t,0) = 1, \ \boldsymbol{S}(t,z) = p(t,z), \ t \in <0,\infty), \qquad (13)$$

and

$$p(t,u) = \boldsymbol{S}(t,u) - \boldsymbol{S}(t, u+1), \ u = 0,1,..., z-1, \qquad (14)$$

$$t \in <0,\infty).$$

Moreover, if

$$\boldsymbol{S}(t,u) = 1 \text{ for } t \leq 0, \ u = 1,2,...,z,$$

then

$$\mu(u) = \int_0^\infty \boldsymbol{S}(t,u)dt, \ u = 1,2,...,z, \qquad (15)$$

is the mean lifetime of the system in the safety state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{n(u) - [\mu(u)]^2}, \ u = 1,2,...,z, \qquad (16)$$

where

$$n(u) = 2\int_0^\infty t\, \boldsymbol{S}(t,u)dt, \ u = 1,2,...,z, \qquad (17)$$

is the standard deviation of the system lifetime in the safety state subset $\{u, u+1,..., z\}$ and moreover

$$\bar{\mu}(u) = \int_0^\infty p(t,u)dt, \ u = 1,2,...,z, \qquad (18)$$

is the mean lifetime of the system in the safety state u while the integrals (15), (17) and (18) are convergent.
Additionally, according to (13), (14), (15) and (18), we get the following relationship

$$\bar{\mu}(u) = \mu(u) - \mu(u+1), \ u = 0,1,..., z-1,$$

$$\overline{\mu}(z) = \mu(z). \qquad (19)$$

Definition 3. A probability

$$\boldsymbol{r}(t) = P(S(t) < r \mid S(0) = z) = P(T(r) \le t),\ t \in \langle 0, \infty),$$

that the system is in the subset of safety states worse than the critical safety state r, $r \in \{1,...,z\}$ while it was in the safety state z at the moment $t = 0$ is called a risk function of the multi-state system [9], [12].
Under this definition, from (11), we have

$$\boldsymbol{r}(t) = 1 - P(S(t) \ge r \mid S(0) = z) = 1 - \boldsymbol{S}(t,r), \qquad (20)$$

$t \in \langle 0, \infty)$,

and if τ is the moment when the system risk exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \qquad (21)$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the system risk function $\boldsymbol{r}(t)$.
The exemplary graph of a four-state system risk function for the critical safety state $r = 2$

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2),\ t \in \langle 0, \infty),$$

corresponding to the safety function illustrated in *Figure 2* is shown in *Figure 3.*

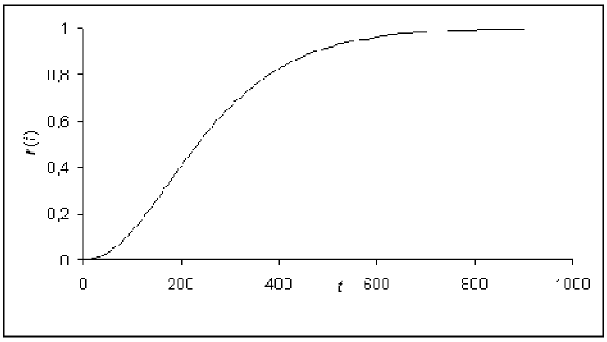

Figure 3. The graph of a four-state system risk function $\boldsymbol{r}(t)$

3. Safety structures of multistate systems

Now, after introducing the notion of the multistate safety analysis, we may define basic multi-state safety structures.

Definition 4. A multistate system is called series if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \le i \le n}\{T_i(u)\},\ u = 1,2,...,z.$$

The number n is called the system structure shape parameter.
The above definition means that a multi-state series system is in the safety state subset $\{u, u+1,..., z\}$ if and only if all its n components are in this subset of safety states. That meaning is very close to the definition of a two-state series system considered in a classical reliability [8], [9], [12] analysis that is not failed if all its components are not failed. This fact can justify the safety structure scheme for a multistate series system presented in *Figure 4.*

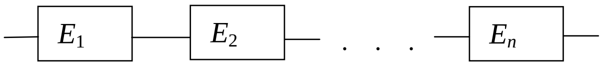

Figure 4. The scheme of a series system safety structure

It is easy to work out that the safety function of the multi-state series system is given by the vector [9], [12]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),..., \boldsymbol{S}(t,z)] \qquad (22)$$

with the coordinates

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{n} S_i(t,u),\ t \in \langle 0, \infty),\ u = 1,2,...,z. \qquad (23)$$

Example 1. We consider an exemplary series system composed of components E_i, $i = 1,2,3,4$, with the safety structure presented in *Figure 5.*

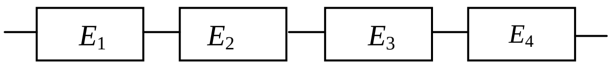

Figure 5. The scheme of the exemplary series system safety structure

We arbitrarily distinguish four safety states of the system components 0, 1, 2, 3, i.e. $z = 3$. We fix that the critical safety state is $r = 2$ and we define the four-state conditional safety functions of the system components E_i, $i = 1,2,3,4$, in the form of the vector

$$S_i(t,\cdot) = [1,\ S_i(t,1), S_i(t,2), S_i(t,3)],\ \ i = 1,2,3,4,$$

with the exponential coordinates
$S_i(t,u) = \exp[-2ut]$, $u = 1,2,3$, $i = 1,2,3,4$.

After direct application the formulae (22)-(23), we get the system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)], \; t \geq 0, \qquad (24)$$

where

$$\boldsymbol{S}(t,u) = \exp[-\sum_{i=1}^{4} \lambda_i(u)t] = \exp[-4 \cdot 2ut]$$

$$= \exp[-8ut], \; u = 1,2,3,$$

and particularly

$$\boldsymbol{S}(t,1) = \exp[-8t], \qquad (25)$$

$$\boldsymbol{S}(t,2) = \exp[-16t], \qquad (26)$$

$$\boldsymbol{S}(t,3) = \exp[-24t]. \qquad (27)$$

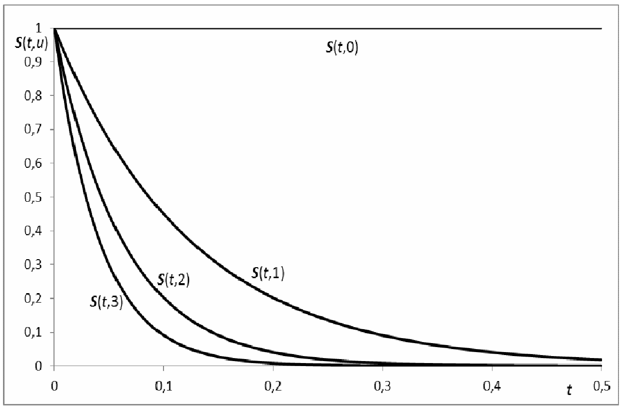

Figure 6. The graph of the exemplary series system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

The expected values and standard deviations of the system unconditional lifetimes in the safety state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, calculated from the results given by (25)-(27), according to (15)-(17), respectively are:

$$\mu(1) = \frac{1}{8} = 0.125, \; \sigma(1) = 0.125, \qquad (28)$$

$$\mu(2) = \frac{1}{16} = 0.0625, \; \sigma(2) \cong 0.0625, \qquad (29)$$

$$\mu(3) = \frac{1}{24} \cong 0.0417, \; \sigma(3) \cong 0.0417. \qquad (30)$$

Consequently, considering (19) and (28)-(30), the mean values of the system lifetimes in the particular safety states 1, 2, 3, respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) = 0.0625,$$

$$\bar{\mu}(2) = \mu(2) - \mu(3) \cong 0.0208,$$

$$\bar{\mu}(3) = \mu(3) \cong 0.0417. \qquad (31)$$

Since the critical safety state is $r = 2$, then the system risk function, according to (20), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2) = 1 - \exp[-16t] \text{ for } t \geq 0. \qquad (32)$$

Hence, by (21), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 0.04. \qquad (33)$$

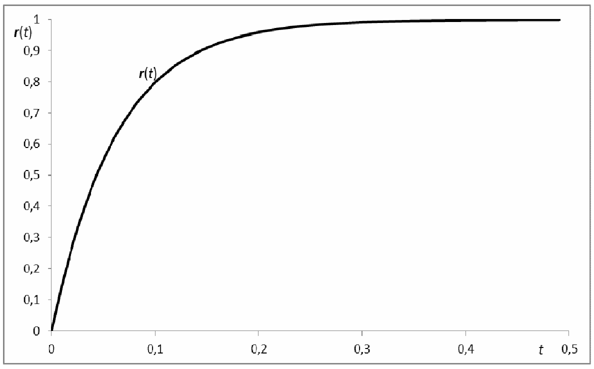

Figure 7. The graph of the exemplary series system risk function $\boldsymbol{r}(t)$

Definition 5. A multistate system is called parallel if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \max_{1 \leq i \leq n}\{T_i(u)\}, \; u = 1,2,...,z.$$

The number n is called the system structure shape parameter.

The above definition means that the multistate parallel system is in the safety state subset $\{u, u+1,..., z\}$ if and only if at least one of its n components is in this subset of safety states. That meaning is very close to the definition of a two-state parallel system in a classical reliability analysis that is not failed if at least one of its components is not failed what can justify the safety structure scheme for a multistate parallel system presented in *Figure 8.*

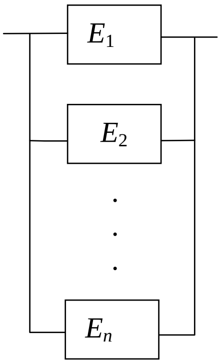

Figure 8. The scheme of a parallel system safety structure

The safety function of the multi-state parallel system is given by the vector [9], [12]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)], \quad (34)$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{n} F_i(t,u), \; t \in <0,\infty), \; u = 1,2,...,z. \quad (35)$$

Example 2. We consider an exemplary parallel system composed of components E_i, $i = 1,2,3,4$, with the safety structure presented in *Figure 9*.

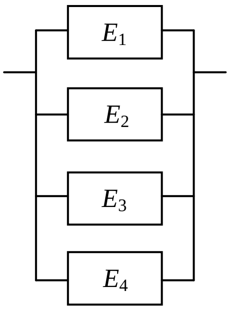

Figure 9. The scheme of the exemplary parallel system safety structure

We arbitrarily distinguish three safety states of the system components 0, 1, 2, i.e. $z = 2$, and we fix that the critical safety state is $r = 1$. We define the three-state safety functions of the system components E_i, $i = 1,2$, in the form of the vector

$$S_i(t,\cdot) = [1, \; S_i(t,1), S_i(t,2)]], \quad i = 1,2,3,4,$$

with the exponential coordinates

$$S_i(t,u) = \exp[-ut], \quad u = 1,2, \; i = 1,2,3,4.$$

After application of the formulae (34)-(35), we get the system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \; \boldsymbol{S}(t,1), \; \boldsymbol{S}(t,2)], \quad t \ge 0, \quad (36)$$

where

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{4}[1-\exp[-\lambda_i(t,u)t]]$$

$$= 1-[1-\exp[-ut]]^4$$

$$= 4\exp[-ut] - 6\exp[-2ut]$$

$$+ 4\exp[-3ut] - \exp[-4ut], \quad u = 1,2,$$

and particularly

$$\boldsymbol{S}(t,1) = 4\exp[-t] - 6\exp[-2t]$$

$$+ 4\exp[-3t] - \exp[-4t], \quad (37)$$

$$\boldsymbol{S}(t,2) = 4\exp[-2t] - 6\exp[-4t]$$

$$+ 4\exp[-6t] - \exp[-8t]. \quad (38)$$

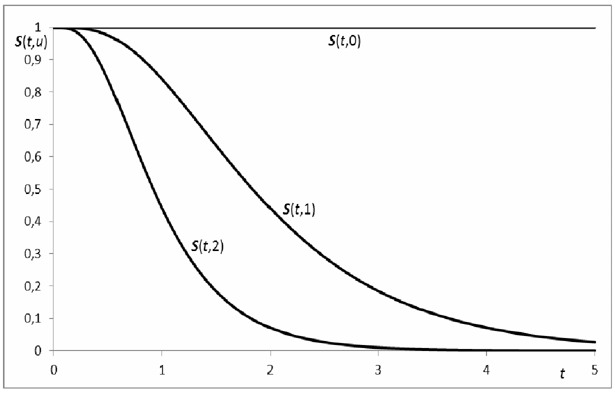

Figure 10. The graph of the exemplary parallel system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

The expected values and standard deviations of the system unconditional lifetimes in the safety state subsets $\{1,2\}$, $\{2\}$, calculated from the results given by (37)-(38), according to (15)-(17), respectively are:

$$\mu(1) = 4\frac{1}{1} - 6\frac{1}{2} + 4\frac{1}{3} - 1\frac{1}{4} \cong 2.083, \quad (39)$$

$$\sigma(1) \cong 1.194,$$

$$\mu(2) = 4\frac{1}{2} - 6\frac{1}{4} + 4\frac{1}{6} - 1\frac{1}{8} \cong 1.042, \quad (40)$$

$$\sigma(2) \cong 0.596,$$

and further, considering (19) and (39)-(40), the mean values of the lifetimes in the particular safety states 1, 2, respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) \cong 1.041,$$

$$\bar{\mu}(2) = \mu(2) \cong 1.041. \qquad (41)$$

Since the critical safety state is r =1, then the system risk function, according to (20), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,1)$$

$$= 1 - 4\exp[-t] + 6\exp[-2t]$$

$$- 4\exp[-3t] + \exp[-4t], \text{ for } t \geq 0. \qquad (42)$$

Hence, by (21), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 1.84. \qquad (42)$$

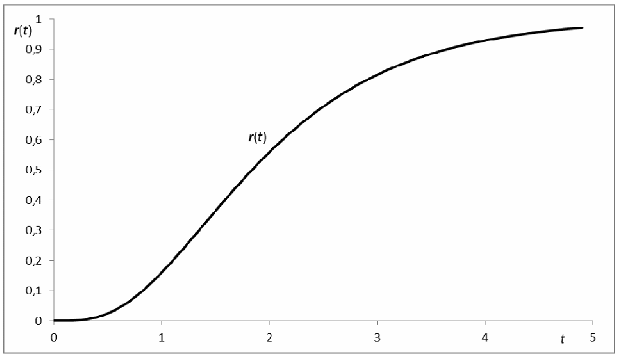

Figure 11. The graph of the exemplary parallel system risk function $\boldsymbol{r}(t)$

Definition 6. A multistate system is called an "*m* out of *n*" system if its lifetime $T(u)$ in the safety state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = T_{(n-m+1)}(u), \; m = 1,2,...,n, \; u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m*-th maximal order statistic in the sequence of the component lifetimes
$T_1(u), T_2(u), ..., T_n(u), \; u = 1,2,...,z.$

The above definition means that the multistate „*m* out of *n*" system is in the safety state subset $\{u, u+1, ..., z\}$ if and only if at least *m* out of its *n* components are in this safety state subset and it is a multistate parallel system if m = 1 and it is a multistate series system if $m = n$. The numbers *m* and *n* are called the system structure shape parameters. The scheme of an "*m* out of *n*" multistate system safety structure, justified in an analogous way as in the case of a multistate series system and a multistate parallel system, is given in *Figure 12*, where $i_1, i_2, \ldots, i_n \in \{1,2,...,n\}$ and $i_a \neq i_b$ for $a \neq b$.

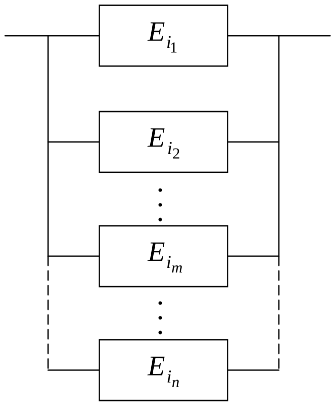

Figure 12. The scheme of an "*m* out of *n*" system safety structure

It can be simply shown that the safety function of the multistate "*m* out of *n*" system is given either by the vector [9], [12]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)], \qquad (43)$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \leq m-1}}^{1} [S_i(t,u)]^{r_i} [F_i(t,u)]^{1-r_i}, \qquad (44)$$

$$t \in \langle 0, \infty), \quad u = 1,2,...,z,$$

or by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)], \qquad (45)$$

with the coordinates

$$\boldsymbol{S}(t,u) = \sum_{\substack{r_1, r_2, ..., r_n = 0 \\ r_1 + r_2 + ... + r_n \leq \bar{m}}}^{1} [F_i(t,u)]^{r_i} [S_i(t,u)]^{1-r_i}, \qquad (46)$$

$$t \in \langle 0, \infty), \quad \bar{m} = n - m, \; u = 1,2,...,z.$$

Example 3. We consider an exemplary "*m* out of *n*" system composed of $n = 5$ identical components $E_1, E_2, ..., E_5$. We arbitrarily assume that $z = 3$, i.e. the system and its components may be in the one of the safety states from the safety state set {0,1,2,3}. Moreover, we assume that the components E_i,

$i = 1,2,...,5,$ in the safety state subsets have the exponential safety functions given by the vector

$$S(t,\cdot) = [1, S(t,1), S(t,2), S(t,3)], \quad (47)$$

$$t \in <0,\infty),$$

with the coordinates

$$S(t,1) = \exp[-0.01t],$$

$$S(t,2) = \exp[-0.03t],$$

$$S(t,3) = \exp[-0.05t], \text{ for } t \geq 0. \quad (48)$$

The system is out of the safety state subset $\{u, u+1,...,3\}$, $u = 1,2,3,$ if at least $m = 2$ of its components are out of this safety state subset. Supposing that the considered system critical safety state is $r = 2$, we conclude that the system is out of the safety state subset $\{2,3\}$, if at least 2 of its components from 5 components are out of the safety state subset $\{2,3\}$. Thus, the considered system is the four-state "2 out of 5" system, and according to formulae (45)-(46), we get the following expression for the system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)], \quad (49)$$

where

$$\boldsymbol{S}(t,u) = 10[S(t,u)]^2[F(t,u)]^3$$
$$+10[S(t,u)]^3[F(t,u)]^2 + 5[S(t,u)]^4 F(t,u)$$
$$+[S(t,u)]^5 \text{ for } t \in <0,\infty), \ u = 1,2,3. \quad (50)$$

In the particular case, when the component E_i, $i = 1,2,...,5,$ in the safety state subsets have the exponential safety functions given by (47)-(48), considering (49)-(50), we get the following formulae for the system safety function coordinates:

$$\boldsymbol{S}(t,1) = 10\exp[-0.02t][1-\exp[-0.01t]]^3$$
$$+10\exp[-0.03t][1-\exp[-0.01t]]^2$$
$$+5\exp[-0.04t][1-\exp[-0.01t]]$$
$$+\exp[-0.05t], \quad (51)$$

$$\boldsymbol{S}(t,2) = 10\exp[-0.06t][1-\exp[-0.03t]]^3$$
$$+10\exp[-0.09t][1-\exp[-0.03t]]^2$$
$$+5\exp[-0.12t][1-\exp[-0.03t]]$$
$$+\exp[-0.15t], \quad (52)$$

$$\boldsymbol{S}(t,3) = 10\exp[-0.10t][1-\exp[-0.05t]]^3$$
$$+10\exp[-0.15t][1-\exp[-0.05t]]^2$$
$$+5\exp[-0.20t][1-\exp[-0.05t]]$$
$$+\exp[-0.25t], \text{ for } t \in <0,\infty). \quad (53)$$

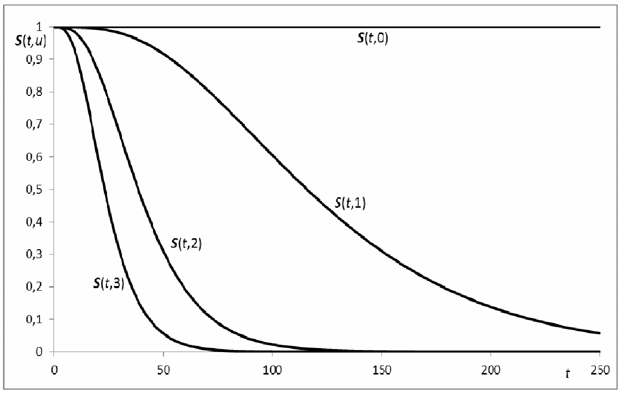

Figure 13. The graphs of the exemplary four-state "2 out of 5" system safety function coordinates

The expected values and standard deviations of the system lifetimes in the safety state subsets $\{1,2,3\}$, $\{2,3\}$ and $\{3\}$, calculated from the results given by (51)-(53), according to (15)-(17), respectively are:

$$\mu(1) \cong 128.330, \ \sigma(1) \cong 68.095, \quad (54)$$

$$\mu(2) \cong 42.780, \ \sigma(2) \cong 22.692, \quad (55)$$

$$\mu(3) \cong 25.670, \ \sigma(3) \cong 13.611, \quad (56)$$

and further, considering (19) and (54)-(56), the mean values of the system lifetimes in the particular safety states 1, 2, 3 respectively are:

$$\overline{\mu}(1) = \mu(1) - \mu(2) \cong 85.550,$$

$$\overline{\mu}(2) = \mu(2) - \mu(3) \cong 17.110,$$

$$\overline{\mu}(3) = \mu(3) \cong 25.670, \quad (57)$$

Since the critical safety state is $r = 2$, then the system risk function, according to (20) and (52), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2)$$

$$= 1 - 10\exp[-0.06t][1-\exp[-0.03t]]^3$$

$$- 10\exp[-0.09t][1-\exp[-0.03t]]^2$$

$$- 5\exp[-0.12t][1-\exp[-0.03t]]$$

$$- \exp[-0.15t], \text{ for } t \in <0,\infty). \quad (58)$$

Hence, by (21), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 38.63. \quad (59)$$

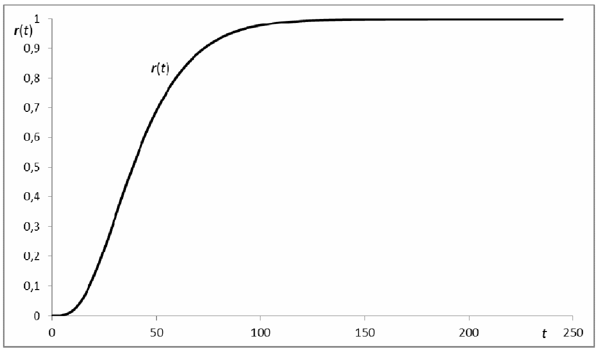

Figure. 14. The graph of the exemplary four-state "2 out of 5" system risk function

Definition 7. A multistate system is called a consecutive "*m* out of *n*: F" system if it is out of the safety state subset $\{u, u+1,..., z\}$ if and only if at least its *m* neighbouring components out of *n* its components arranged in a sequence of E_1, E_2, ..., E_n, are out of this safety state subset. The numbers *m* and *n* are called the system structure shape parameters.

After denoting by

$$\boldsymbol{S}(t,u) = P(S(t) \geq u \mid S(0) = z) = P(T(u) > t), \quad (60)$$

$t \in <0,\infty)$, $u = 0,1,...,z$,

the probability that the consecutive "*m* out of *n*: F" system is in the safety state subset $\{u, u+1,..., z\}$ at the moment *t*, $t \in <0,\infty)$, while it was in the safety state *z* at the moment *t* = 0 and by

$$\boldsymbol{S}(t,u) = P(T(u) \leq t), \; t \in <0,\infty), \; u = 0,1,...,z, \quad (61)$$

the distribution function of the lifetime $T(u)$ of this system in the safety state subset $\{u,u+1,...,z\}$, while it was in the safety state *z* at the moment *t* = 0, we conclude that the safety function of the consecutive "*m* out of *n*: F" system is the given by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \; \boldsymbol{S}(t,1), \; \boldsymbol{S}(t,2), \; ..., \; \boldsymbol{S}(t,z)\,], \quad (62)$$

with the coordinates given by the following recurrent formula [7], [14], [21]

$$\boldsymbol{S}(t,u) = \boldsymbol{S}_n(t,u) = \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n} F_i(t,u) & \text{for } n = m, \\ S_n(t,u)\boldsymbol{S}_{n-1}(t,u) + \sum_{i=1}^{m-1} S_{n-i}(t,u)\boldsymbol{S}_{n-i-1}(t,u) \cdot \prod_{j=n-i+1}^{n} F_j(t,u) & \text{for } n > m, \end{cases}$$

for $t \geq 0$, $u = 1,2,...,z$. (63)

Example 4. We consider the safety of the steel cover composed of $n = 24$ arranged identical sheets $E_1, E_2,..., E_{24}$. We arbitrarily assume that $z = 4$, i.e. the steel cover and the sheets it is composed of may be in the one of the safety states from the safety state set $\{0,1,2,3,4\}$. Moreover, we assume that the sheets E_i, $i = 1,2,...,24$, in the safety state subsets have the exponential safety functions given by the vector

$$S(t,\cdot) = [1, S(t,1), S(t,2), S(t,3), S(t,4)], \quad (64)$$

$t \in <0,\infty)$,

with the coordinates

$$S(t,1) = \exp[-0.01t], \; S(t,2) = \exp[-0.02t],$$

$$S(t,3) = \exp[-0.05t], \; S(t,4) = \exp[-0.10t] \quad (65)$$

for $t \geq 0$.

The cover is out of the safety state subset $\{u, u+1,...,4\}$, $u = 1,2,3,4$, if at least $m = 2$ of its neighbouring sheets is out of this safety state subset. Supposing that the considered steel cover critical safety state is $r = 2$, we conclude that the steel cover is failed, i.e. it is out of the safety state subset $\{2,3,4\}$, if at least 2 of its neighbouring sheets from 24 sheets are out of the safety state subset $\{2,3,4\}$.

Thus, the considered steel cover is the five-state consecutive "2 out of 24: F" system, and according to formulae (62)-(63), we get the following expression for the steel cover safety function ,

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3), \boldsymbol{S}(t,4)], \tag{66}$$

where

$$\boldsymbol{S}(t,u) = \boldsymbol{S}_{24}(t,u)$$
$$= S(t,u)\ \boldsymbol{S}_{23}(t,u)$$
$$+ S(t,u)F(t,u)\ \boldsymbol{S}_{22}(t,u) \tag{67}$$

for $t \in <0,\infty)$, $u = 1,2,3,4$,

and

$$\boldsymbol{S}(t,u) = \boldsymbol{S}_n(t,u) = \begin{cases} 1 & \text{for } n < 2, \\ 1 - \prod_{i=1}^{n} F_i(t,u) & \text{for } n = 2, \\ S(t,u)\boldsymbol{S}_{n-1}(t,u) & \\ + \sum_{i=1}^{1} S(t,u)\boldsymbol{S}_{n-i-1}(t,u) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t,u) & \text{for } n > 2, \end{cases} \tag{68}$$

for $t \geq 0$, $u = 1,2,...,z$, $n = 2,3,...,24$.

In the particular case, when the sheets E_i, $i = 1,2,...,24$, in the safety state subsets have the exponential safety functions given by (64)-(65), considering (67)-(68), we get the following recurrent formulae for the cover safety function coordinates:

- $\boldsymbol{S}_{24}(t,1)$ is determined by the formulae

$$\boldsymbol{S}_2(t,1) = 1 - [1 - \exp[-0.01t]]^2 \text{ for } t \in <0,\infty),$$

$$\boldsymbol{S}_n(t,1) = \exp[-0.01t]\ \boldsymbol{S}_{n-1}(t,1)$$
$$+ \exp[-0.01t]\ [1 - \exp[-0.01t]]\ \boldsymbol{S}_{n-2}(t,1)$$

for $t \in <0,\infty)$, $n = 3,4,...,24$, (69)

- $\boldsymbol{S}_{24}(t,2)$ is determined by the formulae

$$\boldsymbol{S}_2(t,2) = 1 - [1 - \exp[-0.02t]]^2 \text{ for } t \in <0,\infty),$$

$$\boldsymbol{S}_n(t,2) = \exp[-0.02t]\ \boldsymbol{S}_{n-1}(t,2)$$
$$+ \exp[-0.02t]\ [1 - \exp[-0.02t]]\ \boldsymbol{S}_{n-2}(t,2)$$

for $t \in <0,\infty)$, $n = 3,4,...,24$, (70)

- $\boldsymbol{S}_{24}(t,3)$ is determined by the formulae

$$\boldsymbol{S}_2(t,3) = 1 - [1 - \exp[-0.05t]]^2 \text{ for } t \in <0,\infty),$$

$$\boldsymbol{S}_n(t,3) = \exp[-0.05t]\ \boldsymbol{S}_{n-1}(t,3)$$
$$+ \exp[-0.05t]\ [1 - \exp[-0.05t]]\ \boldsymbol{S}_{n-2}(t,3)$$

for $t \in <0,\infty)$, $n = 3,4,...,24$, (71)

- $\boldsymbol{S}_{24}(t,4)$ is determined by the formulae

$$\boldsymbol{S}_2(t,4) = 1 - [1 - \exp[-0.10t]]^2 \text{ for } t \in <0,\infty),$$

$$\boldsymbol{S}_n(t,4) = \exp[-0.10t]\ \boldsymbol{S}_{n-1}(t,4)$$
$$+ \exp[-0.10t]\ [1 - \exp[-0.10t]]\ \boldsymbol{S}_{n-2}(t,4)$$

for $t \in <0,\infty)$, $n = 3,4,...,24$. (72)

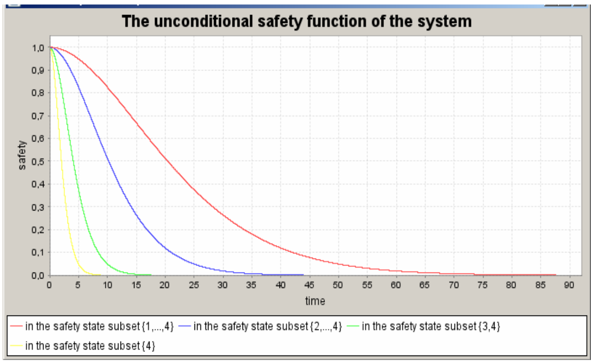

Figure 15. The graphs of the steel cover safety function coordinates

The expected values and standard deviations of the system unconditional lifetimes in the safety state subsets $\{1,2,3,4\}$, $\{2,3,4\}$, $\{3,4\}$ and $\{4\}$, calculated from the results given by (69)-(72), according to (15)-(17), and using the computer programme respectively are:

$$\mu(1) = 22.969,\ \sigma(1) \cong 14.086, \tag{73}$$

$$\mu(2) = 11.485,\ \sigma(2) \cong 7.043, \tag{74}$$

$$\mu(3) = 4.594, \ \sigma(3) \cong 2.817, \qquad (75)$$

$$\mu(4) = 2.297, \ \sigma(4) \cong 1.409, \qquad (76)$$

and further, considering (19) and (73)-(76), the mean values of the unconditional lifetimes in the particular safety states 1, 2, 3, 4 respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) \cong 11.484,$$

$$\bar{\mu}(2) = \mu(2) - \mu(3) \cong 6.891,$$

$$\bar{\mu}(3) = \mu(3) - \mu(4) \cong 2.297,$$

$$\bar{\mu}(4) = \mu(4) \cong 2.297. \qquad (77)$$

Since the critical safety state is $r = 2$, then the system risk function, according to (20) and (70), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}_{24}(t,2) = 1 - \exp[-0.02t]\ \boldsymbol{S}_{23}(t,2)$$

$$- \exp[-0.02t]\ [1 - \exp[-0.02t]]\ \boldsymbol{Cs}_{22}(t,2) \qquad (78)$$

for $t \in <0, \infty)$.

Hence, by (21), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2.5. \qquad (79)$$

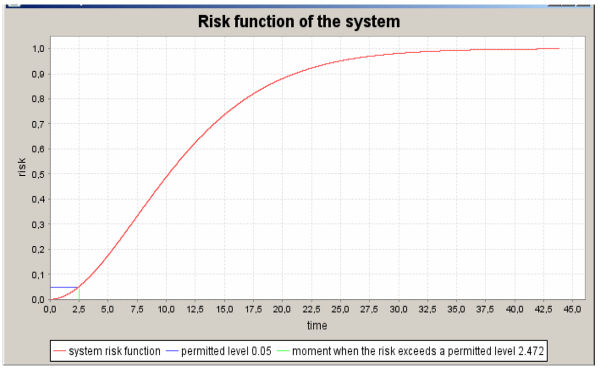

Figure 16. The graph of the steel cover risk function

Other basic multistate safety structures with components degrading in time series-parallel, parallel-series, series-"m out of k", "m_i out of l_i"-series, series-consecutive "m out of k: F" and consecutive "m_i out of l_i: F"-series systems.

To define them, we assume that:

- k is the number of the system subsystems,
- l_i, $i = 1,2,...,k$, are the numbers of the subsystem components,
- E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, k, l_1, l_2,..., l_k $\in N$, are components of a system,
- all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$,
- $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l_i$, k, l_1, l_2,..., l_k $\in N$, are independent random variables representing the lifetimes of components E_{ij} in the safety state subset $\{u, u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$,
- $E_{ij}(t)$ is a component E_{ij} safety state at the moment t, $t \in <0, \infty)$, while they were in the safety state z at the moment $t = 0$,

and proceed in n analogous way as before in defining

4. Conclusion

The proposed in this paper models for safety evaluation and prediction of the considered systems are the basis for the considerations in of the book [18]. These system safety models, together with the models of the system operation process presented in will be used in [18] 3 for constructing the integrated joint general safety models of complex technical systems related to their operation processes. The models applied here, in their particular cases, for the safety analysis and prediction of the exemplary technical systems operating in constant operation conditions will also be applied in [18] to safety analysis and prediction of these systems operating at the variable operation conditions.

References

[1] Aven, T. (1985). Reliability evaluation of multistate systems with multistate components. *IEEE Transactions on Reliability* 34: 473-479.

[2] Aven, T. & Jensen, U. (1999). *Stochastic Models in Reliability*. Springer-Verlag, New York.

[3] Barlow, R.E. & Wu, A.S. (1978). Coherent systems with multi-state components. *Mathematics of Operations Research* 4: 275-281.

[4] Brunelle, R.D. & Kapur, K.C. (1999). Review and classification of reliability measures for multistate and continuum models. *IEEE Transactions on Reliability* 31: 1117-1180.

[5] Huang, J., Zuo, M.J. & Wu, Y. (2000). Generalized multi-state k-out-of-n:G systems. *IEEE Transactions on Reliability* 49: 105-111.

[6] Hudson, J.C. & Kapur, K.C. (1983). Reliability analysis of multistate systems with multistate components. IIE Transactions 15: 127-135.

[7] Hudson, J. & Kapur, K. (1985). Reliability bounds for multistate systems with multistate components. *Operations Research* 33: 735-744.

[8] Kołowrocki, K. (2004). *Reliability of Large Systems*. Elsevier, Amsterdam - Boston -

Heidelberg - London - New York - Oxford - Paris - San Diego - San Francisco - Singapore - Sydney – Tokyo.

[9] Kołowrocki, K. (2008). *Reliability of large systems*. In Encyclopedia of Quantitative Risk Analysis and Assessment, Vol. 4, John Wiley & Sons.

[10] Kołowrocki, K. & Soszyńska, J. (2010). Safety and risk evaluation of a ferry technical system. Summer Safety & Reliability Seminars- SSARS 2010. *Journal of Polish Safety and Reliability Association*, Vol. 4, 159-172.

[11] Kolowrocki, K. &. Soszynska, J. (2011). On safety analysis of complex technical maritime transportation system. *Journal of Risk and Reliability*, 225 (3), 345-354.

[12] Kołowrocki, K. & Soszyńska-Budny, J. (2011). *Reliability and safety of complex technical systems and processes: Modeling – Identification – Prediction – Optimization.* Springer, ISBN 978-0-85729-693-1.

[13] Kołowrocki, K. & Soszyńska-Budny, J. (2012). Preliminary approach to safety analysis of critical infrastructures. Summer Safety and Reliability Seminars – SSARS 2012. *Journal of Polish Safety and Reliability Association*, Vol. 3, No 1, 73-88.

[14] Kołowrocki, K., Soszyńska, J., Judziński, M. & Dziula, P. (2007). On multi-state safety analysis in shipping. *International Journal of Reliability, Quality and Safety Engineering* 14(6): 547-567.

[15] Kossow, A. & Preuss, W. (1995). Reliability of linear consecutively-connected systems with multistate components. *IEEE Transactions on Reliability* 44: 518-522.

[16] Lisnianski, A. & Levitin, G. (2003). *Multi-State System Reliability. Assessment, Optimisation and Applications.* World Scientific Publishing Co. Pte. Ltd., New Jersey, London, Singapore, Hong Kong.

[17] Meng, F. (1993). Component-relevancy and characterisation in multi-state systems. *IEEE Transactions on Reliability* 42: 478-483.

[18] Soszyńska-Budny, J. (2014). Safety of complex technical systems, (in preparation).

[19] Xue, J. (1985). On multi-state system analysis. *IEEE Transactions on Reliability* 34: 329-337.

[20] Xue, J. & Yang, K. (1995). Dynamic reliability analysis of coherent multi-state systems. *IEEE Transactions on Reliability* 44: 683-688.

[21] Xue, J. & Yang, K. (1995). Symmetric relations in multi-state systems. *IEEE Transactions on Reliability* 44: 689-693.