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$\Psi_{\mathcal{I}}$-DENSITY TOPOLOGY

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Abstract. The purpose of this paper is to study the notion of a $\Psi_{\mathcal{I}}$-density point and $\Psi_{\mathcal{I}}$-density topology, generated by it analogously to the classical $\mathcal{I}$-density topology on the real line. The idea arises from the note by Taylor [3] and Terepeta and Wagner-Bojakowska [2].

We introduce the following notation:

$\mathbb{N}$ the set of positive integers,
$\mathbb{R}$ the set of real numbers,
$\mathbb{R}_+$ the set of positive real numbers,
$\mathcal{S}$ σ-algebra of subsets of $\mathbb{R}$ having the Baire property,
$\mathcal{I}$ σ-ideal of subsets of $\mathbb{R}$ of the first category,
$\mathcal{C}$ the family of all nondecreasing continuous functions $\psi : \mathbb{R}_+ \to (0,1]$ such that $\lim_{x\to 0^+} \psi(x) = 0$.

We say that two sets A and B are equivalent ($A \sim B$) if $A \triangle B \in \mathcal{I}$, where $A \triangle B$ is the symmetric difference of A and B. Additionally, if $A \subset \mathbb{R}$, $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, then $-A = \{x \in \mathbb{R} : \ -x \in A\}$, $\alpha \cdot A = \{\alpha \cdot x \in \mathbb{R} : \ x \in A\}$, $A' = \mathbb{R} \setminus A$ and $A - x_0 = \{x \in \mathbb{R} : \ x + x_0 \in A\}$. For each $x \in \mathbb{R}^+$, let $[x] = \max\{n \in \mathbb{N} \cup \{0\} : \quad n \le x \}$.

Definition 1. [1] *We say that 0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if for each increasing sequence of positive integers $\{n_m\}_{m\in\mathbb{N}}$ there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that*

$$\{x: \ \chi_{n_{m_p}\cdot A\cap[-1,1]}(x) \not\longrightarrow 1\} \in \mathcal{I}.$$

A point x_0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if 0 is a point of $\mathcal{I}$-density of the set $A - x_0$. A point x_0 is a point of $\mathcal{I}$-dispersion of a set $A \in \mathcal{I}$ if x_0 is a point of $\mathcal{I}$-density of the set $\mathbb{R} \setminus A$.

Let

$$\Phi(A) = \{x \in \Re: \ \ x \text{ is } \mathcal{I}\text{-density point of } A\}$$

for $A \in \mathcal{S}$, and $\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{S}: \ A \subset \Phi(A)\}$. We recall the following theorems.

Theorem 1. [1] *0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if and only if for each sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = +\infty$ there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ such that*

$$\{x \in [-1,1]: \ \chi_{t_{n_k}\cdot A\cap[-1,1]}(x) \not\longrightarrow 1\} \in \mathcal{I}.$$

Theorem 2. [1] *For any $A \in \mathcal{S}$ and $B \in \mathcal{S}$,*

i) *if $A \subset B$, then $\Phi(A) \subset \Phi(B)$,*

ii) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,

iii) *if $A \sim B$, then $\Phi(A) = \Phi(B)$,*

iv) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

v) $A \sim \Phi(A)$.

Theorem 3. [1] *$\mathcal{T}_{\mathcal{I}}$ is a topology on the real line stronger than the Euclidean topology.*

Definition 2. *Let $\psi \in \mathcal{C}$.*

I. *We say that 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of a set A from $\mathcal{S}$ if for each sequence $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ with the following properties*

- $\{(h_n, m_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}_+ \times (\mathbb{N} \cup \{0\})$,
- *the sequence $\{h_n\}_{n\in\mathbb{N}}$ is decreasing,*
- $\lim\limits_{n\to\infty} h_n = 0$,
- *for each $n \in \mathbb{N}$, $m_n \in \{0, \dots, \left[\frac{1}{\psi(h_n)}\right] - 1\}$*

there exists a subsequence $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ *such that*

$$\{x \in [0,1];\ \chi_{A_k}(x) \not\longrightarrow 0\} \in \mathcal{I},$$

where

$$A_k = \left(\frac{1}{h_{n_k}\psi(h_{n_k})} \cdot A - m_{n_k}\right) \cap [0,1].$$

II. *We say that 0 is a point of left-hand* $\psi_{\mathcal{I}}$*-dispersion of a set* $A \in \mathcal{S}$ *if 0 is a right-hand point of* $\psi_{\mathcal{I}}$*-dispersion of the set* $-A$.

III. *We say that 0 is a point of* $\psi_{\mathcal{I}}$*-dispersion of a set* $A \in \mathcal{S}$ *if 0 is a point of right-hand and left-hand* $\psi_{\mathcal{I}}$*-dispersion of the set* A.

IV. *We say that* $x_0 \in \mathbb{R}$ *is a point of* $\psi_{\mathcal{I}}$*-dispersion of a set* $A \in S$ *if 0 is a point of* $\psi_{\mathcal{I}}$*-dispersion of the set* $A - x_0$.

V. *We say that* $x_0 \in \mathbb{R}$ *is a point of* $\psi_{\mathcal{I}}$*-density of a set* $A \in S$ *if* x_0 *is a point of* $\psi_{\mathcal{I}}$*-dispersion of the set* $\mathbb{R} \setminus A$.

Lemma 1. *Let* $\psi \in \mathcal{C}$ *and* $\{(a_n, b_n)\}_{n\in\mathcal{N}}$ *be a sequence of open intervals such that* $\lim\limits_{n\to\infty} b_n = 0$ *and, for each* $n \in \mathbb{N}$,

i) $0 < a_{n+1} < b_{n+1} < a_n$,

ii) $b_{n+1} \le b_n\psi(b_n)$,

iii) $b_n - a_n \le b_n\psi(b_n)$.

Let $G = \bigcup\limits_{n=1}^{\infty}(a_n, b_n)$. *Then, for each sequence of positive real numbers* $\{h_n\}_{n\in\mathbb{N}}$ *such that* $\lim\limits_{n\to\infty} h_n = 0$ *there exists a subsequence* $\{h_{n_k}\}_{k\in\mathbb{N}}$ *satisfying the condition*

$$\left\{\ x \in [0,1] : \chi_{\frac{1}{h_{n_k}}\cdot G\cap[0,1]}(x) \not\longrightarrow 0\right\} \in \mathcal{I}.$$

Proof. Let $\{h_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers such that $\lim\limits_{n\to\infty} h_n = 0$. We can assume that, for each $n \in \mathbb{N}$, there exists $p_n \in \mathbb{N}$ such that

$$b_{p_n+1} < h_n \le b_{p_n}.$$

We shall consider two cases.

a) There exists positive integer n_0 such that, for each $n \geq n_0$,

$$b_{p_n+1} \leq h_n \leq a_{p_n}.$$

Assume that $n_0 = 1$. We consider a sequence $\left\{\frac{1}{h_n} \cdot b_{p_n+1}\right\}_{n\in\mathbb{N}}$. Then there exist $\alpha \in [0,1]$ and an increasing sequence of positive integers $\{n_k\}_{k\in\mathcal{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot b_{p_{n_k}+1} = \alpha.$$

Hence

$$0 \leq \lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot \left(b_{p_{n_k}+1} - a_{p_{n_k}+1}\right) \leq \lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot b_{p_{n_k}+1} \cdot \psi(b_{p_{n_k}+1}) = 0$$

and

$$\lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot a_{p_{n_k}+1} = \alpha.$$

By the above and

$$0 \leq \lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot b_{p_{n_k}+2} \leq \lim_{k\to\infty} \frac{1}{h_{n_k}} \cdot b_{p_{n_k}+1} \cdot \psi(b_{p_{n_k}+1}) = 0,$$

we infer that

$$\left\{ x \in [0,1] : \chi_{\frac{1}{h_{n_k}}\cdot G\cap[0,1]}(x) \not\longrightarrow 0 \right\} \subset \{0, \alpha, 1\}.$$

b) Now we assume that, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}, k_n \geq n$ such that

$$a_{p_{n_k}} < h_{k_n} < b_{p_{n_k}}.$$

Then

$$1 \leq \lim_{k\to\infty} \frac{1}{h_{k_n}} \cdot b_{p_{n_k}} \leq \lim_{k\to\infty} \frac{1}{a_{p_{n_k}}} \cdot b_{p_{n_k}} \leq \lim_{k\to\infty} \frac{1}{b_{p_{n_k}}(1-\psi(b_{p_{n_k}}))} \cdot b_{p_{n_k}} = 1$$

and

$$\lim_{k\to\infty} \frac{1}{h_{k_n}} \cdot \left(b_{p_{n_k}} - a_{p_{n_k}}\right) \leq \lim_{k\to\infty} \frac{1}{h_{k_n}} b_{p_{n_k}} \psi(b_{p_{n_k}}) = 0.$$

Hence

$$\lim_{k\to\infty} \frac{1}{h_{k_n}} \cdot a_{p_{n_k}} = 1.$$

Additionaly

$$\lim_{k\to\infty}\frac{1}{h_{k_n}}\cdot b_{p_{n_k}+1}\le\lim_{k\to\infty}\frac{1}{h_{k_n}}\cdot b_{p_{n_k}}\psi(b_{p_{n_k}})=0,$$

therefore

$$\left\{x\in[0,1]:\chi_{\frac{1}{h_{n_k}}\cdot G\cap[0,1]}(x)\not\longrightarrow 0\right\}\subset\{0,1\}.$$ □

Theorem 4. *Let $\psi\in\mathcal{C}$. If 0 is a point of right-hand $\psi_{\mathcal{I}}$–dispersion of a set $A\in S$, then it is a point of a right-hand $\mathcal{I}$-dispersion of the set A.*

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $\lim\limits_{n\to\infty}t_n=0$. We may assume that, for each $n\in\mathbb{N}$, there exists a positive h_n such that

$$t_n=h_n\psi(h_n).$$

Then $\lim\limits_{n\to\infty}h_n=0$. Let, for each $n\in\mathbb{N}$, $m_n=0$.

The sequence $\{(h_n,m_n)\}_{n\in\mathbb{N}}$ satisfies the conditions of Definition 2, therefore there exists a sequence $\{(h_{n_k},m_{n_k})\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot A-m_{n_k}\right)\cap[0,1]\in\mathcal{I}.$$

By

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot A-m_{n_k}\right)\cap[0,1]=\limsup_{k\to\infty}\left(\frac{1}{t_{n_k}}\cdot A\right)\cap[0,1],$$

the proof is complete. □

Theorem 5. *Let $\psi\in\mathcal{C}$. There exists an open set G such that 0 is a point of right-hand $\mathcal{I}$-dispersion of the set G and 0 is not a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set G.*

Proof. We shall define a sequence of open intervals $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ such that

i) $0<a_{n+1}<b_{n+1}<a_n$,

ii) $b_{n+1}<\min\{\frac{1}{n},b_n\psi(b_n)\}$,

iii) $b_n-a_n=b_n\psi(b_n)$,

iv) $\frac{1}{\psi(b_n)}\in\mathbb{N}$

for each $n\in\mathbb{N}$.

Let b_1 be a positive real number such that $\psi(b_1) \in \left\{\frac{1}{2}, \frac{1}{3}, ...\right\}$. Let $n \in \mathbb{N}$. Assume that we have defined positive real numbers $b_1, ..., b_n$. Now we shall define a positive b_{n+1} fulfilling the following properties:

$$\psi(b_{n+1}) \in \left\{\frac{1}{2}, \frac{1}{3}, ...\right\} \quad \text{and} \quad b_{n+1} < \min\left\{\frac{1}{n}, b_n\psi(b_n)\right\}.$$

For each $n \in \mathbb{N}$, we put $a_n = b_n - b_n\psi(b_n)$. Then, for each $n \in \mathbb{N}$,

$$a_{n-1} = b_{n-1}(1 - \psi(b_{n-1})) \geq b_{n-1} \cdot \frac{1}{2} \geq b_{n-1} \cdot \psi(b_{n-1}) > b_n.$$

Set $G = \bigcup_{n=1}^{\infty} (a_n, b_n)$.

By Lemma 1, 0 is a point of right-hand $\mathcal{I}$-dispersion of the set G. Now we prove that 0 is not a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set G.

Let $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ be a sequence such that $h_n = b_n$, $m_n = \left[\frac{1}{\psi(h_n)}\right] - 1$ for each $n \in \mathbb{N}$, and let $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ be an arbitrary subsequence of $\{(h_n, m_n)\}_{n\in\mathbb{N}}$. We shall show that

$$(0,1) \subset \limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})} \cdot G - m_{n_k}\right).$$

Let $k \in \mathbb{N}$. Then

$$\frac{1}{h_{n_k}\psi(h_{n_k})} \cdot G - m_{n_k} \supset \frac{1}{h_{n_k}\psi(h_{n_k})} \cdot (a_{n_k}, b_{n_k}) - m_{n_k} =$$

$$\left(\frac{1}{b_{n_k}\psi(b_{n_k})}(b_{n_k} - b_{n_k}\psi(b_{n_k})) - \left[\frac{1}{\psi(b_{n_k})}\right] + 1, \frac{1}{b_{n_k}\psi(b_{n_k})} \cdot b_{n_k} - \left[\frac{1}{\psi(b_{n_k})}\right] + 1\right) =$$

$$= \left(\frac{1}{\psi(b_{n_k})}(1 - \psi(b_{n_k})) - \frac{1}{\psi(b_{n_k})} + 1, \frac{1}{\psi(b_{n_k})} - \frac{1}{\psi(b_{n_k})} + 1\right) = (0,1).$$

By the above, 0 is not a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set G. □

Theorem 6. *Let $\psi \in \mathcal{C}$. There exists an open set G such that 0 is an accumulation point of the set G and 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set G.*

Proof. We define sequences of real positive numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

1) $b_{n+1} \le \frac{1}{n} a_n \psi(a_n)$,

2) $0 < b_n - a_n \le \frac{1}{n} a_n \psi(a_n)$,

for each $n \in \mathbb{N}$, and

3) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

Let b_1 be an arbitrary real positive number. Let $n \in \mathbb{N}$. Assume that we have defined numbers $b_1, ..., b_{n-1}$ and $a_1, ..., a_{n-1}$. Let b_n be a real positive number such that $b_n \le \frac{1}{n-1} a_{n-1} \psi(a_{n-1})$. By the continuity of a function $g(x) = x + \frac{1}{n} x\psi(x)$ and by $b_n < b_n + \frac{1}{n} b_n \psi(b_n)$, there exists a_n such that $a_n < b_n$ and $a_n + \frac{1}{n} a_n \psi(a_n) = b_n$.

Set $G = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$. We shall show that 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of G. Let $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence satisfying the conditions of Definition 2. We consider the following possibilities:

a) Assume that there exists a subsequence $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $m_{n_k} = 0$. Then, in view of Lemma 1, 0 is a point of a right-hand $\mathcal{I}$–dispersion of G. Since $\lim\limits_{k\to\infty} h_{n_k} \psi(h_{n_k}) = 0$, we may choose a subsequence $\left\{h_{n_{k_p}}\right\}_{p\in\mathbb{N}}$ such that

$$\limsup_{p\to\infty} \left(\frac{1}{h_{n_{k_p}} \psi(h_{n_{k_p}})} G - m_{n_{k_p}} \right) \cap [0,1] = \limsup_{p\to\infty} \frac{1}{h_{n_{k_p}} \psi(h_{n_{k_p}})} \cdot G \cap [0,1] \in \mathcal{I}.$$

b) Assume that there exists a subsequence $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ such that

$$[m_{n_k} h_{n_k} \psi(h_{n_k}), (m_{n_k}+1) h_{n_k} \psi(h_{n_k})] \cap G = \emptyset$$

for each $k \in \mathbb{N}$.

Then, for each $k \in \mathbb{N}$, $\left(\frac{1}{h_{n_k}\psi(h_{n_k})} \cdot G - m_{n_k} \right) \cap [0,1] = \emptyset$. Hence

$$\limsup_{k\to\infty} \left(\frac{1}{h_{n_k}\psi(h_{n_k})} \cdot G - m_{n_k} \right) \cap [0,1] = \emptyset.$$

c) If none of the cases a) and b) is true, then there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$, $m_n \ge 1$ and

$$[m_n h_n \psi(h_n), (m_n+1) h_n \psi(h_n)] \cap G \ne \emptyset.$$

We can assume that for each $n \in \mathbb{N}$ there exists $r_n \in \mathbb{N}, r_n > 1$ such that

$$[m_n h_n \psi(h_n), (m_n+1) h_n \psi(h_n)] \cap (a_{r_n}, b_{r_n}) \ne \emptyset.$$

Therefore

$$a_{r_n} \leq (m_n+1)h_n\psi(h_n) \leq \left[\frac{1}{\psi(h_n)}\right] h_n\psi(h_n) \leq h_n$$

and, by

$$b_{r_n+1} \leq \frac{1}{r_n} a_{r_n}\psi(a_{r_n}) \leq 1 \cdot h_n\psi(h_n) \leq m_n h_n \psi(h_n),$$

we have

$$[m_n h_n\psi(h_n), (m_n+1)h_n\psi(h_n)] \cap \bigcup_{j=r_n+1}^{\infty} (a_j, b_j) = \emptyset.$$

Additionaly, by $a_{r_n-1} > h_n$,

$$[m_n h_n\psi(h_n), (m_n+1)h_n\psi(h_n)] \cap \bigcup_{j=1}^{r_n-1} (a_j, b_j) = \emptyset.$$

Let $n \in \mathbb{N}$ and

$$x_n \in [m_n h_n\psi(h_n), (m_n+1)h_n\psi(h_n)] \cap (a_{r_n}, b_{r_n}).$$

Then $\frac{1}{h_n\psi(h_n)} \cdot x_n - m_n \in [0,1]$, for all $n \in \mathbb{N}$. Thus, there exists $\alpha \in [0,1]$ and a subsequence $\left\{\frac{1}{h_{n_k}\psi(h_{n_k})} x_{n_k} - m_{n_k}\right\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left(\frac{1}{h_{n_k}\psi\left(h_{n_k}\right)} x_{n_k} - m_{n_k}\right) = \alpha.$$

By

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \frac{1}{h_{n_k}\psi(h_{n_k})} \cdot (b_{r_{n_k}} - a_{r_{n_k}}) \\ &\leq \lim_{k\to\infty} \frac{1}{h_{n_k}\psi(h_{n_k})} \cdot \frac{1}{r_{n_k}} \cdot a_{r_{n_k}}\psi(a_{r_{n_k}}) \\ &\leq \lim_{k\to\infty} \frac{1}{h_{n_k}\psi(h_{n_k})} \cdot \frac{1}{r_{n_k}} \cdot h_{n_k}\psi(h_{n_k}) \\ &= \lim_{k\to\infty} \frac{1}{r_{n_k}} = 0, \end{aligned}$$

we infer that

$$\lim_{k\to\infty} \left(\frac{1}{h_{n_k}\psi(h_{n_k})} b_{r_{n_k}} - m_{n_k}\right) = \alpha$$

and

$$\lim_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}a_{r_{n_k}} - m_{n_k}\right) = \alpha.$$

Thus

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot G - m_{n_k}\right)\cap[0,1]\subset\{\alpha\}. \qquad \square$$

Theorem 7. *Let $\psi_1 \in \mathcal{C}$. There exist a function $\psi_2 \in \mathcal{C}$ and an open set G such that 0 is a point of right-hand $\psi_{1,\mathcal{I}}$-dispersion of the set G, but 0 is not a point of right-hand $\psi_{2,\mathcal{I}}$-dispersion of the set G.*

Proof. We define a sequence of open intevals $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ such that

1) $0 < a_{n+1} < b_{n+1} < a_n$,
2) $b_{n+1} \le \frac{1}{n}a_n\psi_1(a_n)$,
3) $b_n - a_n \le \frac{1}{n}a_n\psi_1(a_n)$,
4) $\frac{b_n - a_n}{b_n} < \frac{b_{n-1}-a_{n-1}}{b_{n-1}}$,
5) $\frac{b_n}{b_n - a_n} \in \mathbb{N}$,

   for each $n \in \mathbb{N}$, and

6) $\lim\limits_{n\to\infty} b_n = 0$.

Let $b_1 \in (0,1)$ and $k \in \mathbb{N}\setminus\{1\}$. Assume that we have defined numbers $a_1, ..., a_{k-1}$ and $b_1, ..., b_{k-1}$. Let b_k be an arbitrary positive number such that $b_k \le \frac{1}{k-1}a_{k-1}\psi_1(a_{k-1})$.

We consider two functions: $g(x) = x + \frac{1}{k}x\psi_1(x)$ and $h(x) = 1 - \frac{x}{b_k}$. Since $g(b_k) = b_k + \frac{1}{k}b_k\psi_1(b_k) > b_k$, therefore, by continuity of a function g, we have $\alpha \in (0, b_k)$ such that $g(\alpha) = b_k$ and, for each $x \in (\alpha, b_k)$, $g(x) > b_k$. Let p be a positive integer such that

$$\frac{1}{p} < \min\left\{\frac{b_{k-1} - a_{k-1}}{b_{k-1}}, h(\alpha)\right\}.$$

Then

$$0 = h(b_k) < \frac{1}{p} < h(\alpha)$$

and, by continuity of h, we can choose $a_k \in (\alpha, b_k)$ such that $h(a_k) = \frac{1}{p}$.

Set $G = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Let $\psi_2 \in \mathcal{C}$ be a function such that, for each $n \in \mathbb{N}$, $\psi_2(b_n) = \frac{b_n - a_n}{b_n}$. In a similar way as in Theorem 6, one can prove that 0 is a point of right-hand $\psi_{1,\mathcal{I}}$-dispersion of the set G.

We shall show that 0 is not a point of right-hand $\psi_{2,\mathcal{I}}$-dispersion of the set G. Let $h_n = b_n$ for each $n \in \mathbb{N}$ and $m_n = [\frac{1}{\psi_2(b_n)}] - 1$. The sequence $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ satisfies the conditions of Definition 2. Let $\{(h_{n_k}, m_{n_k})\}_{n\in\mathbb{N}}$ be an arbitrary subsequence of $\{(h_n, m_n)\}_{n\in\mathbb{N}}$. Then, for each $k \in \mathbb{N}$,

$$\frac{1}{h_{n_k}\psi_2(h_{n_k})} \cdot G - m_{n_k} \supset \frac{1}{h_{n_k}\psi_2(h_{n_k})} \cdot (a_{n_k}, b_{n_k}) - m_{n_k} = (0, 1).$$

Thus

$$(0, 1) \subset \limsup_{k\to\infty} \left(\frac{1}{h_{n_k}\psi_2(h_{n_k})} \cdot G - m_{n_k} \right) \cap [0, 1].$$

Definition 3. *Let $\psi \in \mathcal{C}$. For a set $A \in \mathcal{S}$, we define $\Phi_\psi(A)$ to be the set of all points of $\psi_\mathcal{I}$-density of the set A.*

Theorem 8. *Let $\psi \in \mathcal{C}$. Then, for any $A, B \in \mathcal{S}$,*

1) $\Phi_\psi(\emptyset) = \emptyset, \Phi_\psi(\mathbb{R}) = \mathbb{R}$,

2) *If $A \subset B$, then $\Phi_\psi(A) \subset \Phi_\psi(B)$,*

3) *If $A \sim B$, then $\Phi_\psi(A) = \Phi_\psi(B)$,*

4) $\Phi_\psi(A \cap B) = \Phi_\psi(A) \cap \Phi_\psi(B)$,

5) $A \sim \Phi_\psi(A)$.

Proof. The conditions 1) and 2) are obvious. Assume that $A \sim B$ and $x \in \Phi_\psi(A)$. Without loss of generality, one can assume that $x = 0$. We only show that if 0 is a point of right-hand $\psi_\mathcal{I}$-dispersion of the set A', then 0 is a point of right-hand $\psi_\mathcal{I}$-dispersion of the set B'.

Let $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence which satisfies conditions of Definition 2. We observe that

$$B' = (B' \cap A') \cup (B' \setminus A'),$$

where $B' \setminus A' = A \setminus B \in \mathcal{I}$, and $B' \cap A' \subset A'$. 0 is a point of right-hand $\psi_\mathcal{I}$-dispersion of the set A', thus it is a point of right-hand $\psi_\mathcal{I}$-dispersion of the set $A' \cap B'$. Therefore, there exists a subsequence $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot(A'\cap B') - m_{n_k}\right)\cap[0,1]\in\mathcal{I}.$$

We define the sets P, P_1, P_2 in the following way:

$$\begin{aligned} P = &\quad \limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot B' - m_{n_k}\right)\cap[0,1],\\ P_1 = &\quad \limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot(A'\cap B') - m_{n_k}\right)\cap[0,1],\\ P_2 = &\quad \limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot(B'\setminus A') - m_{n_k}\right)\cap[0,1]. \end{aligned}$$

Then $P \subset P_1 \cup P_2$. The set P_1 is of the first category, and

$$P_2 = \bigcap_{r=1}^{\infty}\bigcup_{k=r}^{\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot(B'\setminus A') - m_{n_k}\right)\cap[0,1]\in\mathcal{I}.$$

Thus $P \in \mathcal{I}$.

We have proved that $\Phi_\psi(A) \subset \Phi_\psi(B)$. In a similar way, we can prove that $\Phi_\psi(B) \subset \Phi_\psi(A)$.

Now we shall show condition 4). Since $A\cap B \subset A$ and $A\cap B \subset B$, therefore, by condition 2), we have $\Phi_\psi(A\cap B) \subset \Phi_\psi(A)\cap\Phi_\psi(B)$.

Let $x \in \Phi_\psi(A)\cap\Phi_\psi(B)$. We can assume that $x = 0$. Let $\{(h_n, m_n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence which satisfies the conditions of Definition 2. Since 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set A', therefore there exists a subsequence $\{(h_{n_k}, m_{n_k})\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_k}\psi(h_{n_k})}\cdot A' - m_{n_k}\right)\cap[0,1]\in\mathcal{I}.$$

Additionaly, 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set B', thus there exists a subsequence $\{(h_{n_{k_p}}, m_{n_{k_p}})\}_{k\in\mathbb{N}}$, such that

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_{k_p}}\psi(h_{n_{k_p}})}\cdot B' - m_{n_{k_p}}\right)\cap[0,1]\in\mathcal{I}.$$

Then

$$\limsup_{k\to\infty}\left(\frac{1}{h_{n_{k_p}}\psi(h_{n_{k_p}})}\cdot(A\cap B)' - m_{n_{k_p}}\right)\cap[0,1]\subset H,$$

where

$$\begin{aligned} H &= \limsup_{k\to\infty}\left(\frac{1}{h_{n_{k_p}}\psi(h_{n_{k_p}})}\cdot A' - m_{n_{k_p}}\right)\cap[0,1]\cup \\ &\cup \limsup_{k\to\infty}\left(\frac{1}{h_{n_{k_p}}\psi(h_{n_{k_p}})}\cdot B' - m_{n_{k_p}}\right)\cap[0,1]\in\mathcal{I}. \end{aligned}$$

Hence, 0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set $(A\cap B)'$. In a similar way, we can show that 0 is a point of left-hand $\psi_{\mathcal{I}}$-dispersion of the set $(A\cap B)'$.

Now we shall show condition 5). Let $A\in\mathcal{S}$. Then $A=(G\setminus P_1)\cup P_2$, where G is an open set, P_1 i P_2 are sets of the first category and $P_1\subset G$, $P_2\cap G=\emptyset$. By 3), we have $\Phi_\psi(A)=\Phi_\psi(G)$ and $G\subset\Phi_\psi(G)$. Thus

$$A\setminus\Phi_\psi(A)=A\setminus\Phi_\psi(G)\subset A\setminus G\in\mathcal{I}.$$

By Theorem 4, $\Phi_\psi(A)\subset\Phi(A)$ and by Theorem 2, $A\sim\Phi(A)$, therefore $\Phi_\psi(A)\setminus A\subset\Phi(A)\setminus A\in\mathcal{I}$. □

Definition 4. *Let, for $\psi\in\mathcal{C}$,*

$$\mathcal{T}_\psi=\{A\in\mathcal{S}: A\subset\Phi_\psi(A)\}.$$

By theorems 3, 4, 5 and 8 we have the following

Theorem 9. *Let $\psi\in\mathcal{C}$. $\mathcal{T}_\psi$ is a topology on the real line, stronger than the Euclidean topology and weaker than the $\mathcal{I}$–topology.*

Lemma 2. *Assume that we have a sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=0$ and, for each $n\in\mathbb{N}$, $0<b_{n+1}<a_n<b_n$. Then there exists a function $\psi\in\mathcal{C}$ such that 0 is not a point of $\psi_{\mathcal{I}}$-dispersion of the set $G=\bigcup\limits_{n=1}^{\infty}(a_n,b_n)$.*

Proof. First we define values of the finction ψ at points of the sequence $\{b_n\}_{n\in\mathbb{N}}$. Set $\psi(b_1)=\frac{1}{\left[\frac{b_1}{b_1-a_1}\right]+1}$, $a_2'=\max\{a_2,b_2(1-\psi(b_1))\}$ and $\psi(b_2)=\frac{1}{\left[\frac{b_2}{b_2-a_2'}\right]+1}$. Assume that for $n\in\mathbb{N}$ we have defined the points $a_1',\dots,a_n'$ and the real numbers $\psi(b_1),\dots,\psi(b_n)$ in the following way:

- $a_{i+1}'=\max\left\{a_{i+1},b_{i+1}(1-\frac{1}{i}\psi(b_i))\right\}$ if $i\in\{1,\dots,n-1\}$,

- $\psi(b_{i+1}) = \frac{1}{\left[\frac{b_{i+1}}{b_{i+1}-a'_{i+1}}\right]+1}$ if $i \in \{1, \ldots, n-1\}$.

Put $a'_{n+1} = \max\left\{a_{n+1}, b_{n+1}(1 - \frac{1}{n}\psi(b_n))\right\}$ and $\psi(b_{n+1}) = \frac{1}{\left[\frac{b_{n+1}}{b_{n+1}-a'_{n+1}}\right]+1}$.

We observe that $\psi(b_{n+1}) < \frac{1}{n}\psi(b_n)$. Indeed

$$\frac{1}{n}\psi(b_n) \geq 1 - \frac{a'_{n+1}}{b_{n+1}} = \frac{1}{\frac{b_{n+1}}{b_{n+1}-a'_{n+1}}} > \frac{1}{\left[\frac{b_{n+1}}{b_{n+1}-a'_{n+1}}\right]+1} = \psi(b_{n+1}).$$

Let $\psi \in \mathcal{C}$ be a function such that, for any $n \in \mathbb{N}$ and $x \in [a_n, b_n]$, $\psi(x) = \psi(b_n)$. In a similar way as in Theorem 4, we can show that 0 is not a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set G. □

Definition 5. *We denote by $\mathcal{H}$ the Hashimoto topology, where*

$$\mathcal{H} = \{U \setminus P : U - \text{ an open set }, P \in \mathcal{I}\}.$$

Theorem 10. $\bigcap_{\psi \in \mathcal{C}} \mathcal{T}_\psi = \mathcal{H}$.

Proof. It is obvious that $\mathcal{H} \subset \bigcap_{\psi\in\mathcal{C}} \mathcal{T}_\psi$. Let $A \in \mathcal{S}$ and $A \notin \mathcal{H}$. Then $A = (G \setminus P_1) \cup P_2$, where G is an open set, $P_1, P_2 \in \mathcal{I}$, $P_1 \subset G$ and $P_2 \cap G = \emptyset$.

Set $H = \operatorname{Int}(\operatorname{Cl}(G))$ and $R = H \setminus (G \cup P_2)$. By $A \notin \mathcal{H}$, we know that P_2 is not a subset of H. It is easy to see that $\operatorname{Int}(\mathbb{R} \setminus H) \neq \emptyset$ and the set $\mathbb{R} \setminus H$ has no isolated points.

Let $x_0 \in P_2 \cap (\mathbb{R} \setminus H)$ and $\{(c_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of all components of the set $\operatorname{Int}(\mathbb{R} \setminus H)$. We consider the following cases:

a) $x_0 \in \operatorname{Int}(\mathbb{R} \setminus H)$. Then, for an arbitrary function $\psi \in \mathcal{C}$, x_0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set H. Thus, x_0 is a point of right-hand $\psi_{\mathcal{I}}$-dispersion of the set $G \subset H$. Since $\Phi_\psi(A) = \Phi_\psi(G)$, we have $x_0 \notin \Phi_\psi(A)$. Therefore, $A \not\subset \Phi_\psi(A)$, and $A \notin \mathcal{T}_\psi$.

b) There exists $n_0 \in \mathbb{N}$ such that $x_0 = c_{n_0}$ or $x_0 = d_{n_0}$. Then x_0 is a point of right-hand or left-hand $\psi_{\mathcal{I}}$–density of the set $\mathbb{R} \setminus H$ for arbitrary function $\psi \in \mathcal{C}$, respectively, and, as above, $x_0 \in A \setminus \Phi_\psi(A)$.

c) There exists a sequence $\{c_{n_k}\}_{k\in\mathbb{N}}$ which converges to x_0 from the right or there exists a sequence $\{d_{n_k}\}_{k\in\mathbb{N}}$ which converges to x_0 from the left. Then, by Lemma 2, there exists a function $\psi \in \mathcal{C}$ such that x_0 is not a point of $\psi_{\mathcal{I}}$-dispersion of the set $\bigcup_{k=1}^{\infty}(c_{n_k}, d_{n_k})$. Thus, $x_0 \notin \Phi_\psi(A)$ and $A \notin \mathcal{T}_\psi$. Therefore, $A \notin \bigcap_{\psi\in\mathcal{C}} \mathcal{T}_\psi$. □

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