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MONOTONE METHOD FOR RIEMANN-LIOUVILLE MULTI-ORDER FRACTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we develop the monotone method for nonlinear multi-order N-systems of Riemann-Liouville fractional differential equations. That is, a hybrid system of nonlinear equations of orders q_i where $0 < q_i < 1$. In the development of this method we recall any needed existence results along with any necessary changes. Through the method's development we construct a generalized multi-order Mittag-Leffler function that fulfills exponential-like properties for multi-order systems. Further we prove a comparison result paramount for the discussion of fractional multi-order inequalities that utilizes lower and upper solutions of the system. The monotone method is then developed via the construction of sequences of linear systems based on the upper and lower solutions, and are used to approximate the solution of the original nonlinear multi-order system.

Keywords: fractional differential systems, multi-order systems, lower and upper solutions, monotone method.

Mathematics Subject Classification: 34A08, 34A34, 34A45, 34A38.

1. INTRODUCTION

Fractional differential equations have various applications in widespread fields of science, such as in engineering [5], chemistry [7, 14, 16], physics [1, 2, 9], and others [10, 11]. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we construct such a method. For some existence results on fractional differential equations we refer the reader to the papers [6, 8, 15] and the books [10, 17] along with references therein.

Specifically, we construct a technique to approximate solutions to the nonlinear Riemann-Liouville (R-L) fractional differential multi-order N-system. A multi-order system is of the type where the equation in each component is of unique order. That is, a fractional system of the type

$$D^{q_i} x_i = f_i(t, x).$$

This is a generalization of normal R-L systems and yields a type of hybrid system of a fractional type. We note that various complications arise from systems of this type as many known properties used in the study of scalar fractional differential equations and single-order fractional systems require modification, but at the same time multi-order systems present far more possibilities for applications. For example, consider allowing each species in a population model to have their own order of derivative. Though we will not consider any specific applications in this study, we hope this will add to the groundwork of future studies.

The iterative technique we construct will be a generalization of the monotone method for multi-order R-L N-systems of order q_i, where $0 < q_i < 1$. The monotone method, in broad terms, is a technique in which sequences are constructed from the unique solutions of linear differential equations, and initially based off of lower and upper solutions of the original nonlinear equation. These sequences converge uniformly and monotonically, from above and below, to maximal and minimal solutions of the nonlinear equation. If the nonlinear DE considered has a unique solution then both sequences will converge uniformly to that unique solution. The advantage of the monotone method is that it allows us to approximate solutions to nonlinear DEs using linear DEs; further using upper and lower solutions guarantee the interval of existence. For more information on the monotone method for ordinary DEs see [12].

Many complications arise when developing the monotone method for multi-order systems. First of all, as seen in the previous work involving the R-L case in these methods, the sequences we construct, say $\{v_n\}, \{w_n\}$ do not converge uniformly to extremal solutions, but weighted sequences $\{t^{1-q}v_n\}$, $\{t^{1-q}w_n\}$ converge uniformly. Another complication, unique to multi-order systems, involves the well-known result for the fractional derivative of the weighted Mittag-Leffer function, a function which we define below in Section 2. That is, the Mittag-Leffler function has a property similar to that of the natural exponential

$$D_t^q t^{q-1} E_{q,q}(t^q) = t^{q-1} E_{q,q}(t^q).$$

However, this property is dependent on the order of q used, and therefore the weighted Mittag-Leffler function of order q_1 will not have this property with the derivative of order q_2. This issue is present in the proof of Theorem 2.13, and renders it unable to be proven in the same manner as in the single-order case. In order to circumvent this issue we construct a family of generalized Mittag-Leffler functions that operate in much the same way but in a complementary manner to multi-order systems. That is, the q_i-th derivative of this generalized Mittag-Leffler function will yield a linear combination including itself. The construction of this function and properties regarding it are discussed in Definition 2.10, Lemma 2.11 and the neighbouring text.

For the monotone method we generalize our basic system in a way so that we can cover many different cases in a single result. To do this, for each i we rearrange the nonlinear function f to look like $f_i(t, x_i, [x]_r, [x]_s)$ where f is nondecreasing in component r and nonincreasing in component s. We note that the monotone method has been established for the standard nonlinear Riemann-Liouville fractional differential N-systems of order q in [4], and was established for multi-order 2-systems in [3], this study acts as a further generalization of that work.

2. PRELIMINARY RESULTS

In this section, we will first consider basic results regarding scalar Riemann-Liouville differential equations of order q, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to the multi-order case. Then we will consider existence and comparison results for multi-order systems of order $0 < q_i < 1$, with $i \in \{1, 2, 3, \ldots, N\} = D$, which will be used in our main result. In the next section, we will apply these preliminary results to develop the monotone method for these multi-order R-L systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1. Let $p = 1 - q$, a function $\phi(t) \in C(J, R)$ is a C_p function if $t^p\phi(t) \in C(J_0, R)$. The set of C_p functions is denoted $C_p(J, R)$. Further, given a function $\phi(t) \in C_p(J, R)$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2. Let $\phi \in C_p(J, R)$, then $D_t^q\phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3. The Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

For fractional differential equations we utilize the weighted C_p version of the Mittag-Leffler function $t^{q-1}E_{q,q}(t^q)$, since as mentioned previously in Section 1 it is its own q-th derivative. Further, it attains a convergence result we mention in the following remark.

Remark 2.4. The C_p weighted Mittag-Leffler function

$$t^{q-1}E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{kq+q-1}}{\Gamma(kq+q)},$$

where λ is a constant, converges uniformly on compacta of J.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.5. *Let $f \in C_p(J,R)$, then $I_t^q f(t) \in C_p(J,R)$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in R^+$ can be found in [4]. Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \tag{2.1}$$

with initial condition

$$t^p x(t)\big|_{t=0} = x^0,$$

where x^0 is a constant, $x \in C(J_0,R)$, and $z \in C_p(J,R)$, which has unique solution

$$x(t) = x^0\Gamma(q)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)z(s)\,ds.$$

Next, we recall a result we will utilize extensively in our proceeding comparison and existence results, and likewise in the construction of the monotone method. We note that this result is similar to the well known comparison result found in literature, as in [13], but we do not require the function to be Hölder continuous of order $\lambda > q$.

Lemma 2.6. *Let $m \in C_p(J,R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \le 0$ for $t \in (0,t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \ge 0.$$

The proof of this lemma can be found in [4], along with further discussion as to why and how we weaken the Hölder continuous requirement. We use this lemma in the proof of the later main comparison result, which will be critical in the construction of the monotone method.

Now, we will turn our attention to results for the nonlinear R-L fractional multi-order systems, and in doing so we must discuss any changes. First, we will consider systems of orders q_i, $0 \le q_i < 1$. For simplicity we will let $q = (q_1, q_2, q_3, \dots, q_N)$, and when we write inequalities $x \le y$, we mean it is true for all components. Further, from this point on, we will use the subscript i which we will always assume is in $D = \{1, 2, \ldots, N\}$. For defining C_p continuity for multi-order systems we define $p_i = 1 - q_i$ and for simplicity of notation we will define the function x_p such that $x_{p_i}(t) = t^{p_i} x_i(t)$ for $t \in J_0$. We also note that at times it will be convenient to emphasize the product of t^p, therefore we will define $t^p x(t) = x_p(t)$ for $t \in J_0$. Now, we define the set of C_p continuous functions as

$$C_p(J, R^N) = \{x \in C(J, R^N) \,|\, x_p \in C(J_0, R^N)\}.$$

For the rest of our results we will be considering the nonlinear R-L fractional multi-order system

$$\begin{aligned} D^{q_i} x_i &= f_i(t, x), \\ x_{p_i}(0) &= x_i^0, \end{aligned} \tag{2.2}$$

where $f \in C(J_0 \times R^N, R^N)$, and $x^0 \in R^N$. Note that just as in the scalar case, a solution $x \in C_p(J, R^N)$ of (2.2) also satisfies the equivalent R-L integral equation

$$x_i(t) = x_i^0 t^{q_i - 1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i - 1} f_i(s, x(s)) ds. \tag{2.3}$$

Thus, if $f \in C(J_0 \times R^N, R^N)$ then (2.2) is equivalent to (2.3)). See [10, 13] for details. Now we will recall a Peano type existence theorem for equation (2.2).

Theorem 2.7. *Suppose $f \in C(R_0, R^N)$ and $|f_i(t, x)| \le M_i$ on R_0, where*

$$R_0 = \{(t, x) \ : \ |x_p(t) - x^0| \le \eta, t \in J_0\}$$

Then the solution of (2.2) exists on J.

This result is presented for the scalar case in [13], and in [4] it was proven that the solution can be extended to all of J. We note that for multi-order systems it is proved in much the same way. Next we will consider the main Comparison Theorem for multi-order N-systems, which will be utilized extensively in our main results. For this result we will require f to satisfy the following definition.

Definition 2.8. A function $f(t, x) \in C(J_0 \times R^N, R^N)$ is said to be quasimonotone nondecreasing in x if for each i, $x \le y$ and $x_i = y_i$ implies $f_i(t, x) \le f_i(t, y)$. Naturally, f is quasimonotone nonincreasing if we reverse the inequalities.

Further the Comparison Theorem utilizes upper and lower solutions which we give in the following definition.

Definition 2.9. $w, v \in C_p(J, R)$ are upper and lower solutions of system (2.2) if

$$\begin{aligned} D^{q_i} w_i(t) \ge f_i(t, w), \quad w_{p_i}(0) = w_i^0 \ge x_i^0, \\ D^{q_i} v_i(t) \le f_i(t, v), \quad v_{p_i}(0) = v_i^0 \ge x_i^0. \end{aligned}$$

For the Comparison Theorem we will introduce the following function

$$Z = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \frac{c^{u\cdot k-1}\, t^{q\cdot k-1}}{\Gamma(q\cdot k)},$$

where q is defined as above, $k = (k_1, k_2, k_3, \dots, k_N)$, $u = (1, 1, 1, \dots, 1)$, and $c \in R$ is a constant. The development of the Comparison Theorem for multi-order systems will require the construction of various reduced forms of Z, i.e. we will need to define a generalized Z function such that, for example, writing $Z_{1,3,5}$ would give us the construction of Z but with only the components involving q_1, q_3 and q_5. That is,

$$Z_{1,3,5} = \sum_{k_1=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_5=1}^{\infty} \frac{c^{k_1+k_3+k_5-1}\, t^{k_1q_1+k_3q_3+k_5q_5-1}}{\Gamma(k_1q_1 + k_3q_3 + k_5q_5)}.$$

We give a general definition of this concept here.

Definition 2.10. For any subset $A \subset D$, define the function Z_A as

$$Z_A = \sum_{k_i \geq 1, i\in A} \frac{c^{\sum_{j\in A}(k_j)-1} t^{\sum_{j\in A}(k_jq_j)-1}}{\Gamma\big(\sum_{j\in A} k_jq_j\big)}.$$

To shore up our notation further we would like to remove the braces for each A in the subscript of Z_A, therefore we would say that for simplicity that $Z_{1,3,5} = Z_{\{1,3,5\}}$. The set based notation is also complementary to the symmetry of Z_A, since $\{5,3,1\} = \{1,3,5\}$ and $Z_{5,3,1} = Z_{1,3,5}$.

A special case to note here is that for any i

$$\begin{aligned} Z_i = \sum_{k_i=1}^{\infty} \frac{c^{k_i-1} t^{k_iq_i-1}}{\Gamma\big(k_iq_i\big)} &= t^{q_i-1} \sum_{k_i=0}^{\infty} \frac{c^{k_i} t^{k_iq_i}}{\Gamma\big(k_iq_i + q_i\big)} \\ &= t^{q_i-1} E_{q_i,q_i}(ct^{q_i}). \end{aligned}$$

Therefore these Z_A functions generalize the weighted Mittag-Leffler function, and hence from this point we will utilize this notation for it. Before we go much further we need to prove that each Z_A converges uniformly, and we will be employing the q_i-th derivative of Z_A within our later results. We will pursue both of these results in the following Lemma. In these results we will use that the Beta Function

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

is decreasing in x and y for $x, y > 0$.

Lemma 2.11. *For each $A \subset D$, Z_A converges uniformly on compacta of J. Further,*

$$D^{q_a} Z_A = c(Z_{A\setminus\{a\}} - Z_A), \tag{2.4}$$

for each $a \in A$.

Proof. First, we will show that each Z_A converges uniformly on compacta of J. To begin let

$$A = \{a_1, a_2, a_3, \cdots, a_m\} \subset D.$$

Now we note that Z_{a_1} converges uniformly as discussed in Remark 2.4, further in [3] it was proved that $Z_{1,2}$ (Which the authors called Z) converges uniformly on compacta of J, and we can use the same process to show that Z_{a_1,a_2} converges uniformly. Using this as an inductive basis step, now suppose that $Z_{a_1,a_2,a_3,\ldots,a_n}$ converges uniformly up to some $2 \le n < N$. Now we will show that $Z_{a_1,a_2,a_3,\ldots,a_{n+1}}$ converges uniformly. From here we will reduce our notation such that $k_i = k_{a_i}$ and $q_i = q_{a_i}$ for each $1 \le i \le n+1$, and $\sum_j = \sum_{j=1}^n$. Now we note that there exists a $K > 0$ such that for $k_{n+1} \ge K$, that $k_{n+1}q_{n+1} - 1 > 0$. Then, for any $t \in J$, $k_{n+1} \ge K$ and for each other $k_i \ge 1$ we have

$$\begin{aligned}
&\frac{c^{k_{n+1}+\sum_j (k_j)}\, t^{k_{n+1}q_{n+1}+\sum_j (k_j q_j)-1}}{\Gamma\big(k_{n+1}q_{n+1} + \sum_j k_j q_j\big)} \\
&= B\big(k_{n+1}q_{n+1}, \textstyle\sum_j (k_j q_j)\big) \dfrac{c^{k_{n+1}+\sum_j (k_j)-1}\, t^{k_{n+1}q_{n+1}+\sum_j (k_j q_j)-1}}{\Gamma\big(k_{n+1}q_{n+1}\big)\Gamma\big(\sum_j k_j q_j\big)} \\
&\le B\big(q_{n+1}, \textstyle\sum_j q_j\big) \dfrac{c^{k_{n+1}+\sum_j (k_j)-1}\, T^{k_{n+1}q_{n+1}+\sum_j (k_j q_j)-1}}{\Gamma\big(k_{n+1}q_{n+1}\big)\Gamma\big(\sum_j k_j q_j\big)},
\end{aligned}$$

which is obtained by the monotonicity of the Beta function. Now letting $\mathcal{B} = B\big(q_{n+1}, \sum_j q_j\big)$, we note that the series

$$\begin{aligned}
&\mathcal{B} \sum_{k_{n+1}=K, k_i=1}^{\infty} \frac{c^{k_{n+1}+\sum_j (k_j)-1}\, T^{k_{n+1}q_{n+1}+\sum_j (k_j q_j)-1}}{\Gamma\big(k_{n+1}q_{n+1}\big)\Gamma\big(\sum_j k_j q_j\big)} \\
&= \mathcal{B} \sum_{k_i \ge 1} \frac{c^{\sum_j (k_j)-1}\, T^{\sum_j (k_j q_j)-1}}{\Gamma\big(\sum_j k_j q_j\big)} \sum_{k_{n+1}=K}^{\infty} \frac{c^{k_{n+1}} T^{k_{n+1}q_{n+1}}}{\Gamma\big(k_{n+1}q_{n+1}\big)} \\
&\le \mathcal{B} T^{q_{n+1}} Z_{a_1,a_2,a_3,\ldots,a_n}(T) E_{q_{n+1},q_{n+1}}(cT).
\end{aligned}$$

Therefore, by the Weirstrass M-Test, since both $Z_{a_1,a_2,a_3,\ldots,a_n}(T)$ and $E_{q_{n+1},q_{n+1}}(cT)$ converge, $Z_{a_1,a_2,a_3,\ldots,a_{n+1}}$ is a finite sum of $K-1$ weakly singular terms and a series that is uniformly convergent on J_0, thus it is uniformly convergent on compacta of J. By induction we can conclude that Z_A is uniformly convergent on compacta of J. And finally, since A was an arbitrary subset of D we conclude that this will be true for any chosen Z_A.

Now we will consider the q_i-th derivative of each Z_A. For simplicity we will look at the q_1-st derivative and note that computing the others works in the same way. To begin, and to simplify our notation let $A \subset D \setminus \{1\}$, with elements labeled as previously, and let

$$Z_{1,A} = Z_{\{1\}\cup A} = Z_{1,a_1,a_2,\ldots,a_m} = \sum_{k_1, k_i \ge 1, i \in A} \frac{c^{k_1+\sum_{j\in A}(k_j)-1}\, t^{k_1 q_1+\sum_{j\in A}(k_j q_j)-1}}{\Gamma\big(k_1 q_1 + \sum_{j\in A} k_j q_j\big)}.$$

Now, with this notation, we may write statement (2.4) as

$$D^{q_1} Z_{1,A} = c(Z_A + Z_{1,A}).$$

To prove this we will utilize the fact that each $Z_{1,A}$ converges uniformly, thus we can differentiate term by term. Doing so we obtain,

$$D^{q_1} Z_{1,A} = \sum_{k_1, k_i \geq 1, i \in A} \frac{c^{k_1 + \sum_{j\in A}(k_j) - 1} \, t^{(k_1-1)q_1 + \sum_{j \in A}(k_j q_j) - 1}}{\Gamma\big((k_1 - 1)q_1 + \sum_{j\in A} k_j q_j\big)}.$$

Now we can split this series into two cases: the case where $k_1 = 1$ and then the series where $k_1 \geq 2$. So when $k_1 = 1$ we obtain the series

$$\sum_{k_i \geq 1, i \in A} \frac{c^{\sum_{j\in A}(k_j)} \, t^{\sum_{j\in A}(k_j q_j) - 1}}{\Gamma\big(\sum_{j\in A} k_j q_j\big)} = cZ_A.$$

From here we need only renumber the series for $k_1 \geq 2$ to show that

$$\begin{aligned} D^{q_1} Z_{1,A} &= cZ_A + \sum_{k_1, k_i \geq 1, i \in A} \frac{c^{k_1 + \sum_{j\in A}(k_j)} \, t^{k_1 q_1 + \sum_{j\in A}(k_j q_j) - 1}}{\Gamma\big(k_1 q_1 + \sum_{j \in A} k_j q_j\big)} \\ &= cZ_A + cZ_{1,A}. \end{aligned}$$

Using the same argument we can show that $D^{q_i} Z_{i,A} = cZ_A + cZ_{i,A}$ for each $i \in D$ and each $A \subset D \setminus \{i\}$.

□

Now that we have this convergence result it is routine to show that the continuous extensions of Z_A converge uniformly. Specifically, for each i, $t^{p_i} Z_{i,A}$ converges uniformly on J_0. In the following theorem we will need to evaluate this continuous extension at $t = 0$. Therefore, we present the following remark.

Remark 2.12. For each i,

$$t^{p_i} Z_i \big|_{t=0} = E_{q_i, q_i}(0) = \frac{1}{\Gamma(q_i)},$$

and for each nonempty $A \subset D \setminus \{i\}$,

$$t^{p_i} Z_{i,A}\big|_{t=0} = \sum_{k_i, k_j \geq 1, j \in A} \frac{c^{k_i + \sum_{\ell \in A}(k_\ell) - 1} \, t^{(k_i - 1)q_i + \sum_{\ell\in A}(k_j q_j)}}{\Gamma\big(k_i q_i + \sum_{\ell \in A} k_j q_j\big)}\Big|_{t=0} = 0,$$

since $(k_i - 1)q_i + \sum_{\ell\in A}(k_j q_j) > 0$.

We note that the family of functions Z_A retain many properties similar to those of the weighted Mittag-Leffler function for single-order fractional differential systems and the natural exponential for ordinary differential equations. Thus, until a more adequate name is coined, we have taken to calling these functions multi-order generalized Mittag-Leffler functions or multi-order generalized exponentials. We believe that these functions will be paramount in the study of multi-order fractional systems and in the future we plan to turn our attention to various other properties regarding them. For our current result we will use these multi-order functions to construct the comparison theorem for multi-order systems.

Theorem 2.13. *Let $v, w \in C_p(J, R^N)$ be lower and upper solutions of system* (2.2)*. Let $f \in C(J_0 \times R^N, R^N)$ and quasimonotone nondecreasing, and if f satisfies the following Lipschitz condition*

$$f_i(t,x) - f_i(t,y) \le L_i \sum_{k=1}^{N} (x_k - y_k), \tag{2.5}$$

when $x \ge y$, then $v(t) \le w(t)$ on J.

Proof. First we will consider the case when one of the inequalities in Definition 2.9 is strict. So suppose without loss of generality that $D^{q_i} w > f_i(t,w)$ and $w_i^0 > x_i^0$, then we claim that $w > v$ on J. To prove this, suppose to the contrary that the set

$$\omega = \bigcup_{i=1}^{N} \{t \in J \,:\, w_i(t) \le v_i(t)\}$$

is nonempty. Now let $\tau = \inf(\omega)$, then since $w^0 > v^0$ and by the continuity of w and v we can conclude that $w_i(\tau) = v_i(\tau)$ for some $i \in D$, for simplicity and without loss of generality suppose that this $i = 1$. So $w_1 \ge v_1$ on $(0,\tau)$, and thus $v_1 - w_1 \le 0$ on $(0,\tau]$, which by Lemma 2.6 implies that $D^{q_1} v_1 - w_1\big|_{t=\tau} \ge 0$. Further, since τ is the infimum we can also conclude that $w_j \ge v_j$ on $(0,\tau)$ for each $j > 1$. So applying this and the quasimonotonicity of f we have

$$\begin{aligned} f_1(\tau, v(\tau)) \ge D^{q_1} v_1\big|_{t=\tau} &\ge D^{q_1} w_1\big|_{t=\tau} > f_1(\tau, w(\tau)) \\ &= f_1(\tau, v_1(\tau), w_2(\tau), w_3(\tau), \dots, w_N(\tau)) \\ &\ge f_1(\tau, v(\tau)), \end{aligned}$$

which is a contradiction. Therefore $w > v$ on J.

Now we turn our attention to the case when both inequalities are non-strict. To begin we construct a collection of sets in the following manner, let

$$\varphi_i^m = \{A \subset D \,:\, i \in A, |A| = m\}.$$

That is, φ_i^m is the set of all subsets A of D with m components containing the i-th component. And let

$$\zeta_i^m = \sum_{A \in \varphi_i^m} Z_A.$$

That is, ζ_i^m is the sum of all unique possible Z_A functions where $|A| = m$ and each one contains the i-th component. Through this process we are able to eliminate the possibility of redundancies. That is, since $Z_{1,2,3} = Z_{3,2,1}$, using conventional enumerated sum notation would have yielded multiple copies of $Z_{1,2,3}$ in

$$\sum_{a_1=1}^{N}\sum_{a_2=1}^{N} Z_{1,a_1,a_2},$$

but using this ζ notation ensures that $Z_{1,2,3}$ will only appear once in ζ_1^3. We will also consider the sum of all possible unique Z functions made of m components and define it as a ζ with only a superscript as

$$\zeta^m = \sum_{A\subset N, |A|=m} Z_A.$$

Now, we will be utilizing the functions Z_A, with $c = NL$, and where $L = \max\limits_{1\le i\le N}\{L_i\}$. Letting $\varepsilon > 0$ we construct the function

$$\widehat{w}_i = w_i + \varepsilon\zeta_i^*,$$

where

$$\zeta_i^* = Z_i + \zeta_i^2 + \zeta_i^3 + \cdots + \zeta_i^{N-1} + Z_D,$$

that is, ζ_i^* is the sum of all unique possibilities of functions Z_A such that $i \in A$.

For our argument we wish to consider $\sum_{i=1}^N \zeta_i^*$, which we will denote as ζ^*. From here we note that for any $\{a_1, a_2\}$, it is an element of $\varphi_{a_1}^2 \cap \varphi_{a_2}^2$ and so Z_{a_1,a_2} is contained in both $\zeta_{a_1}^2$ and $\zeta_{a_2}^2$. Thus, every 2-component Z_{a_1,a_2} will appear in ζ^* twice. Similarly, for any 3-element $\{a_1, a_2, a_3\} \in \varphi_{a_1}^3 \cap \varphi_{a_2}^3 \cap \varphi_{a_3}^3$, Z_{a_1,a_2,a_3} will appear in ζ^* thrice. And more generally, for any m-component A, $A \in \bigcap_{k=1}^m \zeta_k^m$, and therefore Z_A will appear m-times in ζ^*. Therefore, we can give an explicit representation of ζ^* as

$$\zeta^* = \zeta^1 + 2\zeta^2 + 3\zeta^3 + \cdots + (N-1)\zeta^{N-1} + NZ_D.$$

Now we turn our attention to each q_i-th derivative of ζ_i^*. For simplicity we will only consider the q_1-st derivative of ζ_1^* since the argument for each component will be the same. For any $A \subset D \setminus \{1\}$ with $|A| = m$ we know from Lemma 2.11 that $D^{q_1} Z_{1,A} = NLZ_A + NLZ_{1,A}$, where A and $\{1\} \cup A$ are elements of $\bigcap_{a\in A} \varphi_a^m$ and φ_1^{m+1} respectively. Therefore, when we sum up every unique element Z_A, with A in φ_1^{m+1}, i.e. ζ_1^{m+1}, and compute the q_1-st derivative we will be left with $NL\zeta_1^{m+1}$ plus NL times every element found in ζ^m not including the elements of ζ_1^m. That is,

$$D^{q_1}\zeta_1^{m+1} = NL(\zeta^m - \zeta_1^m + \zeta_1^{m+1}),$$

and since A was arbitrary this will be true for any n, $1 \le m \le N-1$. For $m = 0$, we note that $\zeta_1^1 = Z_1$, and $D^{q_1} Z_1 = NLZ_1$. With this in hand we can show that

$$\begin{aligned} D^{q_1}\zeta_1^* &= NLZ_1 + NL(\zeta^1 - Z_1 + \zeta_1^2) + NL(\zeta^2 - \zeta_1^2 + \zeta_1^3) + \cdots \\ &\quad + NL(\zeta^{N-2} - \zeta_1^{N-2} + \zeta_1^{N-1}) + NL(\zeta^{N-1} - \zeta_1^{N-1} + Z_D) \\ &= NL(\zeta^1 + \zeta^2 + \zeta^3 + \cdots + \zeta^{N-1} + Z_D), \end{aligned}$$

which will hold for each q_i-th derivative.

Now, we wish to show that $\widehat{w}$ satisfies Definition 2.9 with strict inequalities. To begin, from the nature of w and Remark 2.12 we obtain

$$\widehat{w}_{p_i}(0) \geq x_i^0 + \varepsilon/\Gamma(q_i) > x_i^0,$$

for each i. From here we consider

$$\begin{aligned} D^{q_i}\widehat{w}_i &\geq f_i(t,w) + \varepsilon D^{q_1}\zeta_1^* \\ &\geq f_i(t,\widehat{w}) - L_i\sum_{j=1}^{N}(\widehat{w}_j - w_j) + \varepsilon D^{q_1}\zeta_1^* \\ &= f_i(t,\widehat{w}) + \varepsilon(D^{q_1}\zeta_1^* - L_i\zeta^*) \\ &\geq f_i(t,\widehat{w}) + \varepsilon L\sum_{k=1}^{N-1}(N-k)\zeta^k > f_i(t,\widehat{w}). \end{aligned}$$

We conclude the strict inequality in the final step since Remark 2.12 implies that $\zeta^1 > 0$ and each other $\zeta^k \geq 0$. So from our previous work with strict inequalities we have that $\widehat{w} > v$ on J, and letting $\varepsilon \to 0$ we obtain $w \geq v$, which finishes the proof. □

Now, if we know of the existence of lower and upper solutions v and w such that $v \leq w$, we can prove the existence of a solution in the set

$$\Omega = \{(t,y) \,:\, v(t) \leq y \leq w(t),\, t \in J\}.$$

We consider this result in the following theorem.

Theorem 2.14. *Let $v, w \in C_p(J, R^N)$ be lower and upper solutions of* (2.2) *such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, R^N)$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, R^N)$ of* (2.2) *such that $v(t) \leq x(t) \leq w(t)$ on J.*

This theorem is proved in the same way as seen in [4], with only minor additions to apply it to multi-order N-systems.

3. MONOTONE METHOD

In this section we will develop the monotone iterative technique for nonlinear multi-order systems of the type (2.2). In order to cover as many cases as possible we introduce the following generalizing concepts. For each fixed $i \in D$, let r_i, s_i be two nonnegative integers such that $r_i + s_i = N - 1$ so that we can split the vector x into $x = (x_i, [x]_{r_i}, [x]_{s_i})$. Then system (2.2) can be written as

$$D_t^{q_i} x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}), \quad x_{p_i}(0) = x_i^0. \tag{3.1}$$

We do this so that we can consider results in which f, for example, is nondecreasing in $[x]_{r_i}$ and nonincreasing in $[x]_{s_i}$. The specific case we consider for this paper is given in the following definition.

Definition 3.1. A function $f \in C(J_0 \times R^N, R^N)$ possesses a mixed quasimonotone property if for each i, $f_i(t, x_i, [x]_{r_i}, [x]_{s_i})$ is monotone nondecreasing in $[x]_{r_i}$ and monotone nonincreasing in $[x]_{s_i}$.

We note that this definition generalizes quasimonotone monotonicity defined above, since when $r_i = 0$, f is quasimonotone nonincreasing and when $s_i = 0$, f is quasimonotone nondecreasing. Further, this generalization allows us to consider various forms of upper and lower solutions, which we specifically define below.

Definition 3.2. Let $w, v \in C_p(J, R^N)$, w and v are coupled upper and lower quasisolutions of (3.1) if

$$D_t^q w_i \geq f_i(t, w_i, [w]_{r_i}, [v]_{s_i}), \quad w_{p_i}(0) = w_i^0 \geq x_i^0,$$

$$D_t^q v_i \leq f_i(t, v_i, [v]_{r_i}, [w]_{s_i}), \quad v_{p_i}(0) = v_i^0 \leq x_i^0.$$

On the other hand, w and v are coupled quasisolutions of (3.1) if

$$D_t^q w_i = f_i(t, w_i, [w]_{r_i}, [v]_{s_i}), \quad w_{p_i}(0) = x_i^0,$$

$$D_t^q v_i = f_i(t, v_i, [v]_{r_i}, [w]_{s_i}), \quad v_{p_i}(0) = x_i^0.$$

Further, one can define coupled extremal quasisolutions of (3.1) in the usual way.

Next we recall a theoretical existence result via coupled lower and upper solutions of (3.1) when f possesses a mixed quasimonotone property. We omit the proof, but note that it follows along the same line as in [12] with modifications as found in the proof of Theorem 2.14.

Theorem 3.3. *Let $v, w \in C_p(J, R^N)$ be coupled lower and upper quasisolutions of (3.1) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, R^N)$, where*

$$\Omega = \{(t, x) \in J_0 \times R^N \,:\, v_p \leq x_p \leq w_p\}.$$

If f posesses a mixed quasimonotone property, then there exists a solution $x(t)$ of (3.1) such that $v(t) \leq x(t) \leq w(t)$ on J, provided $v^0 \leq x^0 \leq w^0$.

Here we state our main result. Using coupled lower and upper solutions relative to (3.1), we construct sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that $t^p v_n$ and $t^p w_n$ converge uniformly and monitonically to $t^p v$ and $t^p w$ respectively. Where v and w are coupled minimal and maximal solutions of system (3.1).

Theorem 3.4. *Let $f \in C(J \times R^N, R^N)$ possess a mixed quasimonotone property and let v_0, w_0 be coupled lower and upper quasisolutions of system (3.1) such that $v_0 \leq w_0$ on J. Suppose f also satisfies the one-sided Lipschitz condition*

$$f_i(t, x_i, [x]_{r_i}, [x]_{s_i}) - f_i(t, y_i, [x]_{r_i}, [x]_{s_i}) \geq -M_i(x_i - y_i),$$

with $M_i \geq 0$, whenever $v_0^0 \leq x^0 \leq w_0^0$ and $v_0 \leq y \leq x \leq w_0$ on J. Then there exist monotone sequences $\{v_n\}, \{w_n\}$ such that

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w,$$

monotonically and uniformly on J_0, where v and w are coupled minimal and maximal quasisolutions of (3.1) provided $v_0^0 \le x^0 \le w_0^0$. Further if x is any solution of (3.1) such that $v_0 \le x \le w_0$ then $v \le x \le w$ on J.

Proof. To begin we note that the sequences we wish to construct are defined as the unique solutions of the following linear multi-order fractional systems

$$\begin{aligned} D^{q_i} v_{n+1_i} &= f_i(t, v_{ni}, [v_n]_{r_i}, [w_n]_{s_i}) - M_i(v_{n+1_i} - v_{ni}), \\ D^{q_i} w_{n+1_i} &= f_i(t, w_{ni}, [w_n]_{r_i}, [v_n]_{s_i}) - M_i(w_{n+1_i} - w_{ni}), \end{aligned} \tag{3.2}$$

where v^0 and w^0 are defined in our hypothesis. We would like to show that these sequences are monotone and that the weighted sequences converge uniformly. To do so we consider the more general multi-order system

$$\begin{aligned} &D^{q_i} y_i = f_i(t, \xi_i, [\xi]_{r_i}, [\eta]_{s_i}) - M_i(y_i - \xi_i), \\ &y_p(0) = x^0, \end{aligned} \tag{3.3}$$

with $v^0 \le \xi, \eta \le w^0$. We note that since (3.3) is linear that a unique solution exists in $C_p(J, R^N)$ for every particular choice of ξ and η. Therefore, we may construct a mapping F, such that $y = F[\xi, \eta]$ will output the unique solution of (3.3). With this mapping, we can define our sequences as

$$v_{n+1} = F[v_n, w_n], \quad w_{n+1} = F[w_n, v_n].$$

We claim that F is monotone nondecreasing in its first variable and nonincreasing in its second variable. To prove this, suppose that $v^0 \le \xi \le \mu \le w^0$ on J, and let $y = F[\xi, \eta]$ and $z = F[\mu, \eta]$. Now, using the quasimonotone property of f, along with the Lipschitz condition from our hypothesis we have for each i that

$$\begin{aligned} D^{q_i} z_i &\ge f_i(t, \mu_i, [\xi]_{r_i}, [\eta]_{s_i}) - M_i(z_i - \mu_i) \\ &= f_i(t, \xi_i, [\xi]_{r_i}, [\eta]_{s_i}) + f_i(t, \mu_i, [\xi]_{r_i}, [\eta]_{s_i}) - f_i(t, \xi_i, [\xi]_{r_i}, [\eta]_{s_i}) - M_i(z_i - \mu_i) \\ &\ge f_i(t, \xi_i, [\xi]_{r_i}, [\eta]_{s_i}) - M_1(z_1 - \xi_1). \end{aligned}$$

Now, since (3.3) is linear, it is Lipschitz of the form (2.5) and is quasimonotone nondecreasing, so by Theorem 2.13 $y \le z$ on J. This gives us that $F[\xi, \eta] \le F[\mu, \xi]$, implying that F is nondecreasing in its first variable as we claimed. Using a similar argument we can show that F is nonincreasing in its second variable.

From here we can show that the sequences (3.2) are monotone. We will begin by showing that $v_0 \le F[v_0, w_0]$ and $w_0 \ge F[w_0, v_0]$, to do so, let $v_1 = F[v_0, w_0]$, and then note that

$$D^{q_i} v_{1i} = f_i(t, v_{0i}, [v_0]_{r_i}, [w_0]_{s_i}) - M_i(v_{1i} - v_{0i}),$$

and because

$$D^{q_i} v_{0i} \le f_i(t, v_{0i}, [v_0]_{r_i}, [w_0]_{s_i}) - M_i(v_{0i} - v_{0i}),$$

we may apply Theorem 2.13 to show that $v_0 \le v_1$ on J. Similarly, $w_1 \le w_0$ on J. Next, by the monotonicity property of F we have that

$$v_1 = F[v_0, w_0] \le F[w_0, v_0] = w_1.$$

Therefore, $v_0 \le v_1 \le w_1 \le w_0$ on J. Using this as our inductive basis step suppose this is true for up to some $k \ge 1$, that is $v_{k-1} \le v_k \le w_k \le w_{k-1}$. Now, letting $v_{k+1} = F[v_k, w_k]$ and $w_{k+1} = F[w_k, v_k]$ and using the monotone property of F along with our induction hypothesis we have that

$$v_{k+1} = F[v_k, w_k] \ge F[v_{k-1}, w_{k-1}] = v_k,$$

and similarly we have that $w_{k+1} \le w_k$ on J. Finally, we can also show that on J

$$v_{k+1} = F[v_k, w_k] \le F[w_k, v_k] = w_{k+1}.$$

So, by induction we have that $v_{n-1} \le v_n \le w_n \le w_{n-1}$ for all $n \ge 1$ on J.

Now we wish to show that the weighted sequences $\{t^p v_n\}$ and $\{t^p w_n\}$ converge uniformly on J_0. To do so we will apply the Arzelá-Ascoli Theorem; therefore we must show these sequences are uniformly bounded and equicontinuous. For any $n \ge 0$ we submit that

$$|t^{p_i} v_{ni}| \le t^{p_i}\big(|v_{ni} - v_{0i}| + |v_{0i}|\big) \le t^{p_i}\big(|w_{0i} - v_{0i}| + |v_{0i}|\big),$$

implying that the sequence $\{t^p v_n\}$ is uniformly bounded. Noting that we can show a similar result for $\{t^p w_n\}$ we conclude that both weighted sequences are uniformly bounded. Now using this we can show that our weighted sequences are equicontinuous. First, for simplicity let

$$\widetilde{f_i}(t, v_n) = f_i(t, v_{n-1_i}, [v_{n-1}]_{r_i}, [w_{n-1}]_{s_i}) - M_i(v_{ni} - v_{n-1_i}),$$

for all $n \ge 1$, and noting that $\widetilde{f}$ is C_p continuous and that $\{t^p v_n\}$ is uniformly bounded, we can choose a $S \in R^N_+$ such that for each i

$$t^{p_i}\widetilde{f_i}(t, v_n) \le S_i$$

on J_0 for any $n \ge 1$. Now, choose t, τ such that $0 < t \le \tau \le T$. In the following proof of equicontinuity we use the fact that

$$\tau^{p_1}(\tau - s)^{q_1-1} - t^{p_1}(t - s)^{q_1-1} \le 0$$

for $0 < s < t$. To show why this is true, consider the function $\phi(t) = t^{p_1}(t-s)^{q_1-1} = t^{p_1}(t-s)^{-p_1}$ and note that

$$\begin{aligned} \tfrac{d}{dt}\phi(t) &= p_1 t^{p_1-1}(t-s)^{-p_1} - p_1 t^{p_1}(t-s)^{-p_1-1} \\ &= -t^{p_1-1}(t-s)^{-p_1-1} p_1 s \le 0. \end{aligned}$$

This implies that ϕ is nonincreasing, therefore $\phi(\tau)-\phi(t)\le 0$. Now for each i we obtain

$$\begin{aligned}
|\tau^{p_i}v_{ni}(\tau)-t^{p_i}v_{ni}(t)| &\le \frac{1}{\Gamma(q_i)}\int\limits_0^t |\tau^{p_i}(\tau-s)^{q_i-1}-t^{p_i}(t-s)^{q_i-1}||\widetilde{f}_i(t,v_n)|ds\\
&\quad+\frac{\tau^{p_i}}{\Gamma(q_i)}\int\limits_t^\tau (\tau-s)^{q_i-1}|\widetilde{f}_i(t,v_n)|ds\\
&\le \frac{S_i}{\Gamma(q_i)}\int\limits_0^t \left[t^{p_i}(t-s)^{q_i-1}-\tau^{p_i}(\tau-s)^{q_i-1}\right]s^{q_i-1}ds\\
&\quad+\frac{S_i\tau^{p_i}}{\Gamma(q_i)}\int\limits_t^\tau (\tau-s)^{q_i-1}s^{q_i-1}ds\\
&\le \frac{S_i t^{p_i}}{\Gamma(q_i)}\int\limits_0^t (t-s)^{q_i-1}s^{q_i-1}ds-\frac{N_i\tau^{p_i}}{\Gamma(q_i)}\int\limits_0^\tau (\tau-s)^{q_i-1}s^{q_i-1}ds\\
&\quad+\frac{2S_i\tau^{p_i}}{\Gamma(q_i)t^{p_i}}\int\limits_t^\tau (\tau-s)^{q_i-1}ds\\
&=\frac{S_i\Gamma(q_i)}{\Gamma(2q_i)}(t^{q_i}-\tau^{q_i})+\frac{2N_i\tau^{p_i}}{\Gamma(q_i)t^{p_i}}\frac{1}{q_i}(\tau-t)^{q_i}\\
&\le \frac{2N_1\tau^{p_1}}{\Gamma(q_1+1)t^{p_1}}(\tau-t)^{q_1}.
\end{aligned}$$

In the case that $t=0$, we note that

$$|\tau^{p_i}v_{ni}(\tau)-x_i^0/\Gamma(q_i)|\le \frac{S_i}{\Gamma(q_i)}\int\limits_0^\tau (\tau-s)^{q_1-1}ds=\frac{S_i}{\Gamma(q_i+1)}\tau^{q_i}.$$

Now, we can choose $K\in R_+^N$ such that

$$K_i\ge \tfrac{2S_i}{\Gamma(q_i+1)}\tfrac{T^{p_i}}{t^{p_i}}\ge \tfrac{S_i}{\Gamma(q_1+1)},$$

which we note is not dependent on n. Therefore, for each i we have that

$$|\tau^{p_i}v_{ni}(\tau)-t^{p_i}v_{ni}(t)|\le K_i|\tau-t|^{q_i},$$

for $0\le t\le\tau\le T$ and for all $n\ge 1$. This gives us that $\{t^pv_n\}$ is equicontinuous. Likewise, $\{t^pw_n\}$ is also equicontinuous. Therefore, by the Arzelá-Ascoli Theorem, there exist subsequences of both $\{t^pv_n\}$ and $\{t^pw_n\}$ that converge uniformly on J_0, and due to their monotonic nature the full sequences themselves also converge uniformly on J_0. Given this, suppose that $t^pv_n\to t^pv$ and $t^pw_n\to t^pw$ on J_0; we wish to show that v and w are extremal quasi-solutions of (2.2) on J. To do so, first note that $v_n \to v$ pointwise on J, and due to the nature of $\widetilde{f}$ we have that

$$t^{p_i} v_{ni} = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)} \int\limits_0^t (t-s)^{q_i-1} f_i(s, v_{n-1_i}[v_{n-1}]_{r_i}, [w_{n-1}]_{s_i}) ds$$
$$- \frac{t^{p_i}}{\Gamma(q_i)} \int\limits_0^t (t-s)^{q_i-1} M_i(v_{ni}(s) - v_{n-1_i}(s)) ds,$$

which converges uniformly on J_0 to

$$t^{p_i} v_i = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)} \int\limits_0^t (t-s)^{q_i-1} f_i(s, v_i, [v]_{r_i}, [w]_{s_i}) ds,$$

implying that

$$v_i = x_i^0 t^{q_i-1} + \frac{1}{\Gamma(q_i)} \int\limits_0^t (t-s)^{q_i-1} f_i(s, v_i, [v]_{r_i}, [w]_{s_i}) ds$$

on J, and thus that v is a quasisolution to (3.1). By a similar argument w is also a quasisolution to (3.1).

We will use induction to show that v and w are minimal and maximal quasisolutions. First, let x be a solution to (3.1), such that $v_0^0 \leq x^0 \leq w_0^0$. By Theorem 2.14 we know such a solution exists such that $v_0 \leq x \leq w_0$ on J. Given this, and using the monotonicity of F we have that

$$v_1 = F[v_0, w_0] \leq F[x, x] \leq F[w_0, v_0] = w_1,$$

which implies that $v_1 \leq x \leq w_1$ on J since $x = F[x, x]$. Using this as a basis step, we may apply the same steps used above again to inductively show that $v_n \leq x \leq w_n$ on J for all $n \geq 0$, thus implying that $v \leq x \leq w$ on J. This gives us that v and w are extremal quasisolutions and finishes the proof. $\square$

We note that if f satisfies a two-sided Lipshitz condition, then $v = x = w$ which will be the unique solution of (2.2).

In the future, we wish to turn our attention to further generalizations of the monotone method. Further, we note that the construction of numerical applications of this type is quite unwieldy, even for simple illustrative examples, but this is something we would like to pursue for N-systems in the course of time. From here, it would be compelling to study various physical models that would lend themselves to multi-order fractional systems. Further, the multi-order generalized exponentials yield potential for further study into broadening the study of multi-order fractional systems. Our hope is that this initial study may open the doors to further results in multi-order systems beyond the use of the Caputo derivative.

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