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ON SŁOTA-WITUŁA PROBLEM CONCERNING THE VALUE OF SOME DETERMINANTS

Summary. The paper presents the partial solution of a problem, formulated by Słota and Wituła, concerning the form of some determinants.

O PROBLEMIE SŁOTY-WITUŁY O POSTACI PEWNYCH WYZNACZNIKÓW

Streszczenie. W artykule podane jest częściowe rozstrzygnięcie problemu sformułowanego przez Słotę i Witułę o postaci pewnych wyznaczników.

In the paper entitled "On the sum of some alternating series" [1] Wituła and Słota discuss the sums of series of the form

$$S_{r,n} := \sum_{k=0}^{\infty} \left(\frac{1}{kr+n} - \frac{1}{(k+1)r-n} \right),$$

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where $r = 3, 4, \ldots$ and $n = 1, 2, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor$. By using the Fourier series theory for prime number $r = p \geqslant 3$ the Authors obtained in [1] the system of linear equations

$$\frac{\pi}{2p}(p-2k) = \sum_{n=1}^{\lfloor p/2 \rfloor} S_{p,n} \sin \frac{2kn\pi}{p} \tag{1}$$

for $1 \leqslant k \leqslant \lfloor p/2 \rfloor$.

Remark 1. Restricting the discussion on the system of equations (1) only to the prime numbers is associated with the fact that this system is determined only for p being prime numbers.

To obtain the values $S_{p,n}$ from the system of equations (1), according to the Cramer's rule one has to evaluate the expression

$$\det \Delta_p,$$

where

$$\Delta_p := \left[\sin \frac{2ij\pi}{r} \right]_{\lfloor p/2 \rfloor \times \lfloor p/2 \rfloor}.$$

Authors of the cited above paper presented there the formula

$$\det \Delta_p = \begin{cases} \left(-\frac{p}{4}\right)^{(p-1)/4}, & p \equiv 1 \pmod 4, \\ (-1)^{(p-3)/4} \left(\frac{p}{4}\right)^{(p-1)/4}, & p \equiv 3 \pmod 4, \end{cases} \tag{2}$$

which has been positively verified by them, with the aid of computer, for all prime numbers $p < 1051$. However, the following question arises:

Is the above formula correct for every prime number?

Before answering this question let us notice yet that the formula (2) can be presented in the following unified form

$$\det \Delta_p = (-1)^{(p-1)(p-3)/8} \left(\frac{p}{4}\right)^{(p-1)/4}. \tag{3}$$

It results from the fact that for $p = 4k+1$, where $k \in \mathbb{N}$, we have

$$(-1)^{(p-1)(p-3)/8} = (-1)^{k(2k-1)} = (-1)^k = (-1)^{(p-1)/4},$$

and for $p = 4k+3$, where $k \in \mathbb{N}$, we have

$$(-1)^{(p-1)(p-3)/8} = (-1)^{k(2k+1)} = (-1)^k = (-1)^{(p-3)/4},$$

which is compatible with relation (2).

Moreover, we note that the multiplier

$$(-1)^{(p-1)(p-3)/8}$$

is connected with the change of value of determinant of the matrix Δ_p, if we reverse the order of columns of this matrix (or its rows, respectively).

Formulated above question led us to investigate the matrices of form

$$\Delta_r := \left[\sin\frac{2ij\pi}{r}\right]_{\lfloor r/2\rfloor\times\lfloor r/2\rfloor},$$

where $r \geqslant 3$ is an odd number. If r would be an even number, then matrix Δ_r would be singular and this is the reason to restrict the discussion only for odd numbers r.

Our goal is to calculate the value $|\Delta_r|$. To this aim let us evaluate the expression

$$\Delta_r^2 = \left[\sin\frac{2ij\pi}{r}\right]_{\lfloor r/2\rfloor\times\lfloor r/2\rfloor}\cdot\left[\sin\frac{2ij\pi}{r}\right]_{\lfloor r/2\rfloor\times\lfloor r/2\rfloor} = \\ = \left[\sum_{k=1}^{\lfloor r/2\rfloor}\sin\frac{2ik\pi}{r}\sin\frac{2kj\pi}{r}\right]_{\lfloor r/2\rfloor\times\lfloor r/2\rfloor}.$$

We note that

$$\sum_{k=1}^{\lfloor r/2\rfloor}\sin\frac{2ik\pi}{r}\sin\frac{2kj\pi}{r} = \frac{1}{2}\sum_{k=1}^{\lfloor r/2\rfloor}\cos\frac{2(i-j)k\pi}{r} - \frac{1}{2}\sum_{k=1}^{\lfloor r/2\rfloor}\cos\frac{2(i+j)k\pi}{r}.$$

We use now the identity [2]:

$$\sum_{k=1}^{n}\cos k\theta = -\frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \tag{4}$$

which is true for $\theta \neq 2l\pi$, where $l \in \mathbb{Z}$.

Let us suppose that $i \neq j$. From identity (4) we get

$$\sum_{k=1}^{\lfloor r/2\rfloor}\sin\frac{2ik\pi}{r}\sin\frac{2kj\pi}{r} = -\frac{1}{4} + \frac{\sin(i-j)\pi}{4\sin\frac{(i-j)\pi}{r}} + \frac{1}{4} - \frac{\sin(i+j)\pi}{4\sin\frac{(i+j)\pi}{r}} = 0.$$

Now let us assume $i = j$. Then from identity (4) we obtain

$$\sum_{k=1}^{\lfloor r/2\rfloor}\sin\frac{2ik\pi}{r}\sin\frac{2kj\pi}{r} = \frac{r-1}{4} - \frac{1}{2}\sum_{k=1}^{\lfloor r/2\rfloor}\cos\frac{4ik\pi}{r} = \frac{r-1}{4} + \frac{1}{4} - \frac{\sin 2i\pi}{4\sin\frac{2i\pi}{r}} = \frac{r}{4}.$$

The above fact definitely shows that

$$\Delta_r^2 = \frac{1}{4} r \mathbb{1}_{\lfloor r/2 \rfloor},$$

where $\mathbb{1}$ denotes the identity matrix of the respective order.

Thus we may conclude that

$$|\det \Delta_r| = \left(\det \Delta_r^2\right)^{1/2} = \left(\det\left(\frac{1}{4} r \mathbb{1}_{\lfloor r/2 \rfloor}\right)\right)^{1/2} = \\ = \left(\left(\frac{r}{4}\right)^{(r-1)/2} \det \mathbb{1}_{\lfloor r/2 \rfloor}\right)^{1/2} = \left(\frac{r}{4}\right)^{(r-1)/4},$$

which partly confirms the posed above question (accurate to the absolute value).

References

1. Wituła R., Słota D.: *On the sum of some alternating series*. Comput. Math. Appl. **62** (2011), 2658–2664.
2. Knapp M.P.: *Sines and cosines of angles in arithmetic progression*. Math. Mag. **82**, no. 5 (2009), 371–372.