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CONTRACTIVE AND OPTIMAL SETS IN MUSIELAK-ORLICZ SPACES WITH A SMOOTHNESS CONDITION

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Abstract. In this paper we use our recent generalization of a theorem of Jamison-Kamińska-Lewicki (characterizing one-complemented subspaces in Musielak-Orlicz sequence spaces defined by Musielak-Orlicz functions satisfying a general smoothness condition) in order to compare contractive and optimal sets in finite-dimensional Musielak-Orlicz $\ell_\Phi^{(n)}$ spaces in the spirit of Kamińska-Lewicki. We also give an example illustrating the importance of the smoothness assumptions in our theorem.

Keywords: Musielak-Orlicz sequence spaces, one-complemented subspaces, contractive and optimal sets.

Mathematics Subject Classification: 46E30, 46B20.

1. INTRODUCTION

In a recent article [6] we obtained a generalization of the Jamison-Kamińska-Lewicki Theorem characterizing one-complemented subspaces in Musielak-Orlicz sequence spaces. Recall that a subspace Y of a Banach space X is *complemented* if there is a linear bounded projection $P\colon X \to Y$; if P can be chosen with norm 1, then Y is said to be *one-complemented*.

The notion of a one-complemented subspace is closely related to the geometry of the norm in X and norm one projections play a similar role in Banach spaces as orthogonal projections do in Hilbert spaces. One of the first characterization theorems in sequence spaces was obtained for ℓ_p by Baronti and Papini [1]. They showed that a subspace $Y \subset \ell_p$ (where $p \in [1, +\infty) \setminus \{2\}$) of codimension k is one-complemented iff it is the intersection of k hyperplanes defined by functionals having at most two non-zero coordinates. In [11, Theorem 2.7] Jamison, Kamińska and Lewicki obtained a similar characterization of one-complemented subspaces in Musielak-Orlicz ℓ_Φ assuming that the Musielak-Orlicz function Φ satisfies a smoothness condition (S) (Definition 3.1).

This permitted Kamińska and Lewicki to characterize in [12] the contractive sets in Musielak-Orlicz spaces with the condition (S) and to compare these sets with optimal sets. Condition (S), though not really restrictive, excludes such regular functions as t^p for $p \in [1, 2)$ (thus the Jamison-Kamińska-Lewicki Theorem does not not work for ℓ_p). In [6] we generalized Theorem 2.7 from [11] to the case of Musielak-Orlicz functions satisfying a smoothness condition (S') (Definition 3.4) and obtained an analogous characterization with a mixed condition (M).

After providing some necessary background, we present here first an example showing how important the smoothness assumptions in our main theorem from [6] (Theorem 3.7 in the present paper) are. Then we turn to the counterparts of the Kamińska-Lewicki results from [12] concerning contractive and optimal sets in Musielak-Orlicz spaces.

To be more precise, among the problems found in the non-linear theory of Banach spaces is the study of contractive projections, or *contractive sets*, i.e. sets admitting a contractive projection onto them (Definition 5.4). The latter is closely related to the notion of *optimal sets* (Definition 5.2) introduced by P. Enflo (cf. [8, 9] and [7]) and developed by B. Beauzamy in [2]. This is used to study the approximation in norm in Banach spaces. Actually, in this article we will compare contractive and optimal sets defined in a more general way using the *modular*, since our study is devoted to a wide class of Musielak-Orlicz sequence spaces. This will be done precisely using the main result of [6] and some arguments of Kamińska and Lewicki from [12]. Our results complete in some sense the results from [12] providing a characterization of strongly contractive sets in the sense of the modular ρ_Φ in a Musielak-Orlicz space $\ell_\Phi^{(n)}$ when Φ satisfies condition (M).

2. PRELIMINARIES

Let $(X, \|\cdot\|)$ be a real Banach space, X^* its dual. A functional $f \in X^*$ is called *a supporting functional* for $x_0 \in X \setminus \{0\}$ if $f(x_0) = \|x_0\|$ and $\|f\| = 1$. A point $x_0 \in X \setminus \{0\}$ is called *a smooth point* if there is exactly one supporting functional for x_0. If every point of the unit sphere S_X is smooth, then X is called smooth. We denote $Y^\perp := \{f \in X^* \colon f|_Y = 0\}$.

Let $Y \subset X$ be a closed subspace. We denote by $\mathcal{P}(X, Y)$ the space of bounded linear projections from X to Y. Observe that for $Y \neq \{0\}$ we get $\|P\| \geq 1$ for all $P \in \mathcal{P}(X, Y)$.

Definition 2.1. A closed subspace $Y \subset X$ is called *one-complemented* if there exists $P \in \mathcal{P}(X, Y)$ with $\|P\| = 1$.

For all this part we refer the reader to [11].

Definition 2.2. A convex function $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an *Orlicz function* when $\phi(0) = 0$ and ϕ is strictly increasing.

We denote by $\phi^*(t) := \sup_{s>0}\{st - \phi(s)\}$, $t \geq 0$, the Young conjugate of an Orlicz function φ.

Definition 2.3. A sequence $\Phi = (\phi_n)$ of Orlicz functions is called *a Musielak-Orlicz function*, if $\phi_n(1) = 1$ for all $n \in \mathbb{N}$. Then $\Phi^* := (\phi_n^*)$ is called the *conjugate Musielak-Orlicz function.*

If ℓ denotes the space of real sequences, then for a given Musielak-Orlicz function Φ we put

$$\rho_\Phi \colon \ell \ni x = (x_n) \mapsto \sum_{n=1}^{\infty} \phi_n(|x_n|) \in [0, +\infty].$$

Then we define the linear space

$$\ell_\Phi := \Big\{x \in \ell \colon \lim_{\lambda \to 0+} \rho_\Phi(\lambda x) = 0\Big\}.$$

Definition 2.4. The space ℓ_Φ is called *Musielak-Orlicz* (*sequence*) *space.* If $\phi_n = \phi$ for all n, then the space is called the *Orlicz* (*sequence*) *space* and we denote it by ℓ_ϕ.

The condition in the definition of ℓ_Φ is equivalent to

$$\exists \lambda > 0 \colon \rho_\Phi(\lambda x) < +\infty.$$

When we endow ℓ_Φ with the *Luxemburg norm*

$$\|x\|_\Phi = \inf\{\varepsilon > 0 \colon \rho_\Phi(x/\varepsilon) \le 1\},$$

we obtain a Banach space, cf. [17]. Of course $\|x\|_\Phi = \inf\{\varepsilon > 0 \colon x \in \varepsilon B\}$, where $B = \{z \in \ell_\Phi \colon \rho_\Phi(z) \le 1\}$.

We will denote by $\ell_\Phi^{(n)}$ the space defined analoguously to the previous one but taking only $x \in \mathbb{R}^n$. Of course, $\ell_\Phi^{(n)}$ is a subspace of ℓ_Φ. Finally, if (f_i) is a sequence in ℓ_Φ, we write $f_i = (f_{ij})$.

Definition 2.5. The subspace

$$h_\Phi := \{x \in \ell_\Phi \colon \rho_\Phi(\lambda x) < +\infty \text{ for all } \lambda > 0\}$$

is called *the subspace of finite elements.*

Obviously, $\ell_\Phi^{(n)} \subset h_\Phi$ for all $n \in \mathbb{N}$. It is known that h_Φ is closed and separable with canonical base $e_j := (0, \dots, 0, 1_{(j)}, 0, \dots)$. Moreover, for $x \in h_\Phi$, $\|x\|_\Phi = 1$ if and only if $\rho_\Phi(x) = 1$. Besides, $h_\Phi = \ell_\Phi$ exactly when either $\dim \ell_\Phi < +\infty$, or Φ satisfies a *growth condition* called δ_2 ([13, 14, 16]).

Definition 2.6. A Musielak-Orlicz function Φ satisfies condition δ_2, when there are constants $K, \delta > 0$ and a sequence $(c_n) \in \ell_1$ such that for all $n \in \mathbb{N}$ and $t \ge 0$ such that $\phi_n(t) \le \delta$,

$$\phi_n(2t) \le K\phi_n(t) + c_n.$$

This is always satisfied in $\ell_\Phi^{(n)}$. By [16, p. 148] and [10, Theorem 3.1], we have the following theorem.

Theorem 2.7. 1) *ℓ_Φ is reflexive if and only if both Φ and Φ^* satisfy δ_2.*
2) *ℓ_Φ is smooth if and only if Φ satisfies δ_2 and all ϕ_j are differentiable on $[0,1)$.*

For $y \in \ell_{\Phi^*}$ we define a bounded linear functional

$$f_y \colon \ell_\Phi \ni x \mapsto \sum_{n=1}^{\infty} x_n y_n \in \mathbb{R}.$$

Such functionals are called *regular* and their space is denoted $\mathcal{R}_\Phi$. By [10, 18], $\ell_{\Phi^*} \cong \mathcal{R}_\Phi$.

Functionals $f \in (\ell_\Phi)^*$ vanishing on h_Φ are called *singular* and their space is denoted $\mathcal{S}_\Phi$. By Lemma 1.1 and Theorem 2.9 from [10], for all $f \in (\ell_\Phi)^*$ there exist a uniquely determined $r(f) \in \mathcal{R}_\Phi$ and $s(f) \in \mathcal{S}_\Phi$ such that $f = r(f) + s(f)$ and $\|f\| = \|r(f)\| + \|s(f)\|$. The operators r and s are bounded linear projections on $\mathcal{R}_\Phi = \ell_{\Phi^*}$ and $\mathcal{S}_\Phi$, respectively.

Remark 2.8. Note that for $\ell_\Phi^{(m)}$, $\mathcal{S}_\Phi = \{0\}$, whence $(\ell_\Phi^{(m)})^* \cong \mathcal{R}_\Phi \cong \ell_{\Phi^*}^{(m)}$.

We will need a kind of 'normalization' of functionals $f_1, \ldots, f_n$ coming from a closed subspace Y of codimension n (i.e. a particular base of $Y^\perp$).

Definition 2.9. Let $Y \subset \ell_\Phi$ (or $\subset \ell_\Phi^{(m)}$) be a closed subspace of codimension n. Put $k = \dim r(Y^\perp) \leq n$. A base $F = \{f_1, \ldots, f_n\} \subset Y^\perp$ is called a *proper representation of Y*, if:

(1) $r(f_i)_j = \delta_{ij}$, for $i, j = 1, \ldots, k$,
(2) $r(f_i) = 0$, for $i \geq k+1$, when $k < n$.

Remark 2.10. Recall that $Y = \bigcap_{f \in F} \mathrm{Ker} f$. Condition (1) means that

$$r(f_i) = (0, \ldots, 0, 1_{(i)}, 0, \ldots, 0_{(k)}, r(f_i)_{(k+1)}, \ldots).$$

Condition (2) implies that, whenever $k < n$, there is $f_i \in \mathcal{S}_\Phi$ (i.e. $h_\Phi \subset \mathrm{Ker} f_i$) for $i > k$. In other words the first k vectors of the base of $Y^\perp$ when projected on $\mathcal{R}_\Phi$ 'looks like' the canonical base.

When ℓ_Φ coincides with h_Φ, then $\mathcal{S}_\Phi = \{0\}$ and $r = \mathrm{Id}_{(\ell_\Phi)^*}$. In that case a proper representation of Y is a base of $Y^\perp$ such that the first $k = n$ coordinates of its vectors form the canonical base of $\mathbb{R}^n$.

Lemma 1.8 from [11] guarantees the existence of a proper representation up to an isometry.

3. SMOOTHNESS CONDITIONS

The following definition goes back to [11].

Definition 3.1. An Orlicz function ϕ satisfies condition (s), if ϕ is differentiable on $[0, +\infty)$, $\phi(1) = 1$ and both ϕ and ϕ' vanish only at zero. If, moreover, ϕ' is differentiable on $[0,+\infty)$, ϕ'' is continuous and vanishes only at zero, then we say that ϕ satisfies condition (S).

We say that a Musielak-Orlicz function Φ satisfies (s) or (S), whenever all its coordinates satisfy the said condition.

Note that (S) implies that the coordinates of the Musielak-Orlicz function Φ are strictly convex, which in turn means that ℓ_Φ is strictly convex ([12, Theorem 1.2]). However, $\ell_\Phi^{(m)}$ is strictly convex already under weaker assumptions:

Proposition 3.2. *Assume that the Orlicz functions ϕ_j are strictly convex on $(0,1)$ and $\phi_j(1)=1$, $j=1,\dots,m$. Then $\ell_\Phi^{(m)}$ is strictly convex.*

Proof. See [6, Proposition 4.2]. □

Remark 3.3. Theorem 1.5 from [12] implies that in case $\ell_\Phi = h_\Phi$ and Φ satisfies (s), the space ℓ_Φ is smooth (cf. Theorem 2.7). In particular this holds for $\ell_\Phi^{(m)}$ under no other assumptions than (s).

Definition 3.4. We say that an Orlicz function ϕ satisfies condition (S'), if it satisfies (s), is of class $\mathscr{C}^2$ on $(0,+\infty)$ and

$$\lim_{t\to 0^+} \phi''(t) = +\infty.$$

A Musielak-Orlicz function $\Phi = (\phi_1,\phi_2,\dots)$ is said to satisfy (S'), if this condition is satisfied by all the coordinates ϕ_j.

Definition 3.5. We say that an Orlicz function ϕ satisfies condition (w), if it is two times differentiable and $\phi''(t) > 0$ for $t \in (0,\phi^{-1}(1)]$. A Musielak-Orlicz function $\Phi = (\phi_1,\phi_2,\dots)$ satisfies (w), if all the coordinates ϕ_j satisfy it.

Therefore, an Orlicz function ϕ satisfying both conditions (S') and (w) is strictly convex on $(0,1)$ ((s) implies $(0,\phi^{-1}(1)] = (0,1]$).

By (S'), there is an $\varepsilon > 0$ such that $\phi'' > 0$ on $(0,\varepsilon]$, hence (w) is intended to guarantee the possibility of taking $\varepsilon = 1$.

For a given Musielak-Orlicz function $\Phi = (\phi_n)_n$ we may introduce the *mixed* condition (M):

for any $n \in \mathbb{N}$, ϕ_n satisfies either (S), or (S') with (w).

Remark 3.6. In view of Remark 4.9 from [6] may be weakened to be condition (S) with (w) i.e. we assume the class $\mathscr{C}^2$ on $[0,+\infty)$ (with right-hand side derivatives at zero) together with the conditions (s) and (w) as well as the vanishing of the second derivative at zero (but we do not ask it to be non-zero apart from $(0,1)$).

The following theorems are the main result of [6]. They generalize one of the main results from [11] and will be the most important ingredient of the proofs from sections 6 and 7.

Theorem 3.7 ([6]). *Assume that the Musielak-Orlicz function Φ satisfies condition (M) and let $Y \subset \ell_\Phi^{(m)}$ be a codimension $k \leq m-2$ $(m \geq 3)$ one-complemented subspace. Let $f_1, \ldots, f_k \in Y^\perp$ be a proper representation of Y. Then each f_j has at most two non-zero coordinates.*

Of course, for $k > m-2$ the theorem is trivial.

Theorem 3.8 ([6]). *Assume that the Musielak-Orlicz function Φ satisfies the condition (M). If $Y \subset \ell_\Phi$ is a codimension k one-complemented subspace with a proper representation $F \subset Y^\perp$, then for each $f \in F$ there is $f = r(f)$ and this functional has at most two coordinates $\neq 0$.*

4. EXAMPLE

In this part we present an example showing that under the assumption that one of the Orlicz functions ϕ_j vanishes somewhere apart from zero (in particular the given Musielak-Orlicz function $\Phi = (\phi_1, \ldots, \phi_n)$ satisfies neither (S), nor (S')), then the Musielak-Orlicz space can contain one-complemented subspaces defined by functionals whose coordinates are different from zero.

Let $\phi_2, \ldots, \phi_n$ $(n \geq 2)$ be Orlicz functions satisfying $\phi_j(1) = 1$. Assume that $\phi_1 \colon \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and satisfying $\phi_1(1) = 1$ and $\phi_1 \equiv 0$ on $[0, \varepsilon]$ for some $\varepsilon \in (0,1)$. Put $\Phi := (\phi_1, \ldots, \phi_n)$ and consider the Musielak-Orlicz space $\ell_\Phi^{(n)}$. Although $\rho_\Phi(x) = 0$ not only at zero, but also for $x = (x_1, 0, \ldots, 0)$ with $x_1 \in [-\varepsilon, \varepsilon]$, the Luxemburg norm defined by ρ_Φ is an actual norm. Indeed, it is sufficient to check that $\|\cdot\|_\Phi$ vanishes only at zero. Suppose that $\|x\|_\Phi = 0$ and $x_1 \neq 0$. Then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, there is $\lambda|x_1| > 1$. For large λ,

$$1 \geq \rho_\Phi(\lambda x) = \sum_{j=1}^{n} \phi_j(\lambda |x_j|) \geq \phi_1(\lambda |x_1|) > 1,$$

which is a contradiction.

Proposition 4.1. *In the setting introduced above, take $f = (1, f_2, \ldots, f_n)$ a functional for which $\sum_{j=2}^n |f_j| < \varepsilon$. Then* Ker$f$ *is one-complemented in $\ell_\Phi^{(n)}$.*

Proof. Put $P(x) := x - f(x)e_1$, $x \in \ell_\Phi^{(n)}$. It is easy to see that $P \colon \ell_\Phi^{(n)} \to \mathrm{Ker} f$ is a linear projection. Observe that

$$P(x) = \left(-\sum_{j=2}^{n} f_j x_j, x_2, \ldots, x_n \right), \quad x \in \ell_\Phi^{(n)}.$$

The condition $\|x\|_\Phi = 1$ means that $\rho_\Phi(x) \leq 1$. This in turn implies that for all $j = 1, \ldots, n$, $\phi_j(|x_j|) \leq 1$, whence (in view of $\phi_j(1) = 1$, ϕ_j increasing) $|x_j| \leq 1$, $j = 1, \ldots, n$. Thence

$$\phi_1(|P(x)|) \leq \phi_1\left(\sum_{j=2}^{n} |f_j| \cdot |x_j| \right) \leq \phi_1\left(\sum_{j=2}^{n} |f_j| \right) \leq \phi_1(\varepsilon) = 0.$$

Therefore,

$$\rho_\Phi(P(x)) = \sum_{j=2}^{n} \Phi_j(|x_j|) \leq \rho_\Phi(x) \leq 1,$$

i.e. $\|P(x)\|_\Phi \leq 1$, whence $\|P\| \leq 1$. But P being a projection, we get $\|P\| = 1$ which proves the result. □

5. CONTRACTIVE AND OPTIMAL SETS

The following definition was introduced by P. Enflo in order to study approximation in norm in Banach spaces (cf. [5]). Let $(X, \|\cdot\|)$ be a normed space and $A \subset X$ a nonempty set.

Definition 5.1. A point $x \in X$ is *minimal for* A, if there is no other point lying closer to any point of A, i.e. for all $y \in X$,

$$(\forall a \in A : \|y - a\| \leq \|x - a\|) \quad \Rightarrow y = x.$$

The set of minimal points for A is denoted $Min(A)$ and called *the minimal set of* A. Obviously,

$$Min(A) = \{x \in X \mid \forall y \in X \setminus \{x\}\ \exists a \in A : \|x - a\| < \|y - a\|\}.$$

In [5] it is shown that a Banach space X is strictly convex if and only if for any two distinct points $x, y \in X$, $Min(\{x, y\})$ coincides with the segment $[x, y]$. Clearly, $A \subset Min(A)$ always holds.

Definition 5.2. A is called an *optimal set*, if $A = Min(A)$.

Iterating the operation Min usually increases the set (cf. [5]). Minimality can be characterized in the following manner.

Lemma 5.3. *If there is a projection $P\colon X \to A$ (i.e. $P|_A = \mathrm{Id}_A$) such that*

$$\|P(x) - a\| \leq \|x - a\| \quad \textit{for all} \quad x \in X, a \in A,$$

then $A = Min(A)$. The converse holds in reflexive, strictly convex Banach spaces ([3]). In particular, a closed subspace of such a space is one-complemented, if and only if it is an optimal set.

Proof. The second part of the statement can be found in [5]. For the first one take $m \in Min(A)$. If $m \notin A$, then $P(m) \neq m$, whence for some $a \in A$, $\|m - a\| < \|P(m) - a\|$, which is a contradiction. □

The preceding lemma is most useful when coupled with the following definition.

Definition 5.4. A is called a *contractive set*, if there exists a contractive projection $P\colon X \to A$, i.e. a mapping P satisfying $P|_A = \mathrm{Id}_A$ and $\|P(x) - P(y)\| \leq \|x - y\|$ for all $x, y \in X$.

By continuity, a contractive set is closed, $A = \overline{A}$. Actually, we have a more general result.

Proposition 5.5. *Let X be a strictly convex Banach space. Then each contractive set $A \subset X$ is closed and convex.*

Proof. Let $a_1, a_2 \in A$ and $x \in [a_1, a_2]$ and let P be the contractive projection onto A. Then for $j = 1, 2$,

$$\|P(x) - P(a_j)\| = \|P(x) - a_j\| \leq \|x - a_j\|.$$

By [5], $[a_1, a_2] = Min(\{a_1, a_2\})$, whence $P(x) = x$ due to the definition of a minimal set. □

Remark 5.6. Lemma 5.3 implies that each contractive set is optimal (the converse is true in smooth, reflexive, strictly covex Banach spaces, cf. [5]). Of course, one-complemented subspaces are contractive sets.

Definition 5.7. A is called *a set of existence of the best coapproximation* (shortly: *an existence set*), if for all $x \in X$, the set

$$R_A(x) := \{d \in A \,|\, \forall a \in A\colon \|d - a\| \leq \|x - a\|\}$$

is nonempty.

Remark 5.8. It is easy to see that $R_A(x) \neq \varnothing$ for $x \in A$ (then $x \in R_A(x)$). Besides,

$$x \in Min(A) \text{ and } R_A(x) \neq \varnothing \Rightarrow R_A(x) = \{x\}$$

and then $x \in A$.

We present the relations between the introduced notions.

Proposition 5.9. *Let A be a nonempty subset of X. Then*

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4),$$

where:

(1) *A is a one-complemented subspace,*
(2) *A is a contractive set,*
(3) *A is an existence set,*
(4) *A is an optimal set.*

Moreover, if $A \neq \{0\}$ is a closed subspace in a smooth, reflexive, strictly convex X, then by [5, *II.5*] *there is also* $(4) \Rightarrow (1)$.

Proof. The first implication follows from Remark 5.6. For the second one, observe that $P(x) \in R_A(x)$, where P is the contractive projection from (2). For the third one, $x \in Min(A)$ implies by (3) the existence of a point $d \in R_A(x)$. Then the description of this set and the definition of a minimal set yield $x = d \in A$. □

6. AUXILIARY RESULTS AND GENERALIZATIONS

We define the *support* of a sequence $x = (x_i)$ to be

$$\text{supp } x := \{i \in \mathbb{N} \colon x_i \neq 0\}.$$

For the convenience of the reader we recall the following two results from [11] (Theorems 2.5 and 2.6). We stress the fact that both theorems remain true when the original condition (S) is replaced with condition (s); therefore, both assertions hold true when we assume that condition (M) is satisfied.

Theorem 6.1. *Let Φ be a Musielak-Orlicz function satsifying (s) and $Y \subset \ell_\Phi^{(n)}$ a codimension k subspace with proper representation $\{f_1, \ldots, f_k\}$. Set*

$$J := \{i \in \{1, \ldots, k\} \colon \text{supp} f_i = \{i\}\}$$

and for $j \geq k+1$,

$$C_j := \{i \in \{1, \ldots, k\} \colon f_{ij} \neq 0\}, \quad J_1 := \{j \geq k+1 \colon C_j \neq \varnothing\}.$$

Then Y is one-complemented if and only if either $J = \{1, \ldots, k\}$, or for any $j \in J_1$, $Y_j := \bigcap_{i \in C_j} \text{Ker} f_i$ is one-complemented.

Theorem 6.2. *Let Φ be a Musielak-Orlicz function satisfying (s) and $Y \subset \ell_\Phi^{(n)}$ a codimension k subspace with proper representation $\{f_1, \ldots, f_k\}$ and such that $\text{supp} f_i = \{i, k+1\}$, for $i = 1, \ldots, k$. Set $z := e_{k+1} - \sum_{i=1}^k f_{i,k+1} e_i$ and $A := 1/\|z\|_\Phi$. Then Y is one-complemented if and only if there exists numbers $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$ such that for any $t \in [0, A]$ and any $i \in \{1, \ldots, k\}$ the following equation is satisfied:*

$$\left(\phi_{k+1}(t) + \sum_{j=1}^{k} \phi_j(t|f_{j,k+1}|) \right) b_i = \frac{\phi_i(t|f_{i,k+1}|)}{f_{i,k+1}}.$$

In that case, the projection $P(x) = x - \sum_{i=1}^k f_i(x) y_i$, where $y_i := \sum_{j=1}^{k+1} y_{ij} e_j$ with $y_{i,k+1} := b_i$, $y_{ij} := -f_{j,k+1} b_i$, when $i \neq j$, and $y_{ii} = 1 - f_{i,k+1} b_i$, has norm one.

As already said, in $\ell_\Phi^{(n)}$ the norm is constructed using the function ρ_Φ which is a convex *modular* (cf. [12]), i.e. a real-valued, non-negative, symmetric, convex function vanishing only at zero (it is thence a kind of pseudo-distance, however, not even a semi-norm in general). Therefore, such notions as that of an optimal set (Definition 5.2), contractive set (Definition 5.4), or existence set (Definition 5.7), can be generalized by replacing in each definition (in our situation it will be done for the space $\ell_\Phi^{(n)}$) the norm $\|\cdot\|_\Phi$ with the modular ρ_Φ. Then we shall use the following notations for a given subset $A \subset \ell_\Phi^{(n)}$:

$$Min_{\rho_\Phi}(A) := \{x \in \ell_\Phi^{(n)} \mid \forall y \in \ell_\Phi^{(n)} \setminus \{x\}\ \exists a \in A\colon \rho_\Phi(x-a) < \rho_\Phi(y-a)\},$$
$$R_A^{\rho_\Phi}(x) := \{d \in A \mid \forall a \in A\colon \rho_\Phi(d-a) \leq \rho_\Phi(x-a)\}.$$

Of course, analoguous definitions can be introduced in ℓ_Φ spaces. We will now speak of contractive, optimal sets etc. in the sense of the modular, i.e. of ρ_Φ-contractive, ρ_Φ-optimal etc. sets.

In [12] the characterization of contractive sets in ℓ_p spaces ($p \in (1, +\infty)$) given by Davis and Enflo was extended to Musielak-Orlicz spaces with condition (S). This required, however, a strengthening of the previously introduced notions (cf. [12]), namely: let $\varnothing \neq A \subset \ell_\Phi$ or $\subset \ell_\Phi^{(n)}$; we denote by X_Φ the considered space.

Definition 6.3. The set A is said to be *strongly ρ_Φ-optimal*, if there is $A = SMin_{\rho_\Phi}(A)$, where $SMin_{\rho_\Phi}(A)$ is defined to be

$$\{x \in X_\Phi \mid \forall y \in X_\Phi \setminus \{x\}\ \exists a \in A\ \exists t \geq 0\colon\ \rho_\Phi(t(y-a)) > \rho_\Phi(t(x-a))\}.$$

Definition 6.4. The set A is called a *strongly ρ_Φ-existence* set, if for any $x \in X_\Phi$, the set

$$SR_A^{\rho_\Phi}(x) = \{d \in A \mid \forall a \in A\ \forall t \geq 0\colon \rho_\Phi(t(d-a)) \leq \rho_\Phi(t(x-a))\}$$

is nonempty.

Definition 6.5. The set A is said to be *strongly ρ_Φ-contractive*, if there exists a projection $P\colon X_\Phi \to A$ (i.e. $P|_A = \mathrm{Id}_A$) such that

$$\rho_\Phi(t(P(x) - P(y))) \leq \rho_\Phi(t(x-y)) \quad \text{for all} \quad x, y \in X_\Phi, t \geq 0.$$

Such a projection P is called *strongly ρ_Φ-contractive.*

Remark 6.6. Of course, in the case when A is a linear subspace and P is linear too, one can get rid of the parameter t in the preceding definition. In other words, strong ρ_Φ-contractiveness is then identical with ρ_Φ-contractiveness.

Observe that by replacing in any of the preceding definitions the modular ρ_Φ with the norm $\|\cdot\|_\Phi$, we recover the definitions from Section 5. Besides, it follows from the definition of the norm, that if $\rho_\Phi(tx) \leq \rho_\Phi(ty)$, for any $t \geq 0$, then $\|x\|_\Phi \leq \|y\|_\Phi$. Therefore, each set which is strongly ρ_Φ-contractive (or existence) is also a contractive (or existence) set in the sense of the modular or norm. In particular, if A is a linear subspace being strongly ρ_Φ-contractive, then the projection attached to it has norm one. More results from that theory in the most general setting of modular spaces can be found in [12].

Using Theorem 3.7 and the notions introduced above we can adjust the proof of Theorem 3.1 from [11] in order to obtain the following result for Musielak-Orlicz spaces with condition (M).

Theorem 6.7. *Let Φ be a Musielak-Orlicz function satisfying (M) and $Y \subset \ell_\Phi^{(n)}$ a codimension k subspace. Let $g_i \in \ell_{\Phi^*}$ be such that $Y = \bigcap_{i=1}^k \mathrm{Ker} g_i$. Then if for any $i = 1, \ldots, k$, the kernel $\mathrm{Ker} g_i$ is strongly ρ_Φ-contractive in $\ell_\Phi^{(n)}$, then there exists a strongly ρ_Φ-contractive projection onto Y with norm one.*

Before the proof we recall Lemma 1.2 from [11].

Lemma 6.8. *If* $\mathrm{codim} Y = n$*, $\{f_1, \ldots, f_n\}$ is a basis for $Y^\perp$ and $P \in \mathcal{P}(X, Y)$, then there exists a uniquely determined basis $\{w_1, \ldots, w_n\}$ of* $\mathrm{Ker} P$ *such that*

$$f_i(w_j) = \delta_{ij} \quad \text{and} \quad Px = x - \sum_{j=1}^n f_j(x) w_j \ \text{ for } x \in X.$$

Proof of Theorem 6.7. First, we check that in the considered situation $Y_i := \mathrm{Ker} g_i$ are one-complemented. If this is the case, we can adjust the argument from [11] using Theorem 3.7.

Thus, fix $i \in \{1, \ldots, n\}$ and denote by P_i the projection from the definition of the strong ρ_Φ-contractiveness. It can be chosen linear (cf. the proof of Lemma 4.7 from [12]), because in virtue of Lemma III.2 from [5] (The space in consideration is smooth, cf. Remark 3.3), for any $x \in \ell_\Phi^{(n)}$, the set $R_{Y_i}(x)$ consists of a single element and $R_{Y_i}(\alpha x + \beta x') = \alpha R_{Y_i}(x) + \beta R_{Y_i}(x')$ for $\alpha, \beta \in \mathbb{R}$ and points $x, y \in \ell_\Phi^{(n)}$. Due to that, $P_i(x) := z \in R_{Y_i}(x)$ is a linear projection. Now, since $SR_{Y_i}^{\rho_\Phi}(x) \subset R_{Y_i}(x)$ and by the assumptions, $SR_{Y_i}^{\rho_\Phi}(x) \neq \varnothing$ (because Y_i is a strongly ρ_Φ-existence set), then the projection is strongly ρ_Φ-contractive.

For $y = 0$, $t = 1$, we obtain $\rho_\Phi(P_i(x)) \leq \rho_\Phi(x)$ for any $x \in \ell_\Phi^{(n)}$. If $\|x\|_\Phi = 1$, then from the definition of the Luxemburg norm it follows that $\rho_\Phi(x) \leq 1$ (since there exists a sequence $\varepsilon_\nu \to 1^+$ such that $\rho_\Phi(x/\varepsilon_\nu) \leq 1$ and the function $t \mapsto \rho_\Phi(tx)$ is continuous). Therefore, $\rho_\Phi(P_i(x)) \leq 1$, which implies $\|P_i(x)\|_\Phi \leq 1$. Hence $\|P_i\| \leq 1$, but since this is a projection, we finally get $\|P_i\| = 1$.

We have just proved that the assumptions of Theorem 3.1 from [11] are satisfied, and we know this theorem holds true with the condition (M) (because its proof is based either on some results of [11] which require only the condition (s), or on results we know by [6] are true with condition (M)). We can now repeat one part of the proof of this theorem obtaining the linear independence of the vectors w_i which appear in the formula for the projections: $P_i(x) = x - g_i(x) w_i$, $i = 1, \ldots, n$ (cf. Lemma 6.8). This allows us to define a projection onto Y by setting $P(x) := x - \sum_{i=1}^n g_i(x) w_i / n$. Then

$$\begin{aligned}
\rho_\Phi(P(x)) &= \rho_\Phi\left(\frac{nx}{n} - \sum_{i=1}^n \frac{g_i(x) w_i}{n} \right) = \\
&= \rho_\Phi\left(\sum_{i=1}^n \left(\frac{x}{n} - \frac{g_i(x) w_i}{n} \right) \right) = \\
&= \rho_\Phi\left(\sum_{i=1}^n \frac{P_i(x)}{n} \right) \leq \sum_{i=1}^n \frac{\rho_\Phi(P_i(x))}{n},
\end{aligned}$$

by the convexity of ρ_Φ. Since P_i are ρ_Φ-contractive, then

$$\sum_{i=1}^{n} \frac{\rho_\Phi(P_i(x))}{n} \leq \sum_{i=1}^{n} \frac{\rho_\Phi(x)}{n} = \rho_\Phi(x),$$

which means that P is ρ_Φ-contractive, too. But as it is also linear, P is actually strongly ρ_Φ-contractive. $\square$

Remark 6.9. In the proof above we have shown that for a given linear subspace $Y \subset \ell_\Phi^{(n)}$ the following implication holds:

$$\exists P \in \mathcal{P}(\ell_\Phi^{(n)}, Y)\ \forall x\colon \rho_\Phi(P(x)) \leq \rho_\Phi(x) \Rightarrow \exists P \in \mathcal{P}(\ell_\Phi^{(n)}, Y)\colon \|P\| = 1.$$

Moreover, when Y is generated by functionals whose kernels are strongly ρ_Φ-contractive, this implication can be reversed (in general, however, it is impossible, cf. (1) from the Proposition 7.6 presented later on).

7. COUNTERPARTS OF THE KAMIŃSKA-LEWICKI RESULTS

Theorem 7.1. *Let Φ be a Musielak-Orlicz function satisfying (M) and $Y \subset \ell_\Phi^{(n)}$ $(n \geq 2)$ a linear subspace of codimension k. Then Y is strongly ρ_Φ-contractive if and only if there are $f_1, \ldots, f_k \in \ell_{\Phi^*}^{(n)}$ such that $Y = \bigcap_{j=1}^{k} \mathrm{Ker} f_j$ and all the kernels here are ρ_Φ-contractive.*

Proof. The sufficiency of the condition above follows from Theorem 6.7 asserting that Y is strongly ρ_Φ-contractive.

The proof of the necessity is similar to that given in [12] Theorem 4.9. We recall shortly the major steps:
Take a proper representation of Y (this is always possible up to an isometry which does not affect the ρ_Φ-contractivity). In view of the smoothness of the considered space, similarly as in the proof of Theorem 6.7, we can find a linear strongly ρ_Φ-contractive projection P onto Y, with norm 1. By Theorem 3.7, any f_i has at most one non-zero coordinate apart from the 1 appearing on the i-th position. Of course, if $f_i = e_i$, then $\mathrm{Ker} f_i$ is strongly ρ_Φ-contractive.

We suppose thus that $f_i = e_i + f_{ij} e_j$ for some $f_{ij} \neq 0$, $j \geq k+1$. Define now Φ_s for $s > 0$ by putting $\phi_{j,s}(t) := s\phi_j(t)$. In view of Lemma 2.4 (ii) from [12], for any $s > 0$, the projection P has norm $\|\cdot\|_{\Phi_s}$ one. Therefore, by Theorem 6.1, for any $j \in J_1$ there exists a linear projection P_j onto $Y_{C_j} := \bigcap_{i \in C_j} \mathrm{Ker} f_i$ of norm 1, treated as an operator of the space $\ell_{\Phi_s}^{(n)}$ with any $s > 0$. We may assume that $C_j = \{1, \ldots, k\}$. It is easy to check that

$$\lim_{s \to 0^+} \|(-f_{1,k+1}, \ldots, f_{k,k+1}, 0, \ldots)\|_{\Phi_s} = 0$$

and so for $i \in C_j$ there is $b_i \neq 0$ such that the equations from Theorem 6.2 are satisfied with any $t > 0$.

Now we can restrict our considerations to $i = 1$ (the problem being symmetrical); comparing side by side the equations from Theorem 6.2 for $l = 1, \dots, k$, we obtain

$$b_1 f_{1,k+1} \phi_l(t|f_{l,k+1}|) = b_l f_{l,k+1} \phi_1(t|f_{1,k+1}|),$$

which inserted into the first equation yields

$$\big(\phi_1(t|f_{1,k+1}|) + \phi_{k+1}(t)\big) b_1 f_{1,k+1} = \phi_1(t|f_{1,k+1}|)\left(1 - \sum_{r=2}^{k} f_{r,k+1} b_r\right).$$

Hence, applying Theorem 6.2 to the space $\mathrm{Ker} f_1$ we obtain a linear projection P_1 of norm one, from $\ell_\Phi^{(n)}$ onto $\mathrm{Ker} f_1$. But since the equation obtained above is valid for any $t > 0$, then P_1 has norm one also as a projection from $\ell_{\Phi_s}^{(n)}$. Therefore, by Lemma 2.4 (ii) from [12], it is a ρ_Φ-contractive projection. This ends the proof. □

Let X be a linear space. We will call a *half-space* a set $H \subset X$ defined by a hyperplane Y in the following manner: $H = \{x \in X\colon f(x) \geq 0\}$, where $f \in X^* \setminus \{0\}$ is such that $Y = \mathrm{Ker} f$. We recall Theorem 1.2 I from [12].

Theorem 7.2. *The space $\ell_\Phi^{(n)}$ $(n \geq 2)$, or ℓ_Φ, is modularly strictly convex (i.e. ρ_Φ satisfies $\rho_\Phi((x+y)/2) < (\rho_\Phi(x) + \rho_\Phi(y))/2$, for $x \neq y$ such that $\rho_\Phi(x) = \rho_\Phi(y)$) if and only if all the functions ϕ_j, except at most one, are strictly convex.*

The following theorem is a generalization of Theorem 4.10 from [12].

Theorem 7.3. *Let Φ be a Musielak-Orlicz function satisfying (M) and $C \subset \ell_\Phi^{(n)}$ a convex set. Then:*

(1) *If C is a strongly ρ_Φ-existence set, then C is the intersection of at most countably many half-spaces defined by strongly ρ_Φ-contractive hyperplanes.*
(2) *If all the ϕ_j except possibly one are strictly convex, then the following conditions are equivalent:*
   a) *C is a strongly ρ_Φ-existence set;*
   b) *C is a strongly ρ_Φ-contractive;*
   c) *C is the intersection of at most countably many half-spaces defined by strongly ρ_Φ-contractive hyperplanes;*
   d) *C is a strongly ρ_Φ-optimal set.*

Proof. The proof of the first assertion follows the same lines as the proof in [12], Theorem 4.10, of '(a) implies (c)' (we have to use Theorem 7.1). Namely, C is closed and convex ([12] Corollary 2.7). One can assume that zero lies in the interior of C. If we denote by V the linear span of C, then $V = \bigcup_{t \geq 1} C_t$, where $C_t = \{tc\colon c \in C\}$. Of course, each set C_t is a strongly ρ_Φ-existence set. The space $\ell_\Phi^{(n)}$ being reflexive, V is a strongly ρ_Φ-existence set, too (Lemma 2.10 in [12]). Similarly as in the proof of Theorem 6.7 one shows that there is a linear, ρ_Φ-contractive projection P onto V and thus by Theorem 7.1, V is the intersection of k strongly ρ_Φ-contractive kernels of some functionals f_i.

Suppose that $C \neq V$ (otherwise there is nothing to prove). Since V is finite-dimensional, then C in V has empty interior. Repeating the argument from [4] we can show that there is a countable, dense subset of smooth points $Z \subset \partial_V C$, where $\partial_V C$ denotes the border of C in V. Moreover, $C = \bigcap_{z\in Z} T_z$ for T_z tangent half-space to C at z (cf. [5, Lemma 3]). We have that $T_z = \{v \in V\colon g_z(v) \le d_z\}$ for some $g_z \in V^*$ and $d_z \in \mathbb{R}$. Besides, the point $z \in \partial_V C$ being smooth, there is

$$T_z = \{(1-\lambda)z + \lambda c\colon \lambda \ge 1, c \in C\}.$$

Lemmas 2.10 and 4.6 from [12] imply that T_z are strongly ρ_Φ-existence sets, and therefore they are also strongly ρ_Φ-contractive.

By Lemma 4.5 from [12], there exists linear ρ_Φ-contractive projections Q_z from V onto $\mathrm{Ker} g_z$. Therefore, the projections $P \circ Q_z$ are ρ_Φ-contractive, too. By Theorem 7.1 each $\mathrm{Ker} g_z$ is the intersection of $k+1$ strongly ρ_Φ-contractive kernels of functionals h_j^z defined on $\ell_\Phi^{(n)}$. If h_j^z does not vanish on V, then we can assume that $h_j^z|_V = g_z$.

Putting now

$$W_j^z := \{x \in \ell_\Phi^{(n)}\colon h_j^z(x) \le d_z, j = 1, \dots, k+1\}$$

we obtain strongly ρ_Φ-contractive half-spaces (Lemma 4.5 in [12]). Moreover,

$$C = \bigcap_{j=1}^{k} \mathrm{Ker} f_j \cap \bigcap_{z\in Z, i\in J_z} W_i^z,$$

where

$$J_z := \{i \in \{1, \dots, k+1\}\colon h_i^z|_V \not\equiv 0\}.$$

Observe that $J_z \neq \varnothing$ for any $z \in Z$. Recall that Z is at most countable. This ends the proof of the first assertion.

Similarly, proving that conditions (a)–(d) are equivalent is even easier: it is directly the argument in [12] ($\ell_\Phi^{(n)}$ is smooth — due to condition (M) and the finiteness of the dimension, reflexive — because finite-dimensional, and by assumptions modularly strictly convex). Therefore, (c) implies (b) by Lemma 4.5 from [12] together with Corollary 2.19 from [12]. That (a) implies (c) has just been proved above. Finally, note that the implication from (b) to (a) is a direct consequence of the definition, while the equivalence of (a) and (d) is a consequence of [12] Proposition 2.8. □

Before stating the next result we recall that a set $C \subset \ell_\Phi$ which is bounded in the modular (i.e. $\sup_{x\in C} \rho_\Phi(x) < +\infty$) is also bounded in the Luxemburg norm ([12, Lemma 2.4 (i)]).

Theorem 7.4. *Let Φ be a Musielak-Orlicz function satisfying the condition (M) and $C \subset \ell_\Phi$ a bounded set. Assume that ℓ_Φ is reflexive and all the functions ϕ_j, except possibly one of them, are strictly convex. Then the conditions* (a)–(d) *from the preceding theorem are equivalent.*

Proof. We can adjust the proof of [12] Theorem 4.11 making use of Theorem 7.3. Note that by assumptions ℓ_Φ is smooth (since reflexivity means in particular that Φ satisfies the condition δ_2 and due to the condition (M) all the ϕ_j are differentiable, cf. Theorem 2.7). The idea of the proof is as follows (we omit the details since they are alike those in [12]).

It suffices to prove that (a) implies (c) (the implications from (c) to (b) and from (b) to (a), as well as the equivalence of (a) and (d) are proved in the same way as in the preceding theorem). By Lemma 3.7 from [12] (the space in consideration being a Köthe space), C can be written as the closure of an increasing union of compact sets C_k, each of which is a strongly ρ_Φ-existence, convex set. Then in view of Lemma 4.8 from [12] the sets $P_n(C_k) \subset \ell_\Phi^{(n)}$ are strongly ρ_Φ-existence for all $k, n \in \mathbb{N}$ (P_n denotes here the natural projection $\ell_\Phi \to \ell_\Phi^{(n)}$ defined as the truncation of the sequence after the first n coordinates). It is easy to see that the closure of $P_n(C)$ is identical with the closure of the union of the sets $P_n(C_k)$, $k \in \mathbb{N}$, whence it is a strongly ρ_Φ-existence set (by Lemma 2.10 from [12]). Corollary 2.7 in [12] guarantees the convexity and boundedness of the closure of $P_n(C)$.

The space ℓ_Φ being reflexive, the Mazur Theorem implies that C is weakly compact. Therefore, $P_n(C)$ is compact and obviously convex in $\ell_\Phi^{(n)}$. We can thus apply the preceding theorem. This means that each of the sets $P_n(C)$ can be represented as a countable intersection of half-spaces defined by strongly ρ_Φ-contractive hyperplanes $W_{j,n} \subset \ell_\Phi^{(n)}$. We have that $W_{j,n} = \{z \in \mathbb{R}^n : g^{j,n}(z) \le d_{j,n}\}$ for some functional $g^{j,n}$ and some $d_{j,n} \in \mathbb{R}$.

Put

$$F_n := \{x \in \ell_\Phi : x_i = 0, i = 1, \ldots, n\} \text{ and } D_n := P_n(C) \oplus F_n.$$

Then

$$D_n = \bigcap_{j \in \mathbb{N}} V_{j,n}, \text{ where } V_{j,n} := \{x \in \ell_\Phi : g^{j,n}(x) \le d_{j,n}\}$$

($g^{j,n}$ extends in a natural way to a functional on ℓ_Φ, when we assume that $g_i^{j,n} = 0$ for $i > n$). The strong ρ_Φ-contractiveness of $W_{j,n} \subset \ell_\Phi^{(n)}$ is inherited by $V_{j,n} \subset \ell_\Phi$.

Now, since the intersection of all the D_n is identical with the intersection of all the $V_{j,n}$, $j, n \in \mathbb{N}$, then it remains to show that $C = \bigcap_{n \in \mathbb{N}} D_n$. By definition, $C \subset D_n$. On the other hand, if $d \in \bigcap_{n \in \mathbb{N}} D_n$, then for any $n \in \mathbb{N}$ there exist points $c^n \in C$, $d_n \in F_n$ such that $d = P_n(c^n) + d_n$. The set C is weakly compact and thus by Eberlein's Theorem we can assume that $c^n \to c$ weakly. Thence $P_k(c) = \lim_{n \to +\infty} P_k(c^n) = P_k(d)$, which means that $d = c \in C$. This ends the proof. □

Theorem 7.5. *Let Φ be a Musielak-Orlicz function satisfying the condition (M) and such that all the ϕ_j, except possibly one, are strictly convex. Then $C \subset \ell_\Phi^{(n)}$ $(n \ge 2)$ is strongly ρ_Φ-contractive if and only if it is the intersection of half-spaces defined by ρ_Φ-contractive hyperplanes.*

Proof. The necessity of the condition follows from Theorem 7.3 (implication from (b) to (c); we do not need assuming that the space in consideration is modularly strictly convex, since (b) implies (a) by definition, while (a) implies (c) in virtue of the first assertion of that theorem).

The sufficiency can be proved along the same lines as in [12] Theorem 4.14, using Theorem 7.3. We note that arguing as in [4] we can assume that the intersection in consideration is countable, i.e. $C = \bigcap_{n\in\mathbb{N}} Z_n$, where $Z_n = \{x \in \ell_\Phi^{(n)} \colon f_n(x) \leq d_n\}$ is a ρ_Φ-contractive half-space defined by a functional f_n. It is obvious that the sets Z_n and $-Z_n$ are ρ_Φ-optimal, and therefore, by Lemma 2.9 from [12], the intersection of such a pair is ρ_Φ-optimal, too. This in turn implies that $\mathrm{Ker} f_n$ is ρ_Φ-optimal. The space $\ell_\Phi^{(n)}$ is modularly strictly convex, whence each of these kernels is a ρ_Φ-existence set ([12, Proposition 2.8]). But since $\mathrm{Ker} f_n$ is finite-dimensional, then by Theorem 3.3 together with Lemma 4.4 from [12], this kernel is a strongly ρ_Φ-existence set. Lemma 4.5 in [12] guarantees in that case that Z_n is strongly ρ_Φ-contractive. Finally, as these sets are countably many, we obtain the result sought for by applying Theorem 7.3. $\square$

We end this article adding that using the results obtained we can repeat the constructions from Examples 4.12 and 4.13 in [12] to obtain the following proposition:

Proposition 7.6. *Let $\Phi = (\phi_1, \phi_2, \phi_3)$ be a Musielak-Orlicz funciton satisfying (M) and such that all the ϕ_j (except possibly one) are striclty convex. Then:*

(1) *There exists a two-dimensional, one-complemented linear subspace $Y \subset \ell_\Phi^{(3)}$ which is not ρ_Φ-optimal (a fortiori it is not ρ_Φ-contractive).*
(2) *There exists a two-dimensional linear subspace $Y \subset \ell_\Phi^{(3)}$ being a contractive set (in the Luxemburg norm), but which cannot be represented as an intersection of half-spaces defined by strongly ρ_Φ-contractive hyperplanes.*
(3) *There exists a convex and ρ_Φ-contractive set $C \subset \ell_\Phi^{(3)}$ which is not optimal in the Luxemburg norm.*
(4) *there exists a convex set $C \subset \ell_\Phi^{(3)}$ being a ρ_Φ-existence set but not a strongly ρ_Φ-existence set. Moreover, for some $t > 1$, the set tC is not a ρ_Φ-existence set.*
(5) *There exists a convex and ρ_Φ-contractive set $C \subset \ell_\Phi^{(3)}$ which cannot be represented as the intersection of half-spaces defined by ρ_Φ-contractive hyperplanes.*

Proof. Following [12], in (1) and (3) one has to use Theorem 6.2, while in (2) it will be Theorem 7.3. Assertion (4) is a consequence of [12] and (3). The construction of (5) according to [12] is based on Theorem 7.5. $\square$

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