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The magnitude optimum design of the PI controller for plants with complex roots and dead time

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Analytical design of the PID-type controllers for linear plants based on the magnitude optimum criterion usually results in very good control quality and can be applied directly for high-order linear models with dead time, without need of any model reduction. This paper brings an analysis of properties of this tuning method in the case of the PI controller, which shows that it guarantees closed-loop stability and a large stability margin for stable linear plants without zeros, although there are limitations in the case of oscillating plants. In spite of the fact that the magnitude optimum criterion prescribes the closed-loop response only for low frequencies and the stability margin requirements are not explicitly included in the design objective, it reveals that proper open-loop behavior in the middle and high frequency ranges, decisive for the closed-loop stability and robustness, is ensured automatically for the considered class of linear systems if all damping ratios corresponding to poles of the plant transfer function without the dead-time term are sufficiently high.

Key words: dead time, frequency response, magnitude optimum, PID controller, process control, stability margin

1. Introduction

In process control, the PID structure of the feedback controller is the most common in practice due to low number of tuning parameters, low-cost and easy implementation and capability to control even unknown and non-linear processes. The PI variant of the controller is the most frequently used [1], although the full version offers enhanced control quality. On the other hand, using derivative component in the PID controller may not be advantageous for control of processes with significant noise and in these situations the use of the PI controller is preferable. Practical experiences also show that the PI controller is sometimes more successful in control of processes with long dead time [2].

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Simple PID controller tuning methods, which are popular in practice, are often based on simplified models that utilize some kind of approximation of the high-order and dead-time dynamics. The simple models can be obtained by processing process response data or by a model-reduction technique from a high-order model. The tuning rules are often based on the step response damping and maximal overshoot requirements [3, 4], pole placement [5, 6], or internal model principle [7].

The methods capable to work directly with high-order dynamic models enable to achieve enhanced performance, but are usually more complex. Besides the magnitude optimum method, this group includes the dominant pole placement [1], the methods based on minimization of integral criteria in time domain and others. Especially the absolute error areas (IAE, ITAE) provide very convenient evaluation of the control quality, but require a numerical computation in general. In the case of plants without dead time it is possible to utilize prototype transfer functions minimizing the integral criteria for the controller design [8]. In [9] and [10] it is proposed to maximize the integral gain of the controller subject to sensitivity constraints.

This paper deals with the problem of setting-up the PI controller parameters for the class of stable dynamic systems with the transfer function in Laplace transform

$$F(s) = \frac{K}{a_n s^n + \ldots + a_1 s + 1} e^{-\tau s}, \tag{1}$$

where K is the plant static gain and $\tau \geqslant 0$ the dead time parameter. The controller transfer function is considered in the form

$$R(s) = K_C \left(1 + 1/(T_I s)\right), \tag{2}$$

where K_C is the controller gain and T_I is the integral time constant.

The magnitude optimum (MO) or modulus optimum tuning method [11-14] enables to compute the PID-type controller parameters without need of approximation of the dead-time dynamics. The MO criterion requires that

$$\lim_{\omega \to 0} |G_{CL}(i\omega)| = 1, \qquad \lim_{\omega \to 0} \frac{d^k}{d\omega^k} |G_{CL}(i\omega)| = 0, \quad k = 1, 2, \ldots, k_m, \tag{3}$$

where $G_{CL}(s)$ denotes the closed-loop transfer function between the reference signal and the plant output, and k_m is as high as possible for given number of the controller tuning parameters. Eq. (3) is equivalent to the requirement that the closed-loop frequency response modulus is as flat as possible in the low frequency range. This requirement is most natural for the reference tracking control tasks, where the closed-loop system is to be able to respond quickly to changes of the reference input, or to efficiently reject the disturbances affecting the plant output directly.

Since the method works directly with the dead-time dynamics and does not utilize any approximation, it is usable even for systems with long dead time. It is known that the MO method usually gives fast and well damped closed-loop responses for process models frequently used in industry [13]. On the other hand, (3) is a performance-type requirement, which does not explicitly include the stability and robustness aspects. Since the response is prescribed only for low frequencies, it can be expected that the control loop behavior for middle frequencies, decisive for the closed-loop stability, need not be convenient in general. In "normal" situations, the MO settings satisfy

$$L(\omega) \in \{z \,|\mathrm{Re}\, z \geqslant -0.5\}, \quad \forall \omega \geqslant 0, \tag{4}$$

where $L(\omega) = R(i\omega)F(i\omega)$ denotes the open-loop frequency response function. This property guarantees $M_S \leqslant 2$, where

$$M_S = \sup_{\omega \in [0,\infty)} |1 + L(\omega)|^{-1} \tag{5}$$

denotes the closed-loop sensitivity, which is recommended for PI or PID controller tuning in general [1]. The property (4) also ensures the gain margin larger than 2 and the phase margin larger than 60°. However, it is true that the method can fail to produce stabilizing settings for stable plants in some cases, or can give settings with a reduced stability margin. This especially regards plants with zeros, so the method cannot be recommended in such situations in general and it is the reason why zeros are not considered in (1).

The stability margin properties of the MO design for the PID controller have been studied in [15], but only for plants (1) with real roots. It was shown that in this case the open-loop Nyquist plot always comes out from the point $[-0.5,\ \nu]$, where $\nu \to -\infty$, and tends towards the right half-plane for low frequencies. This result was obtained from the observation that the multiplication of the expansions of the terms $(T_k i\omega + 1)^{-1}$ and $e^{-\tau i\omega}$ preserves some relations between the coefficients, but it seems that this approach cannot be extended for plants with complex roots. It reveals that due to the derivative component of the controller a correction of the settings is necessary for plants with very long dead time. The corresponding improvements of the MO tuning for the first-order plant with dead time have been described in [16].

In this paper, the properties of the MO tuning method are analyzed for the general plant (1), in the case of the PI controller. The analysis is important for understanding strengths and weaknesses of this approach. Although the controller (2) depends only on two parameters, the explanation that (4) holds for a specified class of stable plants seems to be far from being simple.

It is true that the MO tuning is predominantly suitable for the reference signal tracking and the output disturbance rejection tasks, but the performance need not be satisfactory in some cases when the disturbance affects the output indirectly. The situation when the disturbance takes effect on the plant input is common in process control. Therefore, in [17] and [18] the MO criterion has been modified to improve the control performance in these cases. A similar modification of the method also has been used in [19] in the case of the SOPDT plant. An alternative approach has been proposed in [20] – instead of modifying the design criterion, the MO-tuned controller is extended with a suitable first-order term, which increases the gain for low frequencies, but preserves the sensitivity level $M_S \leqslant 2$. The corresponding PI controller version has been described in [21].

The paper is organized as follows. Basic known facts about computation of the MO settings of the PI controller for plants (1) are summarized in Section 2. For purposes of Section 4, the MO settings are obtained for the factorized form of $F(s)$ in Section 3. Section 4, which analyses the behavior of $L(\omega)$, is divided into three sub-sections. Section 5 shows simulated responses for a number of plants to verify the conclusions obtained in Section 4 and to demonstrate practical qualities of this approach.

2. Computation of the MO settings

For a simplification, the controller (2) transfer function is rewritten into the form

$$R(s) = K^{-1}\left(r_0 + r_{-1}/s\right), \tag{6}$$

where K is just the plant static gain in (1). In this way the parameter K can be excluded from further considerations. By comparison with (2), $K_C = r_0/K$ and $T_I = r_0/r_{-1}$ are obtained for the actual controller settings, where r_0 and r_{-1} are the parameters to be found. If the Taylor expansion of $F(s)$ at $s = 0$ is considered in the form

$$F(s) = K\left(1 - c_1 s + c_2 s^2 - c_3 s^3 + \ldots\right) \tag{7}$$

it is possible to obtain

$$\begin{aligned}\operatorname{Re} L(\omega) &= r_0 \frac{\operatorname{Re} F(i\omega)}{K} + \frac{r_{-1}}{\omega}\frac{\operatorname{Im} F(i\omega)}{K} \\ &= \left(r_0 - c_1 r_{-1}\right) + \left(-c_2 r_0 + c_3 r_{-1}\right)\omega^2 + \ldots\end{aligned} \tag{8}$$

The requirement (3) is equivalent to

$$\lim_{\omega\to 0} \frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\left(1 + 2\operatorname{Re} L(\omega)\right) = 0, \qquad k = 0, \ldots, k_m\,, \tag{9}$$

where the equations (9) for odd k are satisfied automatically [15]. Conditions equivalent to (9) were obtained in [22] from the fact that to achieve the largest possible closed-loop bandwidth without harmonic overshoot, $L(\omega)$ should lie on the 0 dB M-circle, which is identical to the line $\{z \,|\, \mathrm{Re}\, z = -0.5\}$. Substituting (8) into (9) gives the system of linear equations

$$r_0 - c_1 r_{-1} = -0.5, \qquad -c_2 r_0 + c_3 r_{-1} = 0 \tag{10}$$

for the controller settings. Based on [12], the coefficients c_k can be computed directly from the coefficients of the transfer function (1) as follows:

$$\begin{aligned} c_1 &= a_1 + \tau, \\ &\dots \\ c_k &= (-1)^{k+1} a_k + \frac{\tau^k}{k!} + \sum_{i=1}^{k-1} (-1)^{k+i-1} c_i a_{k-i} \,. \end{aligned} \tag{11}$$

It is an important aspect of this approach that the products $A_k = K c_k$, called characteristic areas, can be computed directly from the plant step response or the response to a general input. Therefore, the knowledge of all coefficients of the transfer function (1) is not needed for computation of the controller settings.

3. The MO settings for $F(s)$ in factorized form

Although the model (1) can be used to compute the controller settings directly, it seems that for higher n it is not suitable for analysis of the behavior of $L(\omega)$. Therefore, for the purposes of the following sections the MO settings are obtained for the factorized form of $F(s)$. Any stable transfer function (1) can be written in the form

$$F(s) = K \prod_{k=1}^{m} \left(\alpha_k T_k^2 s^2 + T_k s + 1 \right)^{-1} e^{-\tau s}, \tag{12}$$

where $m \leqslant n$, $T_k > 0$ and $\alpha_k \geqslant 0$. If $\alpha_k > 1/4$ for a particular k, the corresponding factor in the denominator

$$\alpha_k T_k^2 s^2 + T_k s + 1 = \left(\sqrt{\alpha_k} T_k \right)^2 s^2 + 2\zeta_k \left(\sqrt{\alpha_k} T_k \right) s + 1 \tag{13}$$

has a pair of complex roots with the damping ratio $\zeta_k = 1/\left(2\sqrt{\alpha_k}\right)$, while $\alpha_k \in (0,\ 1/4]$ corresponds to a pair of real roots. The case $\alpha_k = 0$ corresponds to a single real root. Note that $m = n$ in (1) if $\alpha_k = 0$ for all k, otherwise $m < n$.

The purpose of this section is to obtain the MO controller settings for the plant model (12). Denote

$$G(\omega) = \frac{|F(i\omega)|}{K} = \left[\prod_{k=1}^{m} \left(1 - \alpha_k \left(T_k \omega \right)^2 \right)^2 + T_k^2 \omega^2 \right]^{-1/2}, \tag{14}$$

$$H(\omega) = -\angle F(i\omega) = \tau\omega + \sum_{k=1}^{m} \mathrm{atan}_2\left(\mathrm{T_k}\omega,\ 1 - \alpha_\mathrm{k}\left(\mathrm{T_k}\omega\right)^2\right), \tag{15}$$

where the function $\mathrm{atan}_2(\mathrm{y}, \mathrm{x})$ returns φ in the interval $(-\pi, \pi]$ such that $r \sin\varphi = y$ and $r \cos\varphi = x$, $r > 0$.

Proposition 1 *Let*

$$S_j = \sum_{k=1}^{m} \beta_{jk} T_k^j, \qquad T_\Sigma = \sum_{k=1}^{m} T_k + \tau, \tag{16}$$

where the coefficients β_{jk} are for the plant (12) defined in Table 1 in dependence on α_k. The Taylor expansions of $G(\omega)$ and $H(\omega)$ at $\omega = 0$ are in the form

$$G(\omega) = 1 + G_2\omega^2 + G_4\omega^4 + \ldots, \qquad H(\omega) = T_\Sigma\omega + H_3\omega^3 + H_5\omega^5 + \ldots, \tag{17}$$

where

$$G_2 = -\frac{S_2}{2}, \qquad G_4 = \frac{S_4}{4} + \frac{S_2^2}{8}, \qquad G_6 = -\left(\frac{S_6}{6} + \frac{S_2 S_4}{8} + \frac{S_2^3}{48}\right), \tag{18}$$

and

$$H_{2j+1} = (-1)^j \frac{S_{2j+1}}{2j+1}, \qquad j \geqslant 1. \tag{19}$$

Proof. For $\omega \to 0$ it is

$$\mathrm{atan}_2\left(\mathrm{T_k}\omega,\ 1 - \alpha_\mathrm{k}\left(\mathrm{T_k}\omega\right)^2\right) = \arctan\left[\mathrm{T_k}\omega / \left(1 - \alpha_\mathrm{k}\left(\mathrm{T_k}\omega\right)^2\right)\right]. \tag{20}$$

In the case $m = 1$ and $\tau = 0$ it is possible to obtain

$$H_{2j+1} = (-1)^j \frac{\beta_{2j+1,1} T_1^{2j+1}}{2j+1}, \quad j = 0, 1, \ldots \tag{21}$$

by evaluating the derivatives of $H(\omega)$ at $\omega = 0$. For $m > 1$, (19) follows directly from (21). The expansion of

$$-0.5 \ln\left[\left(1 - \alpha_k\left(T_k\omega\right)^2\right)^2 + T_k^2\omega^2\right] = E_{2,k}\omega^2 + E_{4,k}\omega^4 + \ldots \tag{22}$$

for $\omega = 0$ yields

$$E_{2j,k} = (-1)^j \frac{\beta_{2j,k} T_k^{2j}}{2j}. \tag{23}$$

The coefficients E_{2j} corresponding to the general function $G(\omega)$ (14) are sums of $E_{2j,k}$ corresponding to the terms (22). Finally, the expansion of $G(\omega)$ is obtained from

$$\begin{aligned} e^{\ln G(\omega)} &= 1 + \ln G(\omega) + \frac{1}{2}(\ln G(\omega))^2 + \frac{1}{6}(\ln G(\omega))^3 + \ldots \\ &= 1 + E_2\omega^2 + E_4\omega^4 + \ldots + \frac{1}{2}\left(E_2\omega^2 + E_4\omega^4 + \ldots\right)^2 + \ldots \\ &= 1 + E_2\omega^2 + \left(E_4 + \frac{E_2^2}{2}\right)\omega^4 + \left(E_6 + E_2E_4 + \frac{E_2^3}{6}\right)\omega^6 + \ldots \end{aligned} \tag{24}$$

□

Table 1: The coefficients β_{jk} values in dependence on α_k

j	β_{jk}
1	1
2	$1 - 2\alpha_k$
3	$1 - 3\alpha_k$
4	$1 - 4\alpha_k + 2\alpha_k^2$
5	$1 - 5\alpha_k + 5\alpha_k^2$

Figure 1 shows the values of β_{jk} in dependence on α_k in graphical form. Note that for plants with only real poles all β_{jk} are 1.

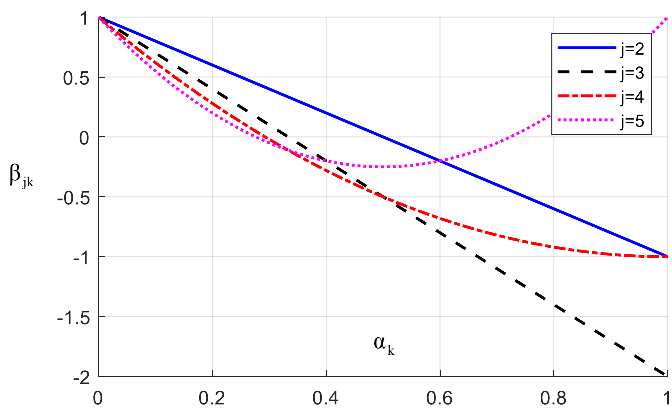

Figure 1: The values of β_{jk} in dependence on α_k

It is useful to make the following transformation of the frequency. If we define $\xi = T_\Sigma\omega$, $g(\xi) = G(\xi/T_\Sigma)$ and $h(\xi) = H(\xi/T_\Sigma)$, the open-loop frequency response function can be expressed in terms of ξ as $L(\omega) = l(\xi)$, where

$$l(\xi) = g(\xi)\left(r_0 + \frac{T_\Sigma r_{-1}}{i\xi}\right)e^{-ih(\xi)}. \tag{25}$$

For the coefficients in the expansions

$$g(\xi) = 1 + g_2\xi^2 + g_4\xi^4 + \ldots, \qquad h(\xi) = \xi + h_3\xi^3 + h_5\xi^5 + \ldots \tag{26}$$

$g_j = G_j T_\Sigma^{-j}$ and $h_j = H_j T_\Sigma^{-j}$ are easily obtained. If we define $s_j = S_j T_\Sigma^{-j}$, it can be easily seen from (18) that

$$g_2 = -\frac{s_2}{2}, \qquad g_4 = \frac{s_4}{4} + \frac{s_2^2}{8}, \qquad g_6 = -\left(\frac{s_6}{6} + \frac{s_2 s_4}{8} + \frac{s_2^3}{48}\right), \tag{27}$$

$$h_{2j+1} = (-1)^j \frac{s_{2j+1}}{2j+1}, \qquad j \geqslant 1. \tag{28}$$

Due to this transformation, $h(\xi) \approx \xi$ for low ξ for any plant $F(s)$. The following result enables to compute the MO settings directly from the plant representation (12).

Proposition 2 *The MO-optimal settings of the PI controller for plant (12) are in the form:*

$$r_{-1} = \frac{3}{4T_\Sigma}\frac{1 + s_2}{1 - s_3}, \qquad r_0 = \frac{3}{4}\frac{1 + s_2}{1 - s_3} - \frac{1}{2}. \tag{29}$$

Proof. First, consider the situation without the real factor $g(\xi)$, i.e. for the plant with the frequency response

$$e^{-ih(\xi)} = 1 - ih(\xi) + \frac{i^2}{2}h(\xi)^2 - \frac{i^3}{6}h(\xi)^3 + \ldots \tag{30}$$

Substituting (26) into (30) yields

$$\begin{aligned} e^{-ih(\xi)} &= 1 - i\left(\xi + h_3\xi^3 + \ldots\right) - \frac{1}{2}\left(\xi^2 + \ldots\right) + \frac{i}{6}\left(\xi^3 + \ldots\right) + \ldots \\ &= 1 - (i\xi) + \frac{1}{2}(i\xi)^2 - \left(\frac{1}{6} - h_3\right)(i\xi)^3 + \ldots \end{aligned} \tag{31}$$

The plant frequency response can be expressed in the dependence on ξ as follows (compare with (7)):

$$F\left(i\xi/T_\Sigma\right) = K\left(1 - \hat{c}_1(i\xi) + \hat{c}_2(i\xi)^2 - \hat{c}_3\left(i\xi\right)^3 + \ldots\right), \tag{32}$$

where $\hat{c}_j = c_j T_\Sigma^{-j}$. The coefficients $\hat{c}_j$ in the expansion (32) then are obtained by multiplying the series (31) and $g(\xi) = 1 - g_2(i\xi)^2 + g_4(i\xi)^4 - \ldots$ as

$$\hat{c}_1 = 1, \quad \hat{c}_2 = 1/2 - g_2, \quad \hat{c}_3 = 1/6 - h_3 - g_2\,. \tag{33}$$

Substituting $c_j = T_\Sigma^{-j}\hat{c}_j$ into (10) and expressing r_{-1}, r_0 yields (29). □

4. The stability margin of the MO tuning method

4.1. General frequency-domain properties of the MO settings

From (9) it follows that $\operatorname{Re} L(\omega) \to -0.5$ for $\omega \to 0$. Let us define ω_u the lowest frequency such that $\angle L(\omega) = -\pi$. Analogously, ξ_u denotes the lowest transformed frequency such that $\angle l(\xi) = -\pi$. Except for the plant $F(s) = K(Ts+1)^{-1}$, it is $\angle L(\omega) \leqslant -\pi$ for $\xi \to \infty$ and if $r_{-1} > 0$, the frequency ω_u exists. If $|L(\omega)|$ is non-increasing for $\omega \geqslant \omega_u$, it is

$$\operatorname{Re} L(\omega) = |L(\omega)| \cos\left(\angle L(\omega)\right) \geqslant -|L(\omega_u)| = \operatorname{Re} L(\omega_u) \tag{34}$$

for $\omega \geqslant \omega_u$. Thus, if $\operatorname{Re} L(\omega)$ is non-decreasing for $\omega \in [0, \omega_u]$, $\operatorname{Re} L(\omega) \geqslant -0.5$ must hold for any $\omega \geqslant 0$. If $r_{-1} > 0$, it is $\operatorname{Im} L(\omega) < 0$ for $\omega < \omega_u$. Since the open loop contains no RHP poles, except for one zero pole, the fact that $\operatorname{Re} L(\omega) \geqslant -0.5$ for all $\omega \geqslant 0$ and $\operatorname{Im} L(\omega) \leqslant 0$ for $\omega \leqslant \omega_u$ guarantees closed-loop stability by the Nyquist criterion. Moreover, there is a large reserve in stability implied by the property (4).

Consequently, the key requirements ensuring (4) are that $\operatorname{Re} L(\omega)$ is non-decreasing for $\omega \in [0, \omega_u]$ and that $|L(\omega)|$ is non-increasing for $\omega > \omega_u$, although these conditions are not necessary. Note that if $\operatorname{Re} L(\omega) < 0$ is non-decreasing in $[0, \omega_u]$, it is sure that $|L(\omega)|$ is decreasing in this interval, so $|L(\omega)|' \leqslant 0$ can be required in full range of ω without loss of generality. Since the frequency transformation $\xi = T_\Sigma\omega$ does not affect the shape of the Nyquist plot, for analyzing properties of the MO settings it is possible to work with $l(\xi)$ instead of $L(\omega)$. The requirement $|L(\omega)|' \leqslant 0$ is thus equivalent to $|l(\xi)|' \leqslant 0$.

Let $A(\xi) = \sin h(\xi)/\xi$, $B(\xi) = \cos h(\xi)$ and

$$\sigma = \frac{r_0}{T_\Sigma r_{-1}} = 1 - \frac{2}{3}\frac{1 - s_3}{1 + s_2}\,. \tag{35}$$

From (25) it follows that $\operatorname{Re} l(\xi) = -T_\Sigma r_{-1} V(\xi)$, where

$$V(\xi) = g(\xi)\left(A(\xi) - \sigma B(\xi)\right). \tag{36}$$

If $s_3 < 1$ and $s_2 > -1$, $r_{-1} > 0$ is always obtained by substitution into (29), which is necessary for the closed-loop stability, as discussed above. The requirement

$s_3 < 1$ ensures bounded values of r_0 and r_{-1}. The lower bound of r_0 then corresponds to $g_2 = -1$, which corresponds to $r_0 = -0.5$. It is then easily seen from (35) that $r_{-1} > 0$ is equivalent to $\sigma < 1$ and $\sigma \geqslant 0$ is equivalent to $r_0 \geqslant 0$.

Unlike $r_{-1} > 0$, $r_0 \geqslant 0$ is not necessary for the closed-loop stability. If $r_0 > 0$, which can be considered as a normal operational mode, the proportional term of the controller increases the closed-loop bandwidth in comparisons to the pure I-controller case. Negative r_0 occurs mostly in the situations when $G(\omega)$ is increasing for $\omega \to 0$.

4.2. The trend of $\operatorname{Re} l(\xi)$ for $\xi \to 0$

Since the MO-optimal trend of $\operatorname{Re} l(\xi)$ is flat for low frequencies, decreasing trend of $\operatorname{Re} l(\xi)$ signalizes that $l(\xi)$ tends towards the point $[-1,\ 0]$, so the stability margin gets reduced. Therefore, non-decreasing trend of $\operatorname{Re} l(\xi)$ for $\xi \to 0$ can be considered as a basic indicator of proper open-loop behavior. Hereafter, $\sigma < 1$ is assumed, so $r_{-1} > 0$.

Proposition 3 *The MO-optimal trend of* $\operatorname{Re} l(\xi)$ *is increasing for sufficiently low* $\xi > 0$ *if*

$$\frac{1-\sigma}{8}\left(2s_4 - s_2^2\right) - \left(\frac{1}{24} + \frac{s_3}{3}\right)\sigma + \frac{s_5}{5} + \frac{s_3}{6} + \frac{1}{120} < 0. \tag{37}$$

Proof. From

$$\begin{aligned} \cos h(\xi) &= 1 - \frac{h^2(\xi)}{2} + \frac{h^4(\xi)}{24} - \ldots, \\ \sin h(\xi) &= h(\xi) - \frac{h^3(\xi)}{6} + \frac{h^5(\xi)}{120} - \ldots \end{aligned} \tag{38}$$

by evaluating the expansions of $(h(\xi))^k$ for $k = 2, \ldots, 5$ using (26), in particular

$$\begin{aligned} h^2(\xi) &= \xi^2\left(1 + 2h_3\xi^2 + \ldots\right), \\ h^3(\xi) &= \xi^3\left(1 + 3h_3\xi^2 + \ldots\right), \end{aligned} \tag{39}$$

we obtain

$$\begin{aligned} A(\xi) - \sigma B(\xi) = (1-\sigma) &+ \left(-\frac{1}{6} + h_3 + \frac{1}{2}\sigma\right)\xi^2 \\ &+ \left[h_5 - \frac{h_3}{2} + \frac{1}{120} - \left(\frac{1}{24} - h_3\right)\sigma\right]\xi^4 + \ldots \end{aligned} \tag{40}$$

Multiplying (40) by the expansion of $g(\xi)$ (26) yields

$$V(\xi) = (1-\sigma) + \left[g_2(1-\sigma) - \frac{1}{6} + h_3 + \frac{1}{2}\sigma\right]\xi^2 + \left[(1-\sigma)g_4 + \left(-\frac{1}{6} + h_3 + \frac{1}{2}\sigma\right)g_2 + h_5 - \frac{h_3}{2} + \frac{1}{120} - \left(\frac{1}{24} - h_3\right)\sigma\right]\xi^4 + \ldots \tag{41}$$

The coefficient at ξ^2 in (41) vanishes due to the MO-optimality, which also means that

$$-\frac{1}{6} + h_3 + \frac{1}{2}\sigma = -g_2(1-\sigma). \tag{42}$$

Consequently, since $\sigma < 1$ is assumed, $\operatorname{Re} l(\xi)$ is locally increasing for $\xi \to 0$ if

$$(1-\sigma)\left(g_4 - g_2^2\right) - \left(\frac{1}{24} - h_3\right)\sigma + h_5 - \frac{h_3}{2} + \frac{1}{120} < 0 \tag{43}$$

which is equivalent to (37). □

It is not difficult to show that the inequality (37) holds if $0 \leqslant s_j$, $s_3 < 1$ and

$$s_{j+1} \leqslant s_j^{(j+1)/j}, \qquad j = 2, \ldots, 4 \tag{44}$$

but this step is omitted. If $\beta_{jk} = 1$ for all $k, j \geqslant 1$, the inequalities (44) are equivalent to

$$\sum_{k=1}^{m} z_k^p \leqslant \left(\sum_{k=1}^{m} z_k\right)^p, \tag{45}$$

where $z_k = (T_k/T_\Sigma)^j$ and $p = (j+1)/j$, which is the triangle inequality in L_p-space. Therefore, the inequality (37) holds if $S_j = \sum_{k=1}^{m} T_k^j$, where $T_k > 0$. This situation corresponds to the plants (12) with only real roots, where $0 \leqslant s_k$ and $s_3 < 1$ are fulfilled if $n > 1$ or $\tau > 0$. Unfortunately, in the case of complex roots the conditions (44) are often not satisfied.

To obtain a more general result, let us write

$$s_j = \sum_{k=1}^{m} \beta_{jk} x_k^j, \quad x_k = T_k/T_\Sigma \tag{46}$$

and divide the factors $\alpha_k T_k^2 s^2 + T_k s + 1$ in the plant (12) denominator into two groups: group A contains all the factors such that $\beta_{3k} \geqslant 0$, i.e. $\alpha_k \leqslant 1/3$, and group B contains the factors where $\beta_{3k} < 0$. Denote I_A and I_B, the sets of indices k corresponding to the group A and B, respectively, so $I_A \cup I_B = \{1, \ldots, m\}$ and $I_A \cap I_B = \emptyset$. The terms s_j are thus decomposed as $s_j = s_j^A + s_j^B$, where

$$s_j^A = \sum_{k \in I_A} \beta_{jk} x_k^j, \qquad s_j^B = \sum_{k \in I_B} \beta_{jk} x_k^j. \tag{47}$$

Proposition 4 *Re $l(\xi)$ is increasing for $\xi \to 0$ if $\alpha_k \leqslant 0.5$ in (12) for all $k = 1, \ldots, m$, and $n > 1$ or $\tau > 0$.*

Proof. Let

$$\psi(z, \varepsilon) = (1-\sigma)\frac{2\left(s_3^A\right)^{4/3} - s_2^2}{8} - \left(\frac{1}{24} + \frac{s_3}{3}\right)\sigma + \frac{\left(s_3^A\right)^{5/3} - \varepsilon}{5} + \frac{s_3}{6} + \frac{1}{120}, \tag{48}$$

where $\varepsilon > 0$ is a small number, $s_3 = s_3^A + s_3^B$, and $\sigma = \sigma(s_2, s_3)$ is given by (35). The symbol z denotes a point in the set containing all the feasible configurations described by the values of s_2^A, s_2^B, s_3^A and s_3^B, which will be specified later.

It can be easily verified that in the interval where $\beta_{jk} \geqslant 0$, $\beta_{j+l,k} \leqslant \beta_{jk}^{(j+l)/j}$ holds for $j \leqslant 3$ and $l = 1, 2$. Note that $\beta_{j+l,k} \approx \beta_{jk}^{(j+l)/j}$ for $\alpha_k \to 0$ and in Fig. 1 it can be seen that always $\beta_{j+l,k} < 0$ if $\beta_{j,k} = 0$. Considering $\beta_{jk} \geqslant 0$ and $s_j^A \geqslant 0$, the inequality (45) yields

$$s_{j+l}^A \leqslant \sum_{k=1}^{m} \beta_{jk}^{(j+l)/j} x_k^{j+l} \leqslant \left(\sum_{k=1}^{m} \left(\beta_{jk}^{1/j} x_k\right)^j\right)^{(j+l)/j} = \left(s_j^A\right)^{(j+l)/j}. \tag{49}$$

If $\alpha_k \leqslant 0.5$ for all k, it is $s_j^B \leqslant 0$ for $j = 3, \ldots, 5$, and $s_2^B \geqslant 0$. Consequently, $s_5 \leqslant \left(s_3^A\right)^{5/3}$ and $s_4 \leqslant \left(s_3^A\right)^{4/3}$. The inequality (37) then holds if $\psi\,(z, 0) < 0$ for $\psi\,(z, \varepsilon)$ given by (48).

By definition $s_1^A + s_1^B \leqslant 1$, and due to (45) $s_2^A \leqslant \left(s_1^A\right)^2$ and $s_3^A \leqslant \left(s_2^A\right)^{3/2}$, so $s_1^B \leqslant 1-\eta$, where $\eta = \left(s_2^A\right)^{1/2}$. If $\alpha_k \leqslant 0.5$, it is $s_2^B \leqslant \left(s_1^B\right)^2/3$ and $s_3^B \geqslant -0.5\left(s_1^B\right)^3$, because $\beta_{2k} \leqslant 1/3$ and $\beta_{3k} \geqslant -0.5$ for $k \in I_B$. Therefore, it is sufficient to verify that $\psi(z, 0) < 0$ for all $z = \left(\eta, s_1^B, s_2^B, s_3^A, s_3^B\right)$ such that $\eta^2 = s_2^A$ and

$$\begin{aligned} &\eta \in [0,\ 1], \quad s_1^B \in [0,\ 1-\eta], \quad s_2^B \in \left[0,\ \left(s_1^B\right)^2/3\right], \\ &s_3^A \in \left[0,\ \eta^3\right], \quad s_3^B \in \left[-0.5\left(s_1^B\right)^3,\ 0\right]. \end{aligned} \tag{50}$$

Since s_1^B does not explicitly participate in (48) and defines only the bounds for s_2^B and s_3^B, it is possible to reduce one free parameter by the choice $s_1^B = 1 - \eta$. Then $z = \left(\eta, s_2^B, s_3^A, s_3^B\right)$, where

$$\begin{aligned} &\eta \in [0,\ 1], \quad s_2^B \in [0,\ (1-\eta)^2/3], \\ &s_3^A \in [0,\ \eta^3], \quad s_3^B \in \left[-0.5(1-\eta)^3, 0\right]. \end{aligned} \tag{51}$$

Denote Σ the set of all z that satisfy (51). It is not difficult to verify the validity of $\psi(z,\varepsilon) < 0$ by generating a dense grid of points in Σ by computer in a systematic manner. On a personal computer it is possible to use the grid with regular step size larger than about $d = 10^{-4}$ for all the components of z. Due to the numerical errors rising from evaluations of the terms $\left(s_k^A\right)^r$ and $\left(s_k^B\right)^r$, it is not possible to simply put $\varepsilon = 0$. The minimal value of ε for the 64-bit floating-point arithmetic, which ensures $\psi(z,\varepsilon) < 0$ in Σ, is about 10^{-16}. The minimum of $\psi(z,\varepsilon)$ is achieved for $s_j^A = 1$, $s_j^B = 0$, $j = 1,\ldots,5$, which corresponds just to $m = n = 1$ and $\tau = 0$. But it is easily seen that in this case actually $\psi(z,0) = 0$, so it is clear that in the other configurations $\psi(z,0) < 0$ holds. The computational program for verification of $\psi(z,\varepsilon) < 0$ in Σ is very simple, but requires a compiled language. The source code in the C++ language is attached in the Appendix. □

The requirement $\alpha_k \leqslant 0.5$ in Proposition 4 corresponds to $\zeta_k \geqslant 0.5$. For higher values of α_k the intervals of s_2^B and s_3^B have to be extended and $\psi(z,\varepsilon) < 0$ in Σ no longer holds. Nevertheless, it was verified that $\psi(z,\varepsilon) < 0$ in Σ if

$$s_2^B \in \left[-\left(s_1^B\right)^2, \left(s_1^B\right)^2/3\right], \quad s_3^B \in \left[-2\left(s_1^B\right)^3, 0\right] \tag{52}$$

with the additional constraints $\sigma \leqslant 1/5$ and $s_2^B \geqslant -\sqrt{-0.5 s_3^B s_1^B}$. The intervals (52) correspond to $\alpha_k \leqslant 1$, where $\beta_{2k} \geqslant -1$ and $\beta_{3k} \geqslant -2$. The second constraint follows from the fact that $\beta_{2k} \geqslant 0.5\beta_{3k}$ if $k \in I_B$ and $\alpha_k \leqslant 1$, and from the inequality

$$\left(\sum_{k=1}^{p} x_k^2\right)^2 \leqslant \left(\sum_{k=1}^{p} x_k^3\right)\left(\sum_{k=1}^{p} x_k\right) \tag{53}$$

which holds for any $x_k \geqslant 0$ and $p \geqslant 1$. The inequality (53) directly follows from the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{p} c_k d_k\right)^2 \leqslant \left(\sum_{k=1}^{p} c_k^2\right)\left(\sum_{k=1}^{p} d_k^2\right) \tag{54}$$

where $c_k = x_k^{3/2}$ and $d_k = x_k^{1/2}$.

However, since β_{5k} is increasing for $\alpha_k > 0.5$ and is positive if $\alpha_k > 0.72$, the validity of (37) cannot be guaranteed by $\psi(z, 0) < 0$ for $\alpha_k > 0.72$ due to the term s_5^B, which is no longer negative, unless

$$(1 - \sigma)s_4^B/4 + s_5^B/5 \leqslant 0. \tag{55}$$

The inequality (55) clearly holds if $\sigma \leqslant 1/5$ and $\alpha_k \leqslant 1$, because $\beta_{5k} = 1$ and $\beta_{4k} = -1$ for $\alpha_k = 1$.

The arguments above show that validity of Proposition 4 can be extended up to $\alpha_k \leqslant 1$, but only under the condition that $\sigma \leqslant 1/5$. $\operatorname{Re} l(\xi)$ is often increasing even if $\alpha_k > 1$, but situations when $\alpha_k \leqslant 1$ and $\operatorname{Re} l(\xi)$ is decreasing for $\xi \to 0$ can be found – in particular, if $F(s)$ contains a dominant real time constant, see example in Section 5. It seems that the actual upper limit of σ for $\alpha_k \leqslant 1$ is a little higher – about 0.25. Since $A(\xi)$ is always decreasing in these cases, a remedy can be achieved by reducing σ to a chosen limit value $\widehat{\sigma}$, keeping $\operatorname{Re} l(0) = -T_\Sigma r_{-1}(1 - \sigma)$ fixed. The corresponding modified settings are easily obtained in the form

$$\widehat{r_{-1}} = r_{-1}\frac{1 - \sigma}{1 - \widehat{\sigma}}, \qquad \widehat{r_0} = T_\Sigma \widehat{r_{-1}}\widehat{\sigma}. \tag{56}$$

From practical point of view, it seems that the correction (56) is necessary for plants with some $\alpha_k \in (0.5,\ 1]$ only if $\sigma > \widehat{\sigma}$, where $\widehat{\sigma} \approx 0.6$, because for lower σ the minimum of $\operatorname{Re} l(\xi)$ remains close to -0.5.

If $\alpha_k \geqslant 1$ for some k, the method can fail, since $r_{-1} < 0$ may results, which implies closed-loop instability. For instance, for the plant

$$F_{\alpha,T}(s) = \left(\alpha T^2 s^2 + Ts + 1\right)^{-1} \tag{57}$$

it is $s_2 = -1$ for $\alpha = 1$, which corresponds to $r_{-1} = 0$ and $\sigma \to -\infty$. The closed-loop system is no longer stable if $\alpha > 1$.

4.3. The behavior of $l(\xi)$ for middle and high frequencies

Even if $\operatorname{Re} l(\xi)$ is increasing for $\xi \to 0$, it is indeed possible that the trend of $\operatorname{Re} l(\xi)$ changes for higher frequencies. Therefore, it is needed to inspect the trend of $\operatorname{Re} l(\xi)$ in extended range, especially in a neighborhood of the open-loop ultimate frequency ξ_u. In this section $\sigma < 1$ is again assumed. Since $l(\xi) = g(\xi)l_0(\xi)$, where

$$l_0(\xi) = T_\Sigma r_{-1}(\sigma - i/\xi)e^{-ih(\xi)} \tag{58}$$

it is easily seen that $\angle l(\xi) < -\pi$ for $h(\xi) = \pi$, which means that $h(\xi_u) < \pi$. Unfortunately, since the expansions of $A(\xi)$ and $B(\xi)$ converge rather slowly in some cases, the partial expansions used in the previous section do not provide sufficient information about the trend of $V(\xi)$ in a neighborhood of ξ_u.

It is common property of all stable plants (12) that $h(\xi) > 0$ is increasing. In addition, if $\sigma < 1$, it is $\lim_{h\to\infty} h(\xi) \geqslant \pi$. This allows to construct the inverse function to $h(\xi)$, denoted $\xi(h)$ in the sequel, which is increasing and bounded in any interval $[0,\ h_1]$, where $h_1 \in [0,\ \pi)$. Denote $g_h(h) = g\left(\xi(h)\right)$ and

$$A_h(h) = \sin h/\xi(h), \qquad B_h(h) = \cos h. \tag{59}$$

$\operatorname{Re} l(\xi)$ is non-decreasing for $\xi \in [0,\ \xi_u]$ if the function $V_h(h) = g_h(h)V_{1h}(h)$, where

$$V_{1h}(h) = A_h(h) - \sigma B_h(h) \tag{60}$$

is non-increasing for $h \in [0,\ h(\xi_u)]$. The function $A_h(h)$ can be increasing for low h in the case of low-damped complex factors in (12), but it is decreasing for $h \geqslant h_A^+$ where $h_A^+ \in [0,\ \pi/2)$, since $A_h'(\pi/2) < 0$ always holds. For $h > h_A^+$ the function $A_h(h)$ has a single flex point, denoted h_A^* in the sequel, which occurs when $A_h(h)$ gets sufficiently close to zero. Figure 2 shows the plots of $A_h(h)$ for several plants $F_{\alpha,T}(s)$ (57). To estimate the minimal value of h_A^*, it seems to be sufficient to inspect only the family of plants $F_{\alpha,T}(s)$ for different values of α, because additional factors in the denominator of $F(s)$ or dead time can only increase h_A^*. The minimal value of h_A^* is about 1.15 and corresponds to $\alpha \approx 0.1$.

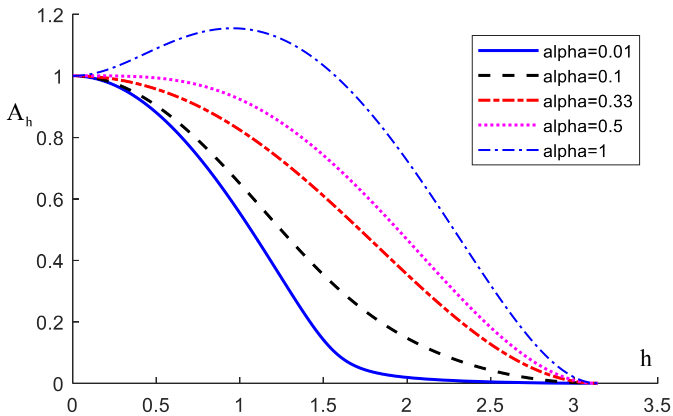

Figure 2: The plots of $A_h(h)$ for the plants $F_{\alpha,T}(s)$ with $T = 1$ and different α

Further, since $V_{1h}(0) = 1 - \sigma > 0$ and $V_{1h}(\pi) = \sigma$, $V_{1h}(h) > 0$ always holds for sufficiently low h, although $V_{1h}(h) < 0$ is possible in general. However, if $V_{1h}(h) \leqslant 0$ for higher h, it is also $V_h(h) \leqslant 0$, but in such a case $V_h(h) < V_h(0)$ and the stability margin (4) cannot be violated. Therefore, it is possible to consider only $V_{1h}(h) > 0$ below for a simplification. The trend of $V_h'(h)$ is analyzed separately for $h < \pi/2$ and $h \in [\pi/2,\ \pi]$ in the following two sub-sections.

Under the assumption that $V_h(h)$ is decreasing for $h \longrightarrow 0$, it is explained that the additional requirement that $|l(\xi)|$ is non-increasing is sufficient.

4.3.1. The trend of $V_h'(h)$ for $h < \pi/2$

Since $g_h(h) > 0$, the requirement $V_h'(h) \leqslant 0$ is equivalent to

$$\frac{g_h'(h)}{g_h(h)} V_{1h}(h) + V_{1h}'(h) = (\ln g_h(h))' V_{1h}(h) + V_{1h}'(h) \leqslant 0. \tag{61}$$

In the cases when $g_h(h)$ is decreasing, the function $(\ln g_h(h))'$ is usually decreasing in $[0,\ \pi/2]$, but for plants with $\alpha_k > 0.5$, $(\ln g_h(h))'$ can be increasing for low h. Fig. 3 shows the plots of $(\ln g_h(h))'$ for the plants $F_{\alpha,T}(s)$ (57) with $T = 1$ and different α. In the situations when $\alpha_k > 0.5$ and σ is close to 1 it is also possible that $(\ln g_h(h))'$ has minimum in $(0,\ \pi/2)$, but these cases are excluded by the requirement that $V_h(h)$ is decreasing for $h \longrightarrow 0$. For instance, for the plant $F(s) = F_{\alpha,T}(s)(s+1)^{-1}$ where $F_{\alpha,T}(s)$ is defined by (57) with $\alpha = 1$, $V_h(h)$ is increasing for $\xi \longrightarrow 0$ if $T < 1$, whereas $(\ln g_h(h))'$ has minimum in $(0,\ \pi/2)$ only if $T < 0.4$.

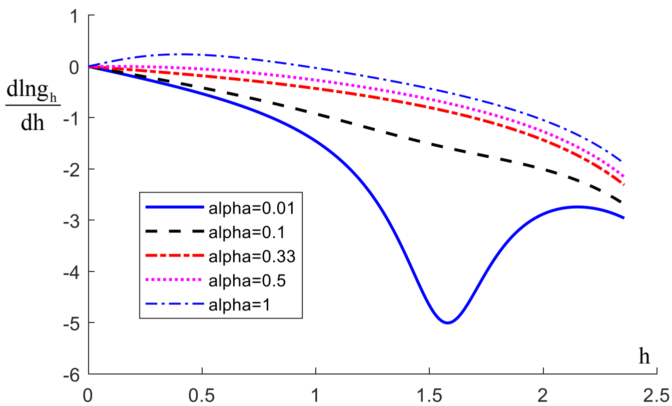

Figure 3: The plots of $(\ln g_h(h))'$ for the plants $F_{\alpha,T}(s)$ with $T = 1$ and different α

A) First, assume that $(\ln g_h(h))'$ is decreasing in $[0,\ \pi/2]$. Since $(\ln g_h(h))' < 0$, the MO optimality and (61) imply $V_{1h}'(h) \geqslant 0$ for low h. Since $V_{1h}'(h) < 0$ means that $V_h'(h) < 0$ in this case, it is possible to consider $V_{1h}'(h) \geqslant 0$ in all the interval $[0,\ \pi/2]$. The condition (61) can be rewritten as

$$\Lambda(h) \stackrel{\text{def}}{=} -(\ln g_h(h))' V_{1h}(h) + \sigma B_h'(h) \geqslant A_h'(h) \tag{62}$$

where the first term in $\Lambda(h)$ is increasing. For $h \longrightarrow \pi/2$ it is $B_h''(h) = 0$, so $\Lambda(h)$ must be increasing in the upper part of $[0, \pi/2]$. Since $A_h'(h)$ is decreasing for $h \leqslant h_A^*$, $\Lambda(h) \geqslant A_h'(h)$ is preserved in $[0, \pi/2]$ if $h_A^* \geqslant \pi/2$.

If $h_A^* < \pi/2$, $A_h'(h)$ is increasing in $\left[h_A^*, \pi/2\right]$. Considering $h_A^* \geqslant 1.15$ as discussed above, $B_h'(h)$ is nearly constant in $\left[h_A^*, \pi/2\right]$, whereas $(\ln g_h(h))'$ can be close to zero for $h = h_A^*$ only in the situations when $h_A^* \geqslant \pi/2$. Therefore, it can be assumed that $\Lambda(h)$ is increasing in $\left[h_A^*, \pi/2\right]$, so $A_h'\left(h_A^*\right) < \Lambda\left(h_A^*\right)$, as discussed in the previous paragraph. Consider

$$A_h'\left(h_A^* + \delta\right) \approx \alpha_0 + \alpha_1\delta + \alpha_2\delta^2, \quad \Lambda\left(h_A^* + \delta\right) \approx \beta_0 + \beta_1\delta + \beta_2\delta^2 \tag{63}$$

in a neighborhood of h_A^*, where $\alpha_0 < \beta_0$. Since $\Lambda'\left(h_A^*\right) > 0$ and $A_h''\left(h_A^*\right) = 0$, it is $\alpha_1 = 0$ and $\beta_1 > 0$. It can be easily seen from (63) that if $A_h'(\pi/2) \leqslant \Lambda(\pi/2)$, then $A_h'(h) \leqslant \Lambda(h)$ for all $h \in \left[h_A^*, \pi/2\right]$. Note that since $\alpha_0 < \beta_0$ and $\alpha_1 < \beta_1$, $A_h'(h) > \Lambda(h)$ is possible in a part of this interval only if $\alpha_2 > \beta_2$, and this situation would be indicated by $A_h'(\pi/2) > \Lambda(\pi/2)$.

However, if $A_h''(h)$ is not monotone and has a maximum in $\left(h_A^*, \pi/2\right)$, $\Lambda(h) < A_h'(h)$ need not hold in the interior of $\left[h_A^*, \pi/2\right]$, even if $V_h'(\pi/2) < 0$. This kind of behavior of $A_h(h)$ should be taken into account, because $A_h'(h)$ can change very fast as $A_h(h)$ gets close to zero, see the plots in Fig. 2. In the cases when $(\ln g_h(h))'$ is decreasing in $[0, \pi/2]$ and $h_A^* < \pi/2$, $\xi'(h)$ is always increasing in this interval. The maximum of $A_h''(h)$ can be close to $h = \pi/2$ only if $\xi(h)$ is increasing significantly faster than h in a neighborhood of $h = \pi/2$. The worst-case situation is therefore represented by the plant $F_{\alpha,T}(s)$ (57) where $\alpha \to 0$, because only in this case $\xi(\pi/2) \to \infty$. For the plant $F_{\alpha,T}(s)$ and $h \in [0, \pi/2)$ it is $\tan h = \xi/(1 - \alpha\xi^2)$ and for $\alpha > 0$

$$A_h(h) = \frac{2\alpha \sin^2 h}{-\cos h + \sqrt{\cos^2 h + 4\alpha \sin^2 h}}. \tag{64}$$

By differentiating (64),

$$\lim_{h\to\pi/2} A_h'(h) = -0.5 \quad \text{and} \quad \lim_{h\to\pi/2} A_h'''(h) = 0.5 \tag{65}$$

were obtained, regardless the value of α. This shows that $A_h''(h)$ is increasing for $h \to \pi/2$, as required in the previous paragraphs. Therefore, if $h_A^* < \pi/2$, $V_h'(h) \leqslant 0$ in $[0, \pi/2]$ is ensured by $V_h'(\pi/2) \leqslant 0$. Consequently, it is sufficient to inspect the behavior of $V_h'(h)$ only for $h \geqslant \pi/2$.

B) If $(\ln g_h(h))'$ is increasing for low h, but decreasing as $h \to \pi/2$, then $V_{1h}'(h)$ is negative and decreasing for $h \to 0$ due to the MO optimality. In these situations it is always $h_A^* > \pi/2$ for the considered class of plants. The fact that $V_{1h}''(\pi/2) = A_h''(\pi/2) < 0$ indicates that $V_{1h}'(h)$ is negative and decreasing in all the interval $[0, \pi/2]$. Since $V_{1h}(h)$ is decreasing in this case, the minimum of the first term in $\Lambda(h)$ is located in $[0, \pi/2)$, which means that $\Lambda(h)$ is increasing as $h \longrightarrow \pi/2$. Since $A'_h(h)$ is decreasing in $[0,\ \pi/2]$, the difference between $\Lambda(h)$ and $A'_h(h)$ grows in the upper part of $[0,\ \pi/2]$ and $V'_h(h) \leqslant 0$ is preserved.

If $F(s)$ contains more factors with $\alpha_k > 0.5$, $(\ln g_h(h))'$ can be increasing in all the interval $[0,\ \pi/2]$. Since $B''_h(h) < 0$ for $h \in (\pi/2,\ \pi)$, the arguments in the previous paragraph can be used if the maximum of $(\ln g_h(h))'$ lies in $[0,\ h^*_A]$, where $h^*_A > \pi/2$. Then $V'_h(h) \leqslant 0$ holds in $[0,\ h^*_A]$.

The fact that $(\ln g_h(h))'$ is positive and increasing in $[0,\ \pi/2]$ means that $\xi'(h)$ is decreasing in this interval. Actually, the interval where $\xi'(h)$ is decreasing usually roughly corresponds to the interval where $(\ln g_h(h))' > 0$. Since $\xi'(h)$ is decreasing, $\xi(h) < h$ and $\xi'(h)/\xi(h) < 1/h$ in $[0,\ \pi/2]$. Let $h_0 = \pi/2$ and $\delta = h - h_0$. Then

$$A_h(h) \approx \frac{\sin h}{\xi(h_0) + \xi'(h_0)\delta} = \frac{1}{\xi(h_0)} \frac{\sin h}{1 + \xi'(h_0)/\xi(h_0)\delta}. \tag{66}$$

The flex point position corresponding to $A_h(h)$ given by (66) is clearly decreasing with respect to the ratio $\xi'(h_0)/\xi(h_0)$, which explains that h^*_A lies beyond the flex point of $\sin h/h$, denoted $h^*_{A0} \approx 2.08$. Since both $s_2 < 0$ and $s_3 < 0$, it is $\sigma < 1/3$, so using (58)

$$\angle l_0\left(\xi\left(h^*_A\right)\right) \leqslant \angle l_0\left(\xi\left(h^*_{A0}\right)\right) \leqslant -\arctan\left(3/h^*_{A0}\right) - h^*_{A0} < -\pi \tag{67}$$

is obtained, which means that $h^*_A > h(\xi_u)$. This shows that $\operatorname{Re} l(\xi)$ is increasing in $[0,\ \xi_u]$ in these cases.

Finally, the function $(\ln g_h(h))'$ also can be increasing in all the interval $[0,\ h^*_A]$. For instance, for the plant $(F_{\alpha,T}(s))^\nu$ where $F_{\alpha,T}(s)$ is given by (57) and $\alpha = 1$, $(\ln g_h(h))'$ is increasing in $[0,\ h^*_A]$ if $\nu \geqslant 6$. For $\nu = 6$ it is $\sigma = 0.16$ and $h^*_A \approx 2.28$. In such cases, although $\Lambda(h) > A'_h(h)$ for $h \longrightarrow 0$, $A'_h(h)$ can start to increase before $\Lambda(h)$ does. But if $\Lambda(h)$ is decreasing in $[0,\ h^*_A]$ and $h(\xi_u) < h^*_A$, $\Lambda(h) < A'_h(h)$ in an upper part of $[0,\ h(\xi_u)]$ would be indicated by $V'_h(h) > 0$ at $h = h(\xi_u)$. Since $(\operatorname{Re} l(\xi))' = -|l(\xi)|'$ at $\xi = \xi_u$, $V'_h(h) \leqslant 0$ in $[0,\ h(\xi_u)]$ therefore holds if $|l(\xi)|$ is non-decreasing.

4.3.2. The trend of $V'_h(h)$ for $h \geqslant \pi/2$

It has been explained above that the behavior of $V'_h(h)$ has to be inspected in $[\pi/2,\ \pi]$ only if $h^*_A < h^*_{A0}$, where $h^*_{A0} \approx 2.08$. In the cases when $h^*_A < \pi/2$, $V'_h(h) < 0$ in $[0,\ \pi/2]$ is ensured by $V'_h(h) < 0$ for $h = \pi/2$. Consider the plant

$$\hat{F}(s) = F_{\alpha,T}(s)e^{-\hat{\tau}s}, \tag{68}$$

where $F_{\alpha,T}(s)$ is given by (57) and $\hat{\tau} \geqslant 0$. The corresponding functions $h(\xi)$ and $g(\xi)$, denoted $\hat{h}(\xi)$ and $\hat{g}(\xi)$, are in the form

$$\begin{aligned} \hat{h}(\xi) &= (1-\vartheta)\xi + \operatorname{atan}_2\Big(\vartheta\xi, 1-\alpha(\vartheta\xi)^2\Big), \\ \hat{g}(\xi) &= \left[\Big(1-\alpha(\vartheta\xi)^2\Big)^2 + (\vartheta\xi)^2\right]^{-1/2}, \end{aligned} \tag{69}$$

where $\vartheta = T/(T+\hat{\tau}) \in [0, 1]$. Analogously denote $\hat{\xi}(h)$ the inverse function to $\hat{h}(\xi)$ and $\hat{A}_h(h)$, $\hat{\xi}(h)$, $\hat{\sigma}$, $\hat{V}_h(h)$ and $\hat{V}_{1h}(h)$ the functions corresponding to $A_h(h)$, $\xi(h)$, σ, $V_h(h)$ and $V_{1h}(h)$, respectively, in the case of the plant $\hat{F}(s)$.

It is assumed hereafter that $T+\hat{\tau} = T_\Sigma$ and that $F(s)$ is not in the form (68), so $m > 1$. At first, consider that the parameters ϑ and α are chosen so that the coefficients s_2 and s_3 of $F(s)$ and $\hat{F}(s)$ are equal. In this case, $\hat{\xi}(h)$ and $\hat{g}_h(h)$ can be viewed as approximations of $\xi(h)$ and $g_h(h)$ for low h. Since the high-order dynamics in $F(s)$ is replaced by the dead time in $\hat{F}(s)$, $1-\vartheta \geqslant \tau/T_\Sigma$ holds. The function $\hat{\xi}'(h)$ is increasing slower than $\xi'(h)$ for $h \to \pi/2$, because the higher-order terms in $F(s)$ cause that $\xi(h)$ is increasing faster for higher frequencies, whereas the trend of $\hat{\xi}(h)$ is more flat.

Given $h_0 \in [\pi/2, \pi)$, it is therefore possible to reduce α, or if $\alpha = 0$, to increase ϑ, so that $\hat{\xi}(h_0) = \xi(h_0)$, $\hat{\xi}(h) > \xi(h)$ in $(0, h_0)$, and $\hat{\xi}'(h_0) \leqslant \xi'(h_0)$ (Fig. 4a). Under the assumption that $\xi'(h)$ is increasing for $h \geqslant \pi/2$, the function

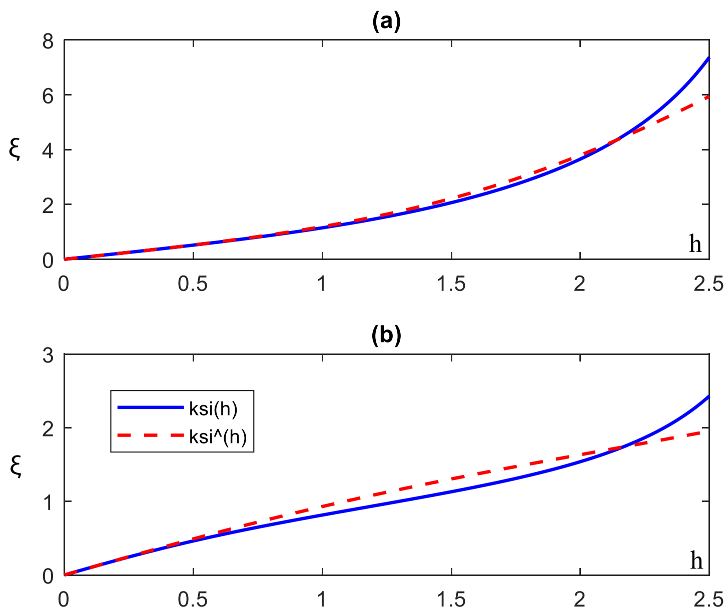

Figure 4: The replacement of $\xi(h)$ by $\hat{\xi}(h)$: (a) $\xi'(h)$ increasing for all $h \geqslant 0$, (b) $\xi'(h)$ decreasing for low h

$\xi(h)$ can be in this way replaced by $\hat{\xi}(h)$ even if $\xi'(h)$ is decreasing for low h (Fig. 4b). From

$$\hat{\sigma} = 1 - \frac{2}{3}\frac{1-(1-3\alpha)\vartheta^3}{1+(1-2\alpha)\vartheta^2} \tag{70}$$

it can be easily seen that the described modifications of α and ϑ, taken in the mentioned order, increase $\hat{\sigma}$, so $\hat{\sigma} \geqslant \sigma$ holds. The situations when $\xi'(h)$ is decreasing in all the interval $[0,\ \pi/2]$ have been discussed in the previous section and can be excluded.

Proposition 5 *Let $h_0 \in [\pi/2,\ \pi)$ and assume that $V_h(h_0) > 0$. Further, consider that the plant $\hat{F}(s)$ parameters α and ϑ are chosen so that $\hat{\xi}(h_0) = \xi(h_0)$, $\hat{\xi}(h) > \xi(h)$ in $(0,\ h_0)$, $\hat{\xi}'(h_0) \leqslant \xi'(h_0)$ and $\hat{\sigma} \geqslant \sigma$. Then $(\ln \hat{V}_h(h_0))' > (\ln V_h(h_0))'$.*

Proof. For the transformed frequency ξ the Bode's gain-phase relationship [23] yields

$$\hat{h}(\xi) - (1-\vartheta)\xi = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\ln \hat{g}(z)}{d\ln z} W(\xi, z)\,\mathrm{d}\ln z, \tag{71}$$

where $W(\xi, z) = \ln|(z+\xi)/(z-\xi)| > 0$ and $W(\xi, z) \to \infty$ for $z \to \xi$. Analogous relation can be written between $h(\xi)$ and $g(\xi)$. Consequently,

$$h(\xi) - \hat{h}(\xi) + (1-\vartheta-\tau/T_\Sigma)\,\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} U(z)W(\xi, z)\,d\ln z \tag{72}$$

where $U(\xi) = d\ln\left(\hat{g}(\xi)/g(\xi)\right)/d\ln\xi$. Since $\hat{h}(\xi) \approx h(\xi) \approx \xi$ for low ξ, and $1-\vartheta \geqslant \tau/T_\Sigma$, $\hat{h}(\xi)-(1-\vartheta)\xi$ is increasing slower than $h(\xi)-\tau\xi/T_\Sigma$ in the low frequency domain. Moreover,

$$\hat{h}(\xi) - (1-\vartheta)\xi < h(\xi) - \tau\xi/T_\Sigma \tag{73}$$

must hold for high frequencies, because the right-hand side of (73) tends to $n\pi/2$, where $n > 2$, while the left-hand side is not larger than π. This shows that the left-hand side of (72) is positive and increasing. Consequently, $U(z) > 0$ for all $\xi \geqslant 0$, which means that $(\ln \hat{g}_h(h))' > (\ln g_h(h))'$, because $(\ln \xi(h))' > 0$. Further, since

$$(\ln V_{1h}(h))' = \frac{V_{1h}'(h)}{V_{1h}(h)} = \frac{A_h'(h) + \sigma\sin h}{A_h(h) - \sigma\cos h} = \frac{\cos h + (\sigma\xi - \xi'/\xi)\sin h}{\sin h - \sigma\xi\cos h} \tag{74}$$

the value of $(\ln V_{1h}(h))'$ at $h = h_0$ is decreasing with respect to ξ'. As regards the influence of σ,

$$\frac{\partial}{\partial\sigma}(\ln V_{1h}(h))' = \frac{\sin h + \cos h\,(\ln V_{1h}(h))'}{V_{1h}(h)} \tag{75}$$

and if $V'_{1h}(h) < 0$, the right-hand side of (75) is always positive, because $\cos h \leqslant 0$ in $[\pi/2, \pi]$. Since $A_h(h)$ is always decreasing for $h \geqslant \pi/2$, $V'_{1h}(h_0) < 0$ if $\sigma \leqslant 0$ for all $h_0 \in [\pi/2, \pi]$. Consequently, it is possible to consider only $V'_{1h}(h) \geqslant 0$ and $\sigma > 0$ below. In this case

$$\cos h \ (\ln V_{1h}(h))' = \cos h \frac{V'_{1h}(h)}{A_h(h) - \sigma \cos h} \geqslant -\frac{V'_{1h}(h)}{\sigma} \tag{76}$$

so

$$\frac{\partial}{\partial \sigma}(\ln V_{1h}(h))' \geqslant \frac{-A'_h(h)}{\sigma V_{1h}(h)} > 0 \tag{77}$$

which means that $(\ln V_{1h}(h_0))'$ is increasing with respect to σ. Since $\xi = \hat{\xi}, \xi' \geqslant \hat{\xi}'$ and $\sigma \leqslant \hat{\sigma}$, this means that $(\ln V_{1h}(h_0))' \leqslant \left(\ln \hat{V}_{1h}(h_0)\right)'$. Since $(\ln V_h(h))' = (\ln g_h(h))' + (\ln V_{1h}(h))'$, $\left(\ln \hat{V}_h(h_0)\right)' > (\ln V_h(h_0))'$ holds. □

Since the requirements of Proposition 5 can be always satisfied, as discussed above, it is sufficient to verify that $\hat{V}'_h(h) \leqslant 0$ in the interval $[\pi/2, \pi)$ for any plant $\hat{F}(s)$ (68). Note that this simplification cannot be used for $h \in [0, \pi/2)$, where $\cos h > 0$. For $\alpha = 0.5$ it is $h^*_A \approx 2.07$, which is very close to h^*_{A0}. Since $h^*_A > h^*_{A0}$ for higher α, it is possible to consider α only in the interval $[0, 0.5]$. Validity of $\hat{V}'_h(h) \leqslant 0$ in $h \in [\pi/2, \pi)$ can be for the class of plants (68) where $\alpha \in [0, 0.5]$ and $\vartheta \in [0, 1]$ directly verified, except for the case $\alpha = 0$, $\vartheta = 1$, corresponding to $F(s) = K(Ts + 1)^{-1}$. Since $\hat{g}_h(h) > 0$, the requirement $\hat{V}'_h(h) < 0$ can be written as

$$\theta_{\alpha,\vartheta}(h) \stackrel{\text{def}}{=} \gamma(h)\left(\frac{\sin h}{\hat{\xi}(h)} - \hat{\sigma}\cos h\right) + \frac{\cos h}{\hat{\xi}(h)} + \left(\hat{\sigma} - \frac{\hat{\xi}'(h)}{\hat{\xi}(h)^2}\right)\sin h < 0, \tag{78}$$

where $\hat{\xi}'(h) = 1/\hat{h}'(\hat{\xi}(h))$ and

$$\gamma(h) = \hat{g}'_h(h)/\hat{g}_h(h) = (\ln \hat{g}(\xi))'\,\hat{\xi}'(h). \tag{79}$$

Differentiating (69) yields

$$\hat{h}'(\xi) = 1 - \vartheta + \vartheta\left(1 + \alpha(\vartheta\xi)^2\right)/D, \qquad D = \left(1 - \alpha(\vartheta\xi)^2\right)^2 + (\vartheta\xi)^2. \tag{80}$$

The value of $\gamma(h)$ is obtained by substituting $\xi = \hat{\xi}(h)$ into

$$\frac{d \ln \hat{g}(\xi)}{d\xi} = \vartheta^2 \xi \left[2\alpha\left(1 - \alpha(\vartheta\xi)^2\right) - 1\right]/D. \tag{81}$$

Although it is difficult to express $\hat{\xi}$ from (69) directly for $\hat{\tau} > 0$, the value of $\hat{\xi}(h)$ can be for $h \in [\pi/2, \pi - \varepsilon]$, where $\varepsilon > 0$ is arbitrarily small, easily obtained iteratively by bisection, because the function on the right-hand side of (69) is increasing and it is $\hat{h}(\xi) > \pi - \varepsilon$ for $\xi \to \infty$ if $\hat{\sigma} < 1$.

Validity of $\theta_{\alpha,\vartheta}(h) < 0$ was verified for 20 values of $h \in [\pi/2, \pi]$. Figure 5a shows the plots of $\theta_{\alpha,\vartheta}(\pi/2)$ in dependence on $\alpha \in [0, 0.5]$, for discrete steps of $\vartheta \in [0, 1]$. It can be seen that $\theta_{\alpha,\vartheta}(h) \to 0$ only if $\alpha \to 0$ and $\vartheta \to 0$, but if $\alpha = 0$ and $\vartheta = 0$, it is $\hat{\sigma} = 1$ and $\hat{\xi}(\pi/2)$ is not defined. In addition, Fig. 5b shows the plots of $\partial\theta_{\alpha,\vartheta}/\partial h$ at $h = \pi/2$ for $\alpha \in [0, 0.5]$ and $\vartheta \in [0, 1]$ obtained by numerical differentiation. The fact that $\partial\theta/\partial h < 0$ means that $\theta_{\alpha,\vartheta}(h)$ tends to decrease with respect to h in a neighborhood of $h = \pi/2$.

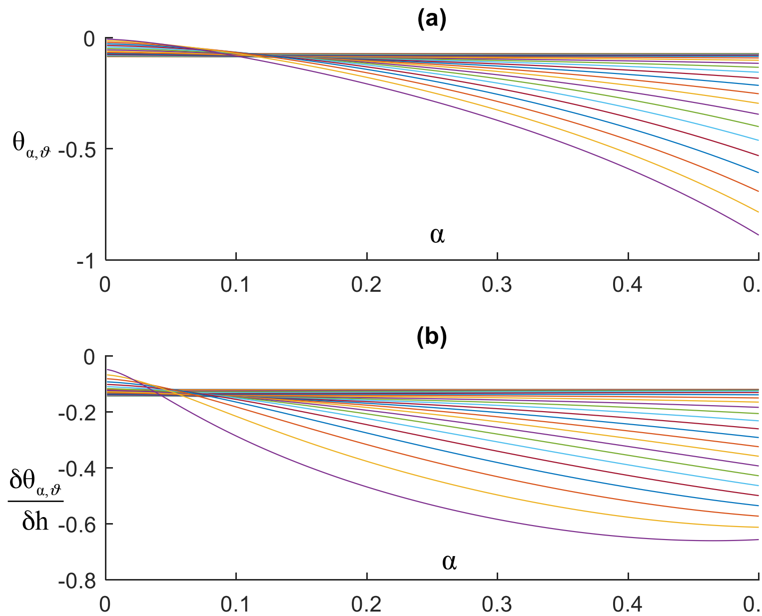

Figure 5: The values of (a) $\theta_{\alpha,\vartheta}(\pi/2)$ and (b) $\partial\theta_{\alpha,\vartheta}/\partial h$ at $h = \pi/2$, for discrete steps of ϑ in $[0, 1]$ and variable $\alpha \in [0, 0.5]$

4.3.3. The monotone trend of $|l(\xi)|$

As discussed in Section 4.1, the requirement $|l(\xi)|' \leqslant 0$ guarantees preservation of the property (4) for all ξ such that $\angle l(\xi) < -\pi$. In addition, this condition is needed to ensure the increasing trend of $\operatorname{Re} l(\xi)$ for $\xi \leqslant \xi_u$ in the situations when $(\ln g_h(h))'$ is increasing in all the interval $[0, h_A^*]$, as discussed in Section 4.3.1.

If $\alpha_k \in [0, 0.5]$, which is equivalent to $\zeta_k \geqslant 1/\sqrt{2}$, the requirement $|l(\xi)|' \leqslant 0$ is always satisfied, because $|l(\xi)| = g(\xi)|l_0(\xi)|$, where $l_0(\xi)$ is given by (58) and both $g(\xi)$ and $|l_0(\xi)|$ are non-increasing. Note that the stabilizing settings with the property (4) are well guaranteed even for the pure dead-time plant $F(s) = e^{-s}$. If there are factors in $F(s)$ such that $\alpha_k > 0.5$, $g(\xi)$ can be increasing for $\xi \geqslant \xi_u$

and $|l(\xi)|' \leqslant 0$ need not hold in general. But if $\alpha_k \leqslant 1$, this happens only if $F(s)$ contains a high number of complex factors with $\alpha_k > 0.5$. For instance, for the plant $(F_{\alpha,T}(s))^\nu$ (57), where $\alpha = 1$, it was verified that $|l(\xi)|' \leqslant 0$ for all $\xi \geqslant 0$ if $\nu \leqslant 7$ and the stability margin is reduced only if $\nu > 8$.

5. Simulated results

In this section, simulated responses are shown for the family of plants in the form

$$F_k(s) = \frac{e^{-\tau s}}{\left(\alpha T^2 s^2 + Ts + 1\right)^\nu (T_0 s + 1)} \tag{82}$$

for different values of the parameters T, α, ν, T_0 and τ. The plants $F_1(s)$ to $F_4(s)$ can be considered as low-order and well damped ones, and with a moderate τ, while $F_k(s)$ for $k \geqslant 5$ represent oscillating plants of higher order or with important dead time. The goal is to verify the conclusions obtained analytically in the previous sections and to demonstrate practical qualities of this method, especially for plants with complex roots. Used combinations of these parameter values are listed in Table 2.

Table 2: The plants $F_k(s)$ parameters

k	T	α	ν	T_0	τ
1	1	0	1	0	0.1
2	1	0	1	0.2	0
3	0.85	1/3	1	0	0.05
4	0.7	0.5	1	0	0.2
5	0.4	1	1	1	0
6	1	0.5	1	0	4
7	1	1	1	0	0.2
8	1	2	2	1	0

Expressions (29) for computation of the MO settings can be for plants (12) rewritten into the following general formula:

$$r_{-1} = \frac{3}{4} \frac{T_\Sigma^2 + \sum_{k=1}^{m} (1 - 2\alpha_k) T_k^2}{T_\Sigma^3 - \sum_{k=1}^{m} (1 - 3\alpha_k) T_k^3}, \qquad r_0 = T_\Sigma r_{-1} - \frac{1}{2}, \tag{83}$$

where T_Σ is given by (16), which gives the PI controller parameters $K_C = r_0/K$ and $T_I = r_0/r_{-1}$ for the plants $F_k(s)$ readily.

For a comparison, the AMIGOf tuning method [24] was chosen with the recommended settings $M_s = 1.4$ and $\varphi = 130°$. This method, based on performance optimization with the sensitivity constraint given by the parameter M_s, gives the controller (2) settings in the form

$$K_C = K_\varphi^{-1} \frac{0.4126}{1 + 1.6516\, K_\varphi/K}, \qquad T_I = \frac{2\pi}{\omega_\varphi} \frac{0.8526}{\left(1 + 1.7051 K_\varphi/K\right)^2}, \tag{84}$$

where ω_φ is the frequency such that $H(\omega_\varphi) = \varphi$ and $K_\varphi = \left|F(i\omega_\varphi)\right|$. The value of ω_φ was obtained iteratively. Classical tuning methods, such as [3–5], often give slow or oscillatory responses for plants with significant dead time.

Figures 6 and 8 show the MO-optimal Nyquist plots and corresponding closed-loop reference signal step responses for the plants $F_k(s)$. Figures 7 and 9 show the corresponding results in the case of the AMIGOf method. The MO tuning usually

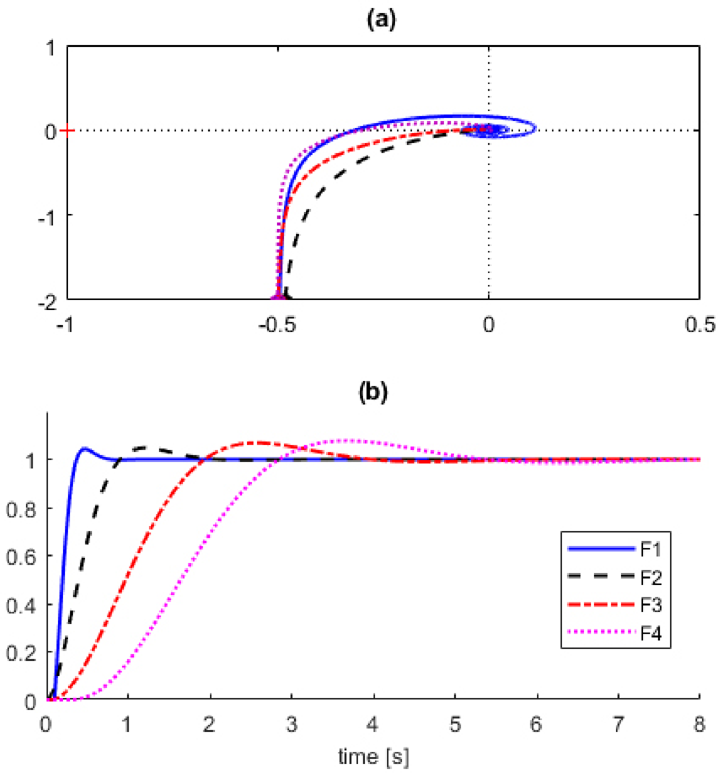

Figure 6: The MO method: (a) the open-loop Nyquist plots and (b) corresponding closed-loop reference signal step responses for the plants $F_k(s)$, $k = 1, .., 4$

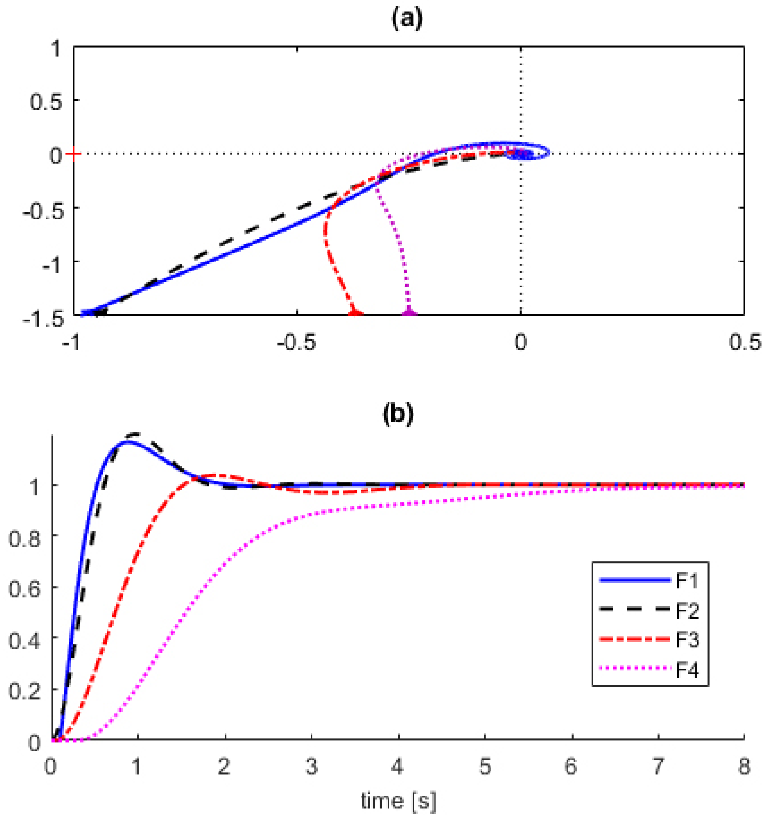

Figure 7: The AMIGOf method: (a) the open-loop Nyquist plots, (b) the closed-loop reference signal step responses for the plants $F_k(s)$, $k = 1, .., 4$

results in a little faster and more damped responses, especially in the presence of dead time. However, for plants with a strongly dominant real time constant, such as $F_1(s)$, the MO method may produce very large values of r_0 and r_{-1}.

For $k = 7$, where $\sigma = -3.7$, the controller achieves flatness of $\mathrm{Re}\, l(\xi)$ by means of the open-loop RHP zero σ^{-1}. In this operational mode the stability margin is preserved, but the response is delayed with undershoot. In general, the performance drops strongly for about $\sigma < -4$, even though it can be seen that in such cases $L(\omega)$ is very close to the line $\{z \,|\, \mathrm{Re}\, z = -0.5\}$ for $\omega \in [0, \omega_u]$.

In the case of $F_5(s)$, $\sigma = 0.68$ is obtained though $\alpha > 0.5$, which explains why $\mathrm{Re}\, l(\xi)$ is not increasing for low frequencies. Note that the stability margin defined by (4) is violated, even though both $g(\xi)$ and $|l(\xi)|$ are decreasing.

The responses corresponding to $F_8(s)$, where $\sigma = -1$, show that the MO method is able to provide fast responses in comparison to other methods even if $\alpha_k > 1$, under the condition that $|L(\omega)|$ is monotone and σ not too low.

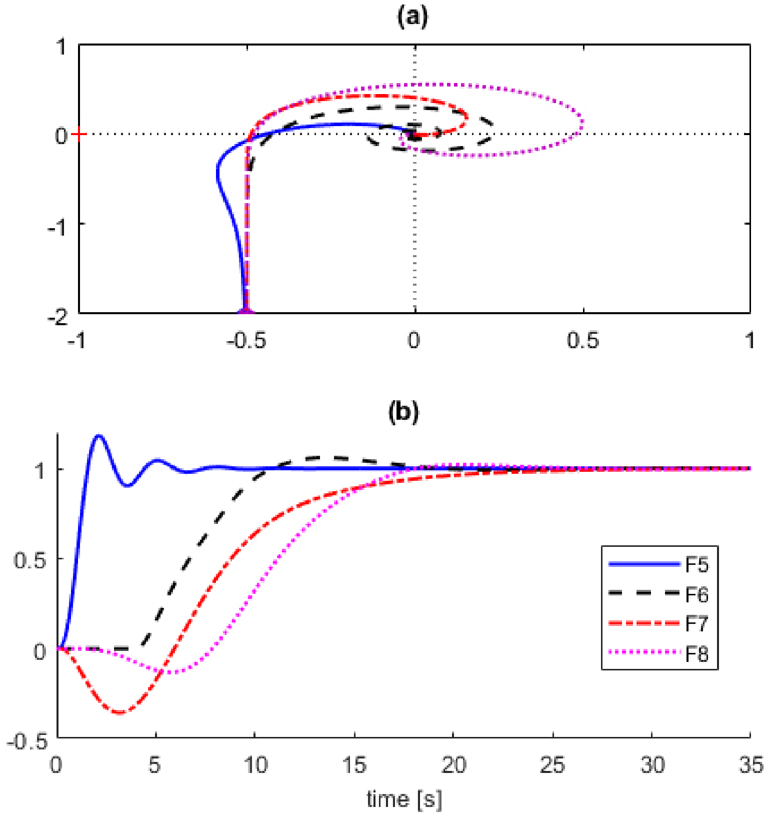

Figure 8: The MO method: (a) the open-loop Nyquist plots and (b) corresponding closed-loop reference signal step responses for the plants $F_k(s)$, $k = 5, .., 8$

6. Conclusions

The MO tuning method for the PI controller can be applied directly for high-order models with dead time, without need of any model reduction, and provides the controller settings in the form of analytical formulas for given parameters of the plant transfer function. It is well known that this method usually provides fast and well damped responses, but for some stable plants it fails to produce stabilizing settings or gives settings with a reduced stability margin. This paper analyses properties of this method for the family of stable plants (1). The analysis consists of inspecting the trend of $\operatorname{Re} L(\omega)$ for low frequencies and in the middle and high frequency ranges, where $L(\omega)$ denotes the loop frequency response.

It reveals that if $\alpha_k \leqslant 0.5$ in (12) for all $k = 1, \ldots, m$, and $n > 1$ or $\tau > 0$, the MO settings guarantee the property (4), which implies the sensitivity level $M_S \leqslant 2$ (the case $n = 1, \tau = 0$ is excluded). This fact does not directly follow from

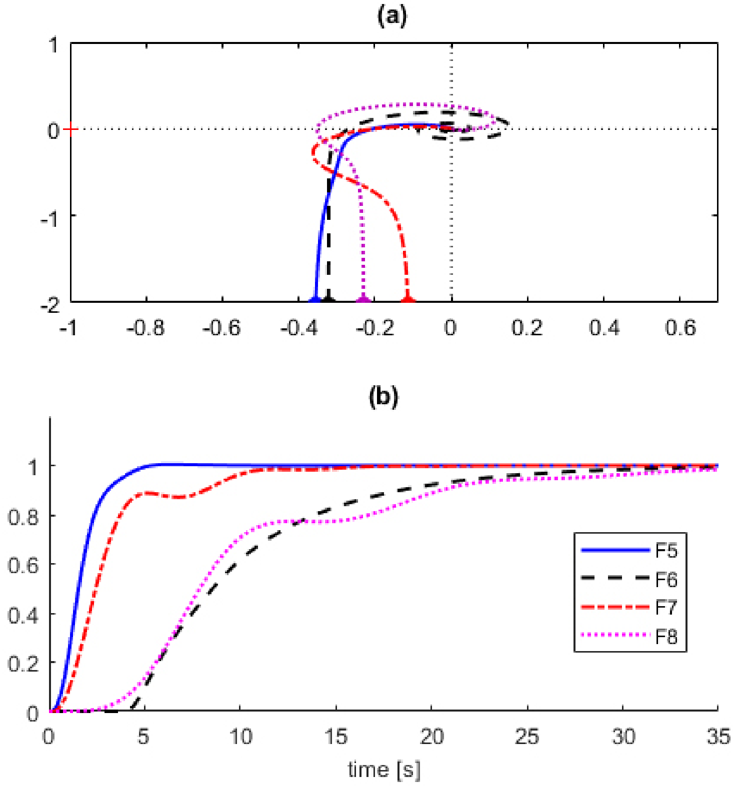

Figure 9: The AMIGOf method: (a) the open-loop Nyquist plots, (b) the closed-loop reference signal step responses for the plants $F_k(s)$, $k = 5, .., 8$

the requirement (9), which characterizes the closed-loop response for $\omega \to 0$ and cannot be explained only by the decreasing loop magnitude $|L(\omega)|$. The increasing trend of $\mathrm{Re}\, L(\omega)$ for low and middle frequencies, which is necessary for preservation of the stability margin, is more likely enabled by a favorable gain-phase relation in the case of plants (1).

These conclusions can be extended up to $\alpha_k < 1$, but only under the condition that σ given by (35) is not higher than about 0.25 and that $|L(\omega)|$ is non-increasing. If $\sigma > 0.25$ and $F(s)$ contains some factors with $\alpha_k \in (0.5,\ 1]$, the stability margin may be reduced due to decreasing trend of $\mathrm{Re}\, L(\omega)$ for low ω, which may result in oscillatory response, although from practical point of view it seems that the settings can be left without modifications up to $\sigma \approx 0.6$. To remove this defect for higher values of σ a simple correction of the settings has been proposed.

For plants (12) with $\alpha_k \geqslant 1$ the method has to be used with caution, since the stability margin may be reduced severely in the situations when $\mathrm{Re}\, L(\omega)$ is decreasing for low ω. In addition, $\sigma \geqslant 1$ may result, which implies the closed-loop instability, and the stability margin also may be violated due to non-monotonic behavior of $|L(\omega)|$ for high frequencies. The performance is not satisfactory in the cases when roughly $\sigma < -4$ due to delayed response with undershoot.

The performance of the method was compared with the AMIGOf method [24] by means of simulations. The simulations confirm the results obtained analytically and show that the MO method provides very good control quality for the plants (12) with $\alpha_k \leqslant 0.5$, especially if the dead-time dynamics is important, but has limitations for plants containing factors with $\alpha_k > 0.5$, as discussed above.

Appendix

The appendix contains the program code to verify $\psi(z, \varepsilon) < 0$ in the proof of Proposition 4. The program below written in standard C++ language displays maximal value of $\psi(z, \varepsilon)$ found in the area Σ defined by (51). The meaning of variables corresponds to the proof of Proposition 4, except for η, which is stored in the variable named s1A. Note that the grid is constructed so that the boundary points of the intervals in (51) are included.

```
#include <stdio.h>
#include <math.h>

typedef double NUM;
NUM d=1.0/1000; //grid step size
NUM eps=1e-15, Vmax=-1e+6;

void main()
{
   bool bE1A,bE1B,bE2B,bE3A,bE3B;

   for(NUM s1A=0,bE1A=0;!bE1A;s1A+=d)
   {
      if(s1A>=1) {s1A=1;bE1A=1;}

      NUM s1B=1-s1A, s2A=s1A*s1A;

      for(NUM s2B=0,bE2B=0;!bE2B;s2B+=d)
      {
         if(s2B>=s1B*s1B/3) {s2B=s1B*s1B/3; bE2B=1;}

         NUM s2=s2A+s2B, p13=pow(s1A,3.0);

         for(NUM s3A=0,bE3A=0;!bE3A;s3A+=d)
         {
            if(s3A>=p13) {s3A=p13;bE3A=1;}
            for(NUM s3B=0,bE3B=0;!bE3B;s3B-=d)
            {
               if(s3B<=-0.5*pow(s1B,3))
                  {s3B=-0.5*pow(s1B,3); bE3B=1;}
```

```
          NUM s3=s3A+s3B, s4=pow(s3A,4.0/3);
          NUM s5=pow(s3A,5.0/3)-eps;
          NUM sig=1-2.0/3*(1-s3)/(1+s2);
          NUM V=(1-sig)/8*(2*s4-s2*s2);
          V+=-(1.0/24+s3/3)*sig+s5/5+s3/6+1.0/120;
          if(V>Vmax) Vmax=V;
        }
      }
    }
    printf("s1A=%g: Vmax=%g\n",s1A,Vmax);
  }
}
```

References

[1] K.J. Åström and T. Hägglund: *PID Controllers: Theory, Design, and Tuning*. 2nd edition. Instrument Society of America, 1995.

[2] T.K. Kiong, W. Quing-Guo, H.Ch. Chiech, and T.J. Hägglund: *Advances in PID Control*. Springer-Verlag, 1999.

[3] J.G. Ziegler and N.B. Nichols: Optimum settings for automatic controllers. *Transactions of the ASME*, **64**, (1942), 759–768.

[4] K.L. Chien, J.A. Hrones, and J.B. Reswick: On the automatic control of generalised passive systems. *Transactions of the ASME*, **52**, (1952), 175–185.

[5] G.H. Cohen and G.A. Coon: Theoretical consideration of retarded control. *Transactions of the ASME*, **75**, (1953), 827–834.

[6] L. Trybus: A set of PID tuning rules. *Archives of Control Sciences*, **15**(1), (2005), 5–17.

[7] S. Skogestad: Simple analytic rules for model reduction and PID controller tuning. *Journal of Process Control*, **13**(4), (2003), 291–309. DOI: 10.1016/S0959-1524(02)00062-8.

[8] D. Graham and R.C. Lathrop: The synthesis of ‘optimum’ transient response: Criteria and standard forms. *Transactions AIEE*, **72**, (1953), 273–288.

[9] K.J. Åström, H. Panagopoulos, and T. Hägglund: Design of PI Controllers based on Non-Convex Optimization. *Automatica*, **34**(5), (1998), 585–601. DOI: 10.1016/S0005-1098(98)00011-9.

[10] K.J. Åström and T. Hägglund: Revisiting the Ziegler-Nichols step response method for PID control. *Journal of Process Control*, **14**(6), (2004), 635–650. DOI: 10.1016/j.jprocont.2004.01.002.

[11] R.C. Oldenbourg and H. Sartorius: A uniform approach to the optimum adjustments of control loops. *Transactions of the ASME*, **76**, (1954), 1265–1279.

[12] D. Vrančić, Y. Peng, and S. Strmčnik: A new PID controller tuning method based on multiple integrations. *Control Engineering Practice*, **7**(5), (1999), 623–633. DOI: 10.1016/S0967-0661(98)00198-1.

[13] D. Vrančić, S. Strmčnik, and D. Juričić: A magnitude optimum multiple integration tuning method for filtered PID controller. *Automatica*, **37**(9), (2001), 1473–1479. DOI: 10.1016/S0005-1098(01)00088-7.

[14] K.G. Papadopoulos: *PID Controller Tuning Using the Magnitude Optimum Criterion*. Springer, 2015. DOI: 10.1007/978-3-319-07263-0.

[15] J. Cvejn: The design of PID controller for non-oscillating time-delayed plants with guaranteed stability margin based on the modulus optimum criterion. *Journal of Process Control*, **23**(4), (2013), 570–584, DOI: 10.1016/j.jprocont.2013.01.008.

[16] J. Cvejn: PID control of FOPDT plants with dominant dead time based on the modulus optimum criterion. *Archives of Control Sciences*, **26**(1), (2016), 5–17. DOI: 10.1515/acsc-2016-0001.

[17] D. Vrančić, S. Strmčnik, and J. Kocijan: Improving disturbance rejection of PI controllers by means of the magnitude optimum method. *ISA Transactions*, **43**(1), (2004), 73–84. DOI: 10.1016/S0019-0578(07)60021-4.

[18] D. Vrančić, S. Strmčnik, J. Kocijan, and P.M. Oliviera: Improving disturbance rejection of PID controllers by means of the magnitude optimum method. *ISA Transactions*, **49**(1), (2010), 47–56. DOI: 10.1016/j.isatra.2009.08.002.

[19] J. Fişer, P. Zítek, and T. Vyhlídal: Magnitude Optimum Design of PID Control Loop with Delay. *IFAC-PapersOnLine*, **48**(12), (2015), 446–451. DOI: 10.1016/j.ifacol.2015.09.419.

[20] J. Cvejn and D. Vrančić: The magnitude optimum tuning of the PID controller: Improving load disturbance rejection by extending the controller. *Transactions of the Institute of Measurement and Control*, **40**(5), (2018), 1669–1680. DOI: 10.1177/0142331216688749.

[21] J. Cvejn: Enhancing disturbance rejection performance for the magnitude-optimum-tuned PI controller. *2017 21st International Conference on Process Control (PC)*, Štrbské Pleso, Slovakia. (2017), 155–160. DOI: 10.1109/PC.2017.7976206.

[22] R. Hanus: Determination of controllers parameters in the frequency domain. *Journal A*, **16**, (1975), 128–132.

[23] J.L. Douce, W.D. Widanage, and K.R. Godfrey: Evaluation of the relationship between gain and phase using extrapolation techniques. *IET Control Theory & Applications*, **1**(4), (2007), 1122–1130. DOI: 10.1049/iet-cta:20060333.

[24] T. Hägglund and K.J. Åström: Revisiting the Ziegler-Nichols step response method for PI control – Part II The Frequency Response Method. *Asian Journal of control*, **6**(4), (2004), 469–482. DOI: 10.1111/j.1934-6093.2004.tb00368.x.