Opuscula Math. 36, no. 1 (2016), 5–23
http://dx.doi.org/10.7494/OpMath.2016.36.1.5 Opuscula Mathematica

VERTEX-WEIGHTED WIENER POLYNOMIALS OF SUBDIVISION-RELATED GRAPHS

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Communicated by Dalibor Fronček

Abstract. Singly and doubly vertex-weighted Wiener polynomials are generalizations of both vertex-weighted Wiener numbers and the ordinary Wiener polynomial. In this paper, we show how the vertex-weighted Wiener polynomials of a graph change with subdivision operators, and apply our results to obtain vertex-weighted Wiener numbers.

Keywords: vertex-weighted Wiener numbers, vertex-weighted Wiener polynomials, subdivision graphs.

Mathematics Subject Classification: 05C76, 05C12, 05C07.

1. INTRODUCTION

In this paper, we are concerned with connected finite graphs without loops or multiple edges. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$. The shortest-path distance between vertices u and v in G is denoted by $d_G(u,v)$. The degree of a vertex u in G is denoted by $d_G(u)$. If there is no ambiguity on G, we omit the subscript G in $d_G(u,v)$ and $d_G(u)$. We denote by $|S|$ the cardinality of a set S.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based *structure-descriptors*, which are also referred to as *topological indices* [9, 26]. The *Wiener number* (or Wiener index), introduced by Wiener in 1947 [27], is the first reported distance-based topological index. This index was used for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener number of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v),$$

where the summation is taken over all unordered pairs of vertices u and v. Details on the Wiener index, and its theory and applications can be found in [7, 8, 10, 12, 14, 20, 24].

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The (unweighted) *Wiener polynomial* of G is defined as

$$P_0(G;x) = \sum_{\{u,v\}\subseteq V(G)} x^{d(u,v)},$$

with x a dummy variable. This coincides with definitions of Hosoya [17] and Sagan *et al.* [23]. Some authors prefer the name *Hosoya polynomial.*

A corresponding *singly vertex-weighted* Wiener polynomial of G is defined as [19]

$$P_v(G;x) = \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} [d(u)+d(v)]x^{d(u,v)+1}.$$

A *doubly vertex-weighted* Wiener polynomial of G is defined as [19]

$$P_{vv}(G;x) = \sum_{\{u,v\}\subseteq V(G)} [d(u)d(v)]x^{d(u,v)+2}.$$

The following relationship between the Wiener number and the Wiener polynomial of G was noted in [17]:

$$W(G) = P_0'(G;1).$$

The corresponding generalizations for the singly and doubly vertex-weighted cases were given in [19]:

$$W_v(G) = \left[\frac{1}{x}P_v(G;x)\right]'_{x=1}, \quad W_{vv}(G) = \left[\frac{1}{x^2}P_{vv}(G;x)\right]'_{x=1}.$$

Here, $W_v(G)$ denotes the singly vertex-weighted Wiener number of G,

$$W_v(G) = \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} [d(u)+d(v)]d(u,v).$$

Also, $W_{vv}(G)$ denotes the doubly vertex-weighted Wiener number of G,

$$W_{vv}(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u)d(v)]d(u,v).$$

The *Zagreb indices* were introduced by Gutman and Trinajstić in 1972 [16]. The first Zagreb index $M_1(G)$ of G is defined as

$$M_1(G) = \sum_{u\in V(G)} d(u)^2.$$

It can also be expressed as a sum over edges of G,

$$M_1(G) = \sum_{uv\in E(G)} [d(u)+d(v)].$$

The second Zagreb index $M_2(G)$ of G is defined as

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

Obviously, the Zagreb indices can be viewed as the contributions of pairs of adjacent vertices to vertex-weighted Wiener numbers. We refer the reader to [2, 4, 13, 15, 21, 22], for more information about these indices.

It is well-known that many graphs of general, and in particular of chemical interest, arise from simpler graphs via various *graph operators*. It is, hence, important to understand how certain invariants of a graph change under graph operators. In this paper, we show how the singly and doubly vertex-weighted Wiener polynomials change with subdivision operators, and apply our results to singly and doubly vertex-weighted Wiener numbers. Readers interested in more information on computing topological indices and polynomials of graph operations are referred to [1, 3, 5, 6, 11, 18, 25, 28].

2. DEFINITIONS AND PRELIMINARIES

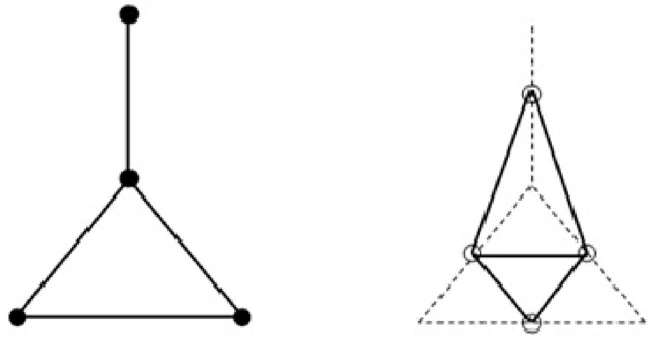
In this section, we recall the definitions of subdivision related graphs from the reference [28], and state some preliminary results about them.

Suppose $G = (V(G), E(G))$ is a connected graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $V(e)$ denote the set of two end vertices of an edge e of G. Related to the graph G, the *line graph* $L(G)$, the *subdivision graph* $S(G)$, and the *total graph* $T(G)$ are defined as follows:

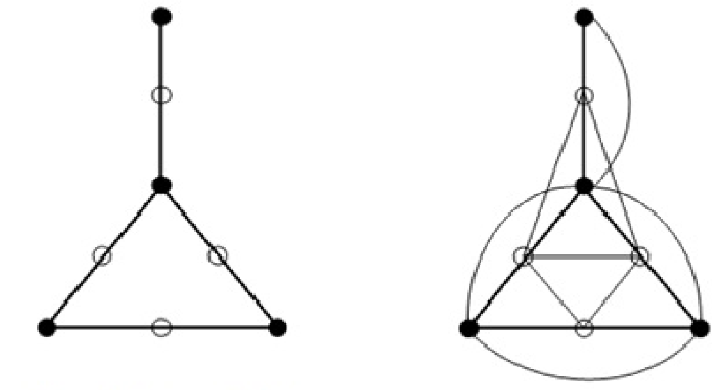
- **Line graph:** $L(G)$ is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common; see Figure 1(b).
- **Subdivision graph:** $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex of degree 2 into each edge of G; see Figure 1(c).
- **Total graph:** $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident; see Figure 1(d).

Two extra subdivision operators named $R(G)$ and $Q(G)$ are defined as follows:

- $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G, and by joining each new vertex to the end vertices of the edge corresponding to it; see Figure 2(a).
- $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G, and by joining with edges those pairs of new vertices which lie on adjacent edges of G; see Figure 2(b).

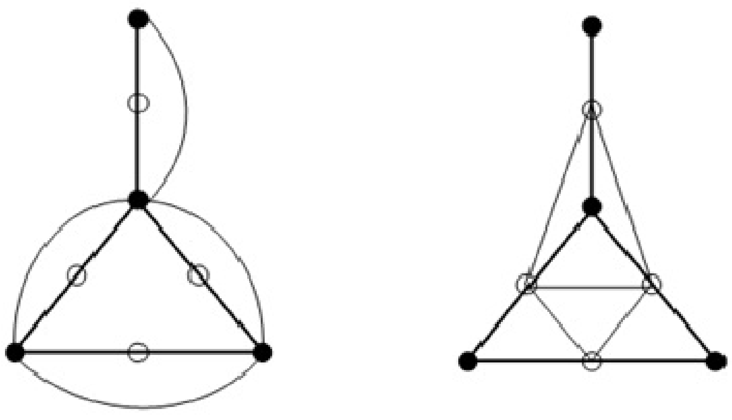


(a) A graph G (b) The line graph $L(G)$



(c) The subdivision-graph $S(G)$ (d) The total graph $T(G)$

Fig. 1. A graph G and the subdivision operators $L(G)$, $S(G)$, and $T(G)$



(a) The graph $R(G)$ (b) The graph $Q(G)$

Fig. 2. The two additional subdivision operators $R(G)$ and $Q(G)$

Now, consider the sets $EE(G)$ and $EV(G)$ for the graph $G = (V(G), E(G))$ as follows:

$$EE(G) = \{ee'|e, e' \in E(G), |V(e) \cap V(e')| = 1\}, \quad EV(G) = \{ev|e \in E(G), v \in V(e)\}.$$

It is easy to see that

$$|EE(G)| = \sum_{u \in V(G)} \binom{d(u)}{2} = \frac{1}{2}M_1(G) - |E(G)|, \quad |EV(G)| = 2|E(G)|.$$

Based on the definitions of these sets, we may write the subdivision-related graphs as

$$\begin{aligned}
L(G) &= (E(G), EE(G)),\\
S(G) &= (V(G) \cup E(G), EV(G)),\\
T(G) &= (V(G) \cup E(G), E(G) \cup EE(G) \cup EV(G)),\\
R(G) &= (V(G) \cup E(G), E(G) \cup EV(G)),\\
Q(G) &= (V(G) \cup E(G), EE(G) \cup EV(G)).
\end{aligned}$$

Obviously,

$$|V(L(G))| = |E(G)|,$$
$$|V(S(G))| = |V(T(G))| = |V(R(G))| = |V(Q(G))| = |V(G)| + |E(G)|.$$

Also,

$$|E(S(G))| = 2|E(G)|, \ |E(R(G))| = 3|E(G)|, \ |E(T(G))| = \frac{1}{2}M_1(G) + 2|E(G)|,$$
$$|E(Q(G))| = \frac{1}{2}M_1(G) + |E(G)|, \ |E(L(G))| = \frac{1}{2}M_1(G) - |E(G)|.$$

In the following lemma, we find the relationship among the degrees of vertices in subdivision-related graphs.

Lemma 2.1. *For any vertex $v \in V(G)$,*

$$d_{T(G)}(v) = d_{R(G)}(v) = 2d_{S(G)}(v) = 2d_{Q(G)}(v) = 2d_G(v),$$

and for any edge $e = uv \in E(G)$,

$$d_{S(G)}(e) = d_{R(G)}(e) = 2, \quad d_{T(G)}(e) = d_{Q(G)}(e) = d_{L(G)}(e) + 2 = d_G(u) + d_G(v).$$

Proof. By definition of the subdivision-related graphs, the proof is obvious. □

We define the third Zagreb index $M_3(G)$ of the graph $G = (V(G), E(G))$ as follows:

$$M_3(G) = \sum_{u \in V(G)} d(u)^3 = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2].$$

Now, we use Lemma 2.1 to determine the first Zagreb index of subdivision-related graphs.

Theorem 2.2.

(i) $M_1(L(G)) = M_3(G) + 2M_2(G) - 4M_1(G) + 4\,|E(G)|$,
(ii) $M_1(S(G)) = M_1(G) + 4\,|E(G)|$,
(iii) $M_1(T(G)) = M_3(G) + 2M_2(G) + 4M_1(G)$,
(iv) $M_1(R(G)) = 4M_1(G) + 4\,|E(G)|$,
(v) $M_1(Q(G)) = M_3(G) + 2M_2(G) + M_1(G)$.

Proof. (i) By definition of the line graph $L(G)$ and Lemma 2.1,

$$\begin{aligned} M_1(L(G)) &= \sum_{e\in V(L(G))} d_{L(G)}(e)^2 = \sum_{uv\in E(G)} [d_G(u) + d_G(v) - 2]^2 \\ &= \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2] + 4\,|E(G)| + 2 \sum_{uv\in E(G)} d_G(u)d_G(v) \\ &\quad - 4 \sum_{uv\in E(G)} [d_G(u) + d_G(v)] \\ &= M_3(G) + 2M_2(G) - 4M_1(G) + 4\,|E(G)|\,. \end{aligned}$$

(ii) By definition of the subdivision graph $S(G)$ and Lemma 2.1,

$$M_1(S(G)) = \sum_{u\in V(G)} d_G(u)^2 + \sum_{e\in E(G)} 2^2 = M_1(G) + 4\,|E(G)|\,.$$

(iii) By definition of the total graph $T(G)$ and Lemma 2.1,

$$\begin{aligned} M_1(T(G)) &= \sum_{u\in V(G)} (2d_G(u))^2 + \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2 \\ &= 4 \sum_{u\in V(G)} d_G(u)^2 + \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2] + 2 \sum_{uv\in E(G)} d_G(u)d_G(v) \\ &= M_3(G) + 2M_2(G) + 4M_1(G). \end{aligned}$$

(iv) By definition of $R(G)$ and Lemma 2.1,

$$M_1(R(G)) = \sum_{u\in V(G)} (2d_G(u))^2 + \sum_{e\in E(G)} 2^2 = 4M_1(G) + 4\,|E(G)|\,.$$

(v) By definition of $Q(G)$ and Lemma 2.1,

$$\begin{aligned} M_1(Q(G)) &= \sum_{u\in V(G)} d_G(u)^2 + \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2 \\ &= M_1(G) + \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2] + 2 \sum_{uv\in E(G)} d_G(u)d_G(v) \\ &= M_3(G) + 2M_2(G) + M_1(G). \end{aligned}$$

□

In the following lemma, we summarize the relations among distances between vertices in subdivision-related graphs.

Lemma 2.3 ([28]).

(i) *For any two vertices* $v, v' \in V(G)$,

$$\frac{1}{2}d_{S(G)}(v,v') = d_{T(G)}(v,v') = d_{R(G)}(v,v') = d_{Q(G)}(v,v') - 1 = d_G(v,v').$$

(ii) *For any two edges* $e, e' \in E(G)$,

$$\frac{1}{2}d_{S(G)}(e,e') = d_{T(G)}(e,e') = d_{R(G)}(e,e') - 1 = d_{Q(G)}(e,e') = d_{L(G)}(e,e').$$

(iii) *For any vertex* $v \in V(G)$ *and edge* $e \in E(G)$,

$$\frac{1}{2}(d_{S(G)}(e,v)+1) = d_{T(G)}(e,v) = d_{R(G)}(e,v) = d_{Q(G)}(e,v).$$

3. MAIN RESULTS

In this section, we prove several interesting relationships among vertex-weighted Wiener polynomials of subdivision operators. Then, by taking the first derivative of these relations at $x = 1$, we get the corresponding relationships for vertex-weighted Wiener numbers. Throughout this section, let G be a simple connected graph with n vertices and m edges. We start this section with the following simple lemma. Results follow easily from the definitions, so their proofs are omitted.

Lemma 3.1.

(i) $P_0(G;1) = \binom{n}{2}$,
(ii) $P_v(G;1) = m(n-1)$,
(iii) $P_{vv}(G;1) = 2m^2 - \frac{1}{2}M_1(G)$,
(iv) $P'_v(G;1) = W_v(G) + m(n-1)$,
(v) $P'_{vv}(G;1) = W_{vv}(G) + 4m^2 - M_1(G)$.

Theorem 3.2.

$$\begin{aligned} P_v(S(G);x) = \Big(\frac{1}{x} - \frac{1}{x^2}\Big)P_v(G;x^2) + \frac{1}{2x^2}P_v(R(G);x^2) - \frac{1}{2}P_0(G;x^2) \\ + \frac{1}{2}P_0(T(G);x^2) + \Big(2x - x^2 - \frac{1}{2}\Big)P_0(L(G);x^2). \end{aligned} \tag{3.1}$$

Proof. By definition of the singly vertex-weighted Wiener polynomial and Lemma 2.1, the polynomial $P_v(S(G);x)$ can be obtained by adding three polynomials as follows:

$$\begin{aligned} P_v(S(G);x) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} [d_{S(G)}(u)+d_{S(G)}(v)]x^{d_{S(G)}(u,v)+1} \\ +\frac{1}{2}\sum_{\{e,f\}\subseteq E(G)} [2+2]x^{d_{S(G)}(e,f)+1} \\ +\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)} [d_{S(G)}(u)+2]x^{d_{S(G)}(u,e)+1}. \end{aligned} \tag{3.2}$$

Now, we use Lemma 2.1 and Lemma 2.3 to compute each polynomial, separately.

The first polynomial is computed as follows:

$$\begin{aligned} &\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} [d_{S(G)}(u)+d_{S(G)}(v)]x^{d_{S(G)}(u,v)+1} \\ &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} [d_G(u)+d_G(v)]x^{2d_G(u,v)+1} = \frac{1}{x}P_v(G;x^2). \end{aligned}$$

The second polynomial is computed as follows:

$$2\sum_{\{e,f\}\subseteq E(G)} x^{d_{S(G)}(e,f)+1} = 2\sum_{\{e,f\}\subseteq E(G)} x^{2d_{L(G)}(e,f)+1} = 2xP_0(L(G);x^2).$$

The third polynomial is computed as follows:

$$\begin{aligned} &\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)} [d_{S(G)}(u)+2]x^{d_{S(G)}(u,e)+1} = \frac{1}{4}\sum_{u\in V(G)}\sum_{e\in E(G)} [d_{R(G)}(u)+4]x^{2d_{R(G)}(u,e)} \\ &= \frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)} x^{2d_{T(G)}(u,e)} + \frac{1}{4}\sum_{u\in V(G)}\sum_{e\in E(G)} [d_{R(G)}(u)+2]x^{2d_{R(G)}(u,e)} \\ &= \Big[\frac{1}{2}P_0(T(G);x^2) - \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} x^{2d_{T(G)}(u,v)} - \frac{1}{2}\sum_{\{e,f\}\subseteq E(G)} x^{2d_{T(G)}(e,f)}\Big] \\ &\quad + \Big[\frac{1}{2x^2}P_v(R(G);x^2) - \frac{1}{4}\sum_{\{u,v\}\subseteq V(G)} [d_{R(G)}(u)+d_{R(G)}(v)]x^{2d_{R(G)}(u,v)} \\ &\quad - \frac{1}{4}\sum_{\{e,f\}\subseteq E(G)} [2+2]x^{2d_{R(G)}(e,f)}\Big] \end{aligned}$$

$$= \Big[\frac{1}{2}P_0(T(G);x^2) - \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} x^{2d_G(u,v)} - \frac{1}{2}\sum_{\{e,f\}\subseteq E(G)} x^{2d_{L(G)}(e,f)}\Big]$$
$$+ \Big[\frac{1}{2x^2}P_v(R(G);x^2) - \frac{1}{4}\sum_{\{u,v\}\subseteq V(G)} [2d_G(u) + 2d_G(v)]x^{2d_G(u,v)}$$
$$- \sum_{\{e,f\}\subseteq E(G)} x^{2d_{L(G)}(e,f)+2}\Big]$$
$$= \Big[\frac{1}{2}P_0(T(G);x^2) - \frac{1}{2}P_0(G;x^2) - \frac{1}{2}P_0(L(G);x^2)\Big]$$
$$+ \Big[\frac{1}{2x^2}P_v(R(G);x^2) - \frac{1}{x^2}P_v(G;x^2) - x^2P_0(L(G);x^2)\Big]$$
$$= \frac{1}{2x^2}P_v(R(G);x^2) - \frac{1}{x^2}P_v(G;x^2) - \frac{1}{2}P_0(G;x^2) + \frac{1}{2}P_0(T(G);x^2)$$
$$- \Big(x^2 + \frac{1}{2}\Big)P_0(L(G);x^2).$$

Now, Eq. (3.1) is obtained by adding the above three polynomials and simplifying the resulting expression. □

By taking the first derivative from Eq. (3.1) with respect to x, and then by substituting $x = 1$, we can prove the following corollary. We also use Lemma 3.1 to simplify the relation.

Corollary 3.3.

$$W_v(S(G)) = W_v(R(G)) + W(T(G)) + W(L(G)) - W(G) - m(n + 2m - 1).$$

By rearranging the terms in the proof of Theorem 3.2, we can obtain an alternative expression for $P_v(S(G);x)$.

Theorem 3.4.

$$\begin{aligned} P_v(S(G);x) = {} & \Big(1 + \frac{1}{x} - \frac{2}{x^2}\Big)P_v(G;x^2) - \frac{1}{x^2}P_v(Q(G);x^2) + \frac{1}{x^2}P_v(T(G);x^2) \\ & - P_0(G;x^2) + P_0(T(G);x^2) + (2x-1)P_0(L(G);x^2). \end{aligned} \tag{3.3}$$

Proof. Using Eq. (3.2) and the proof of Theorem 3.2, we have

$$\begin{aligned} P_v(S(G);x) = {} & \frac{1}{x}P_v(G;x^2) + 2xP_0(L(G);x^2) \\ & + \frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)} [d_{S(G)}(u) + 2]x^{d_{S(G)}(u,e)+1}. \end{aligned} \tag{3.4}$$

By Lemma 2.1 and Lemma 2.3, the last polynomial in Eq. (3.4) can also be computed as follows:

$$
\begin{aligned}
&\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{S(G)}(u)+2]x^{d_{S(G)}(u,e)+1}\\
&=\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{T(G)}(u)-d_{Q(G)}(u)]x^{d_{S(G)}(u,e)+1}+\sum_{u\in V(G)}\sum_{e\in E(G)}x^{d_{S(G)}(u,e)+1}\\
&=\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{T(G)}(u)+d_{T(G)}(e)]x^{2d_{T(G)}(u,e)}\\
&\quad-\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{Q(G)}(u)+d_{Q(G)}(e)]x^{2d_{Q(G)}(u,e)}+\sum_{u\in V(G)}\sum_{e\in E(G)}x^{2d_{T(G)}(u,e)}\\
&=\Big[\frac{1}{x^2}P_v(T(G);x^2)-\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[d_{T(G)}(u)+d_{T(G)}(v)]x^{2d_{T(G)}(u,v)}\\
&\quad-\frac{1}{2}\sum_{\{e,f\}\subseteq E(G)}[d_{T(G)}(e)+d_{T(G)}(f)]x^{2d_{T(G)}(e,f)}\Big]\\
&\quad-\Big[\frac{1}{x^2}P_v(Q(G);x^2)-\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[d_{Q(G)}(u)+d_{Q(G)}(v)]x^{2d_{Q(G)}(u,v)}\\
&\quad-\frac{1}{2}\sum_{\{e,f\}\subseteq E(G)}[d_{Q(G)}(e)+d_{Q(G)}(f)]x^{2d_{Q(G)}(e,f)}\Big]\\
&\quad+\Big[P_0(T(G);x^2)-\sum_{\{u,v\}\subseteq V(G)}x^{2d_{T(G)}(u,v)}-\sum_{\{e,f\}\subseteq E(G)}x^{2d_{T(G)}(e,f)}\Big]\\
&=\Big[\frac{1}{x^2}P_v(T(G);x^2)-\sum_{\{u,v\}\subseteq V(G)}[d_G(u)+d_G(v)]x^{2d_G(u,v)}\Big]\\
&\quad-\Big[\frac{1}{x^2}P_v(Q(G);x^2)-\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[d_G(u)+d_G(v)]x^{2(d_G(u,v)+1)}\Big]\\
&\quad+\Big[P_0(T(G);x^2)-\sum_{\{u,v\}\subseteq V(G)}x^{2d_G(u,v)}-\sum_{\{e,f\}\subseteq E(G)}x^{2d_{L(G)}(e,f)}\Big]\\
&=\Big(1-\frac{2}{x^2}\Big)P_v(G;x^2)-\frac{1}{x^2}P_v(Q(G);x^2)+\frac{1}{x^2}P_v(T(G);x^2)-P_0(G;x^2)\\
&\quad+P_0(T(G);x^2)-P_0(L(G);x^2).
\end{aligned}
$$

Now, using Eq. (3.4), we can get the desired result. □

By taking the first derivative from Eq. (3.3) with respect to x, and then by substituting $x = 1$, we easily arrive at:

Corollary 3.5.

$$W_v(S(G)) = 2W_v(T(G)) - 2W_v(Q(G)) + 2W(T(G)) + 2W(L(G)) - 2W(G) \\ + m(n - m - 2).$$

By eliminating the term $P_v(S(G); x)$ in Eq. (3.1) and Eq. (3.3), we can obtain a formula for $P_v(R(G); x)$ similar to Eq. (3.3).

Corollary 3.6.

$$P_v(R(G); x) = 2(x - 1)P_v(G; x) - 2P_v(Q(G); x) + 2P_v(T(G); x) - xP_0(G; x) \\ + xP_0(T(G); x) + (2x^2 - x)P_0(L(G); x). \tag{3.5}$$

From Eq. (3.5), we obtain the following relationship among vertex-weighted Wiener numbers.

Corollary 3.7.

$$W_v(R(G)) = 2W_v(T(G)) - 2W_v(Q(G)) + W(T(G)) + W(L(G)) - W(G) \\ + m(2n + m - 3).$$

By combining Eqs. (3.3) and (3.5), we get a relation among the singly vertex-weighted Wiener polynomials of $S(G)$, $R(G)$, $Q(G)$, $T(G)$, and G.

Corollary 3.8.

$$x^2P_v(S(G); x) + (x^2 - x)P_v(G; x^2) - P_v(Q(G); x^2) \\ + P_v(T(G); x^2) - P_v(R(G); x^2) + (2x^4 - 2x^3)P_0(L(G); x^2) = 0. \tag{3.6}$$

Proof. By Eq. (3.5),

$$x^2[P_0(T(G); x^2) - P_0(G; x^2)] = P_v(R(G); x^2) - 2(x^2 - 1)P_v(G; x^2) + 2P_v(Q(G); x^2) \\ - 2P_v(T(G); x^2) - (2x^4 - x^2)P_0(L(G); x^2).$$

Also, by multiplying the Eq. (3.3) by x^2, we have

$$x^2P_v(S(G); x) = (x^2 + x - 2)P_v(G; x^2) - P_v(Q(G); x^2) + P_v(T(G); x^2) \\ + x^2(2x - 1)P_0(L(G); x^2) + x^2[P_0(T(G); x^2) - P_0(G; x^2)].$$

Now, by eliminating the term $x^2[P_0(T(G); x^2) - P_0(G; x^2)]$ between the above two equations, we can get the desired result. □

From Eq. (3.6) we obtain the following relation for the vertex-weighted Wiener numbers of four subdivisions.

Corollary 3.9.

$$2W_v(Q(G)) + 2W_v(R(G)) - 2W_v(T(G)) - W_v(S(G)) = m(3n + 3m - 4).$$

We notice that the result does not depend neither on $W(G)$ nor on $W_v(G)$.

Now, we turn our attention toward doubly vertex-weighted Wiener polynomials. In the next theorem, we obtain a relation among doubly vertex-weighted Wiener polynomials of $S(G)$, $R(G)$, and G.

Theorem 3.10.

$$\begin{aligned} P_{vv}(S(G); x) = &\Big(\frac{1}{x^2} - \frac{2}{x^3}\Big)P_{vv}(G; x^2) + \frac{1}{2x^3}P_{vv}(R(G); x^2) \\ &+ (4x^2 - 2x^3)P_0(L(G); x^2). \end{aligned} \tag{3.7}$$

Proof. By definition of the doubly vertex-weighted Wiener polynomial and Lemma 2.1, the polynomial $P_{vv}(S(G); x)$ can be obtained by adding three polynomials as follows:

$$\begin{aligned} P_{vv}(S(G); x) = &\sum_{\{u,v\}\subseteq V(G)} [d_{S(G)}(u)d_{S(G)}(v)]x^{d_{S(G)}(u,v)+2} \\ &+ \sum_{\{e,f\}\subseteq E(G)} [2 \times 2]x^{d_{S(G)}(e,f)+2} \\ &+ \sum_{u\in V(G)} \sum_{e\in E(G)} [d_{S(G)}(u) \times 2]x^{d_{S(G)}(u,e)+2}. \end{aligned} \tag{3.8}$$

Now, we use Lemma 2.1 and Lemma 2.3 to compute these polynomials.

The first polynomial is computed as follows:

$$\begin{aligned} &\sum_{\{u,v\}\subseteq V(G)} [d_{S(G)}(u)d_{S(G)}(v)]x^{d_{S(G)}(u,v)+2} \\ &= \sum_{\{u,v\}\subseteq V(G)} [d_G(u)d_G(v)]x^{2d_G(u,v)+2} = \frac{1}{x^2}P_{vv}(G; x^2). \end{aligned}$$

The second polynomial is computed as follows:

$$4 \sum_{\{e,f\}\subseteq E(G)} x^{d_{S(G)}(e,f)+2} = 4 \sum_{\{e,f\}\subseteq E(G)} x^{2d_{L(G)}(e,f)+2} = 4x^2P_0(L(G); x^2).$$

The third polynomial is computed as follows:

$$\begin{aligned}
&\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{S(G)}(u)\times 2]x^{d_{S(G)}(u,e)+2}\\
&=\frac{1}{2}\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{R(G)}(u)\times 2]x^{2d_{R(G)}(u,e)+1}\\
&=\frac{1}{2x^3}P_{vv}(R(G);x^2)-\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[d_{R(G)}(u)d_{R(G)}(v)]x^{2d_{R(G)}(u,v)+1}\\
&\quad-\frac{1}{2}\sum_{\{e,f\}\subseteq E(G)}[2\times 2]x^{2d_{R(G)}(e,f)+1}\\
&=\frac{1}{2x^3}P_{vv}(R(G);x^2)-\frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}[2d_G(u)\times 2d_G(v)]x^{2d_G(u,v)+1}\\
&\quad-2\sum_{\{e,f\}\subseteq E(G)}x^{2d_{L(G)}(e,f)+3}\\
&=\frac{1}{2x^3}P_{vv}(R(G);x^2)-\frac{2}{x^3}P_{vv}(G;x^2)-2x^3P_0(L(G);x^2).
\end{aligned}$$

Now, Eq. (3.7) is obtained by adding the above three polynomials, and simplifying the resulting expression. □

By taking the first derivative from Eq. (3.7) with respect to x, and then by substituting $x = 1$, we can prove the following corollary. We also use Theorem 2.2 and Lemma 3.1, to simplify the relation.

Corollary 3.11.

$$W_{vv}(S(G)) = W_{vv}(R(G)) - 2W_{vv}(G) + 4W(L(G)) + 2m(1 - 3m).$$

In the following theorem, we find a relation between the Wiener polynomials and the singly and doubly vertex-weighted Wiener polynomials of $S(G)$ and G.

Theorem 3.12.

$$\begin{aligned}
P_{vv}(S(G);x)-\frac{1}{x^2}P_{vv}(G;x^2)&=4[xP_v(S(G);x)-P_v(G;x^2)]\\
&\quad-4x^2[P_0(S(G);x)-P_0(G;x^2)].
\end{aligned}\tag{3.9}$$

Proof. Using Eq. (3.8) and the proof of Theorem 3.10, we have

$$\begin{aligned}
P_{vv}(S(G);x)&=\frac{1}{x^2}P_{vv}(G;x^2)+4x^2P_0(L(G);x^2)\\
&\quad+\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{S(G)}(u)\times 2]x^{d_{S(G)}(u,e)+2}.
\end{aligned}\tag{3.10}$$

By Lemma 2.1 and Lemma 2.3, the last polynomial in Eq. (3.10) can also be computed as follows:

$$\begin{aligned}
&\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{S(G)}(u)\times 2]x^{d_{S(G)}(u,e)+2}\\
&= 2\sum_{u\in V(G)}\sum_{e\in E(G)}[d_{S(G)}(u)+2]x^{d_{S(G)}(u,e)+2} - 4\sum_{u\in V(G)}\sum_{e\in E(G)}x^{d_{S(G)}(u,e)+2}\\
&= \Bigg[4xP_v(S(G);x) - 2\sum_{\{u,v\}\subseteq V(G)}[d_{S(G)}(u)+d_{S(G)}(v)]x^{d_{S(G)}(u,v)+2}\\
&\quad - 2\sum_{\{e,f\}\subseteq E(G)}[2+2]x^{d_{S(G)}(e,f)+2}\Bigg]\\
&\quad - \Bigg[4x^2P_0(S(G);x) - 4\sum_{\{u,v\}\subseteq V(G)}x^{d_{S(G)}(u,v)+2} - 4\sum_{\{e,f\}\subseteq E(G)}x^{d_{S(G)}(e,f)+2}\Bigg]\\
&= \Bigg[4xP_v(S(G);x) - 2\sum_{\{u,v\}\subseteq V(G)}[d_G(u)+d_G(v)]x^{2d_G(u,v)+2}\\
&\quad - 8\sum_{\{e,f\}\subseteq E(G)}x^{2d_{L(G)}(e,f)+2}\Bigg]\\
&\quad - \Bigg[4x^2P_0(S(G);x) - 4\sum_{\{u,v\}\subseteq V(G)}x^{2d_G(u,v)+2} - 4\sum_{\{e,f\}\subseteq E(G)}x^{2d_{L(G)}(e,f)+2}\Bigg]\\
&= \big[4xP_v(S(G);x) - 4P_v(G;x^2) - 8x^2P_0(L(G);x^2)\big]\\
&\quad - \big[4x^2P_0(S(G);x) - 4x^2P_0(G;x^2) - 4x^2P_0(L(G);x^2)\big]\\
&= 4[xP_v(S(G);x) - P_v(G;x^2)] - 4x^2[P_0(S(G);x) - P_0(G;x^2)]\\
&\quad - 4x^2P_0(L(G);x^2).
\end{aligned}$$

Now, by Eq. (3.10), we can get the desired result. □

From Eq. (3.9) we have the following relation for vertex-weighted Wiener numbers.

Corollary 3.13.

$$W_{vv}(S(G)) - 2W_{vv}(G) = 4W_v(S(G)) - 8W_v(G) - 4W(S(G)) + 8W(G).$$

The following theorem is similar to Theorem 3.12 and gives a relationship between the Wiener polynomials and the singly and doubly vertex-weighted Wiener polynomials of $R(G)$ and G.

Theorem 3.14.

$$\begin{aligned}
P_{vv}(R(G);x) - 4P_{vv}(G;x) &= 4x[P_v(R(G);x) - 2P_v(G;x)]\\
&\quad - 4x^2[P_0(R(G);x) - P_0(G;x)].
\end{aligned} \tag{3.11}$$

Proof. Using the same argument as in the proof of Theorem 3.12, we can get Eq. (3.11). □

By taking the first derivative of Eq. (3.11) at $x = 1$, we easily obtain the following result.

Corollary 3.15.

$$W_{vv}(R(G)) - 4W_{vv}(G) = 4W_v(R(G)) - 8W_v(G) - 4W(R(G)) + 4W(G).$$

In the next theorem, we prove a relation among doubly vertex-weighted Wiener polynomials of $L(G)$, $Q(G)$, $T(G)$, and G.

Theorem 3.16.

$$\begin{aligned} P_{vv}(Q(G); x) = (x-2)P_{vv}(G; x) + \frac{1}{2}P_{vv}(T(G); x) + \frac{1}{2}P_{vv}(L(G); x) \\ + 2xP_v(L(G); x) + 2x^2P_0(L(G); x). \end{aligned} \tag{3.12}$$

Proof. By definition of the doubly vertex-weighted Wiener polynomial and Lemma 2.1, the polynomial $P_{vv}(Q(G); x)$ can be obtained by adding three polynomials as follows:

$$\begin{aligned} P_{vv}(Q(G); x) = &\sum_{\{u,v\}\subseteq V(G)} [d_{Q(G)}(u)d_{Q(G)}(v)]x^{d_{Q(G)}(u,v)+2} \\ &+ \sum_{\{e,f\}\subseteq E(G)} [d_{Q(G)}(e)d_{Q(G)}(f)]x^{d_{Q(G)}(e,f)+2} \\ &+ \sum_{u\in V(G)} \sum_{e\in E(G)} [d_{Q(G)}(u)d_{Q(G)}(e)]x^{d_{Q(G)}(u,e)+2}. \end{aligned}$$

The first polynomial is computed as follows:

$$\begin{aligned} &\sum_{\{u,v\}\subseteq V(G)} [d_{Q(G)}(u)d_{Q(G)}(v)]x^{d_{Q(G)}(u,v)+2} \\ &= \sum_{\{u,v\}\subseteq V(G)} [d_G(u)d_G(v)]x^{d_G(u,v)+3} = xP_{vv}(G; x). \end{aligned}$$

The summation of the second and third polynomials is equal to

$$
\begin{aligned}
&\sum_{\{e,f\}\subseteq E(G)} [d_{Q(G)}(e)d_{Q(G)}(f)]x^{d_{Q(G)}(e,f)+2} \\
&\quad + \sum_{u\in V(G)} \sum_{e\in E(G)} [d_{Q(G)}(u)d_{Q(G)}(e)]x^{d_{Q(G)}(u,e)+2} \\
&= \frac{1}{2} \sum_{\{e,f\}\subseteq E(G)} [(d_{L(G)}(e)+2)(d_{L(G)}(f)+2)]x^{d_{L(G)}(e,f)+2} \\
&\quad + \frac{1}{2} \sum_{\{e,f\}\subseteq E(G)} [d_{T(G)}(e)d_{T(G)}(f)]x^{d_{T(G)}(e,f)+2} \\
&\quad + \frac{1}{2} \sum_{u\in V(G)} \sum_{e\in E(G)} [d_{T(G)}(u)d_{T(G)}(e)]x^{d_{T(G)}(u,e)+2} \\
&= \left[\frac{1}{2}P_{vv}(L(G);x) + 2xP_v(L(G);x) + 2x^2P_0(L(G);x)\right] \\
&\quad + \left[\frac{1}{2}P_{vv}(T(G);x) - \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} [d_{T(G)}(u)d_{T(G)}(v)]x^{d_{T(G)}(u,v)+2}\right] \\
&= \left[\frac{1}{2}P_{vv}(L(G);x) + 2xP_v(L(G);x) + 2x^2P_0(L(G);x)\right] \\
&\quad + \left[\frac{1}{2}P_{vv}(T(G);x) - \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} [2d_G(u)\times 2d_G(v)]x^{d_G(u,v)+2}\right] \\
&= \frac{1}{2}P_{vv}(L(G);x) + 2xP_v(L(G);x) + 2x^2P_0(L(G);x) \\
&\quad + \frac{1}{2}P_{vv}(T(G);x) - 2P_{vv}(G;x).
\end{aligned}
$$

Now, Eq. (3.12) is obtained by adding the above polynomials and simplifying the resulting expression. □

From Eq. (3.12), we get the following relationship among the considered vertex-weighted Wiener numbers.

Corollary 3.17.

$$
\begin{aligned}
2W_{vv}(Q(G)) &= W_{vv}(T(G)) - 2W_{vv}(G) + W_{vv}(L(G)) + 4W_v(L(G)) \\
&\quad + 4W(L(G)) - M_1(G) + 4m^2.
\end{aligned}
$$

Finally, by combining Eqs. (3.7) and (3.12), we can get an interesting relation among doubly vertex-weighted Wiener polynomials of the graph G and all subdivision operators.

Corollary 3.18.

$$\frac{1}{x-2}[2x^3P_{vv}(S(G);x) - P_{vv}(R(G);x^2)] = 2(x^3-2x+1)P_{vv}(G;x^2) \\ - 2xP_{vv}(Q(G);x^2) + xP_{vv}(T(G);x^2) + xP_{vv}(L(G);x^2) + 4x^3P_v(L(G);x^2). \tag{3.13}$$

Proof. By Eq. (3.12),

$$2x^4P_0(L(G);x^2) = P_{vv}(Q(G);x^2) - (x^2-2)P_{vv}(G;x^2) - \frac{1}{2}P_{vv}(T(G);x^2) \\ - \frac{1}{2}P_{vv}(L(G);x^2) - 2x^2P_v(L(G);x^2).$$

Now, the result follows by eliminating the term $P_0(L(G);x^2)$ in the above relation and Eq. (3.7). □

The corresponding relationship for vertex-weighted Wiener numbers is given in the following corollary.

Corollary 3.19.

$$W_{vv}(S(G)) - W_{vv}(R(G)) = 2W_{vv}(Q(G)) - W_{vv}(T(G)) - W_{vv}(L(G)) - 4W_v(L(G)) \\ + M_1(G) - 2m(5m-1).$$

Acknowledgments
The authors would like to thank the anonymous referees for their careful reading and valuable suggestions. Partial support by the Center of Excellence of Algebraic Hyper-structures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the second author (AI). Partial support of the Croatian Science Foundation (research project BioAmpMode (IP-11-2013)) is gratefully acknowledged by the third author.

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Received: December 18, 2014.
Revised: June 14, 2015.
Accepted: June 16, 2015.