MONTE CARLO METHOD IN ANALYSIS OF ROAD ACCIDENTS VERSUS INTERPRETATION OF CALCULATION RESULTS

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Summary

In the article, the Monte Carlo method (MCM) has been characterized from the point of view of road accident reconstruction. This method lies in making repeated calculations with the use of the same deterministic mathematical model, but with picking out the values of specific parameters on a pseudo-random basis from within predefined ranges of uncertainty. The calculation results have been presented in the form of a probability density function similar, in terms of its graphical representation, to a bell-shaped curve; such a form facilitates the statistical interpretation of data and the uncertainty analysis. In particular, it is possible to narrow the range of results by rejecting the extreme areas of low probability. Examples have been presented, focused on the issues concerning the calculation of pre-impact velocities, location of the collision point on the road, and kinematic analysis (referred to as "time-distance analysis") of the pre-impact phase of a pedestrian accident. In the collision analysis, both the reconstruction methods (based on the momentum conservation principle and on Marquard models of calculating the post-impact velocities) and simulation techniques (simulation of the impact and the dynamics of motion in the PC-Crash program) were employed. It has been shown that the area of the largest concentration of the Monte Carlo simulation results is actually the area of most common responses of the deterministic model used for the data ranges adopted, but not necessarily a reflection of the truth. The crucial point is to develop an adequate mathematical model of the physical phenomenon.

Keywords: Monte Carlo method, collision, pedestrian accident, uncertainty

1. Introduction

An important problem inherent in the accident reconstruction lies in the relatively scant set of the data having been collected and the necessity to introduce many parameters with a wide range of tolerance of the parameter values, which raises the uncertainty of calculation results. Among various uncertainty analysis methods, comprehensively studied by, inter alia, Brach [3] or Guzek and Lozia [10], the one that deserves special

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attention is the Monte Carlo method, which makes it possible to present the result in the form of probability density function for the parameter calculated.

The Monte Carlo method was invented by a Polish mathematician of the Lvov school, Stanisław Ulam, in cooperation with John von Neumann and Nicholas Metropolis when they worked on the Manhattan project [14]. Kost and Werner [12] as well as Wood and O'Riordain [27] presented advantages of analysing the uncertainty with the use of the Monte Carlo method. In the former study, attention was paid to the possibility of loading the input data in the form of various probability density functions and in the latter one, the authors highlighted the easier analysing of susceptibility of calculation results to the scatter of data and the possibility of reducing the number of equations describing all the circumstances by verifying some additional criteria related to partial problems. Bartlett [1] described methods of carrying out Monte Carlo calculations with the use of the tools available in the MS Excel spreadsheet program. Kimbrough [11] considered the possibility of using a Monte Carlo simulation for analysing the likelihood ratio of two opposing versions of the pre-accident situation. Fleck and Daily [7] examined the sensitivity of the Monte Carlo method in the reconstruction of a vehicle collision.

The Monte Carlo method lies in making repeated calculations with the use of the same deterministic mathematical model, but with picking out every time the values of specific parameters on a pseudo-random basis from within predefined ranges of uncertainty. At this method, an assumption is made that the input data are statistically independent; simultaneously, the probability density functions of the data must be known or assumed a priori [1, 7]. The procedure is cyclically repeated until the result reaches a form at which its probability density function becomes close to a bell-shaped curve, because this will enable further processing of the results with statistical methods.

In the PC-Crash road accident simulation program, the Monte Carlo method can be used for searching for the optimum parameters of vehicle motion just before the impact and/or for a collision model by varying any of the parameters that characterize the vehicles involved and/or the controlling of the vehicles in the pre-impact motion [17].

In the article, the issues have been addressed that concern the calculation of vehicle collision parameters and the kinematic analysis (referred to as "time-distance analysis") of the pre-impact phase of a pedestrian accident, where the Monte Carlo method was employed. For this method to be more easily understood and for its practical advantages in the reconstruction of road accidents to be highlighted, a few examples have been shown. For a collision of vehicles, three examples have been presented with a model using the momentum conservation principle and Marquard's methods of calculation of the separation velocities (i.e. the vehicle velocities just after the impact) in the accident-reconstruction approach; this has been supplemented with one example based on the simulation of a collision and post-impact motion. The calculations were made with the use of a computer program (Marlo.exe) developed by the author and the PC-Crash simulation program. A more in-depth study on the reliability of the reconstruction of a road accident, where the uncertainty of calculations is one of the important factors, was presented in publication [22].

2. Example 1: Introduction into the method

Let us consider a simple example, where the Monte Carlo method will be used to calculate the initial vehicle velocity. What is known is the length s = 20 m of the skid marks left by a vehicle braking on a horizontal dry and clean asphalt road surface. The average value of the coefficient of friction has been assumed as μ = 0.8, according to literature tables. Based on a formula:

$$v = \sqrt{2\mu g s}, \tag{1}$$

where g = 9.81 m/s^2 – acceleration of gravity, the vehicle velocity at the beginning of the skid marks is to be calculated, inclusive of the uncertainty of the result. The values of uncertainty of the input data have been assumed as Δs = ±0.9 m for the length measured and $\Delta\mu$ = ±0.06 for the coefficient of friction. The maximum and root-mean-square (rms) uncertainty values (see e.g. [3, 10]) are given by the following formulas:

$$\Delta v_{max} = \frac{\partial v}{\partial \mu}\Delta\mu + \frac{\partial v}{\partial s}\Delta s \quad \mathrm{i} \quad \Delta v_{sqr} = \sqrt{\left(\frac{\partial v}{\partial \mu}\Delta\mu\right)^2 + \left(\frac{\partial v}{\partial s}\Delta s\right)^2} \tag{2}$$

where:

$$\frac{\partial v}{\partial \mu} = \frac{gs}{v}, \qquad \frac{\partial v}{\partial s} = \frac{g\mu}{v}. \tag{3}$$

Having substituted the numerical input data and having made the calculations, we obtain $v_0 \pm \Delta v_{max}$ = 17.7 ± 1,1 m/s and $v_0 \pm \Delta v_{sqrt}$ = 17.7 ± 0.78 m/s.

Every velocity value within the ranges $v_0 \pm \Delta v_{max}$ or $v_0 \pm \Delta v_{sqrt}$ is equally likely to occur, i.e. the probability distribution is uniform and the corresponding probability density function is given by a formula

$$f(v) = \begin{cases} \dfrac{1}{2\cdot\Delta v} & dla \quad v \in (v - \Delta v, v + \Delta v) \\ 0 & dla \quad v \notin (v - \Delta v, v + \Delta v) \end{cases}, \tag{4}$$

where Δv represents either Δv_{max} or Δv_{sqr}. These distributions have been shown in Fig. 1.

A Monte Carlo simulation was also carried out, with calculations to formula (1) being repeated 20 000 times for uniform distributions of data within the ranges of $s\pm\Delta s$ and $\mu\pm\Delta\mu$. The distribution of results in the form of a bell-shaped curve has been marked in blue in the graph in Fig. 1. Then, a similar simulation was carried out, but with the data being sampled at random according to the normal distributions for:

$\sigma_\mu = \Delta\mu/3$ =0.02 – standard deviation of the coefficient offriction;

$\sigma_s = \Delta s/3$=0.3 m – standard deviation of the length measured.

The v calculation results formed a pattern very close to normal distribution with parameters v_0 = 17.7 m/s and σ_v = 0.36 m/s (black curve in the graph). The theoretical normal distribution for such parameters is slightly shifted leftwards (Pearson product-moment correlation coefficient R^2 = 0.983) and it has been represented by the black dashed curve. In each case, the expected value v_0 is identical; the same applies to the span of results, except for the span related to Δv_{sqr}). However, the Monte Carlo method made it possible to narrow the range of the most likely results. In the case of Δv_{sqr}, the span of results covered approximately the range $v_0 \pm 2\sigma_v$, which translates in practice into cutting off the rarest extreme results.

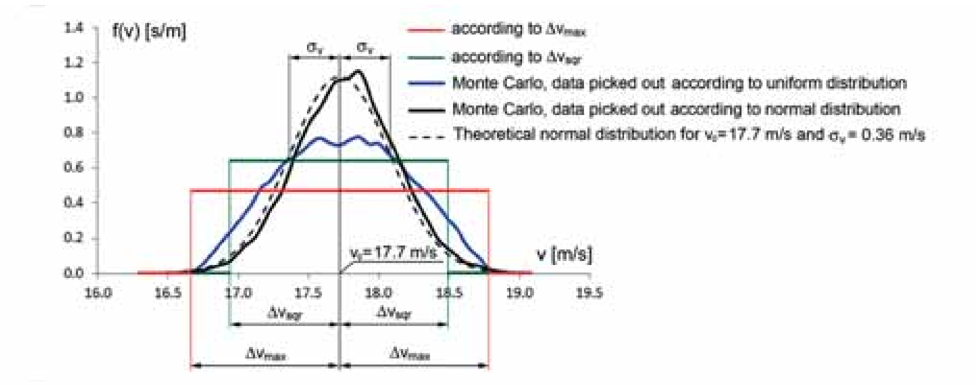

Fig. 1. Comparison of probability density functions

At the reconstruction of a specific case for forensic investigation purposes with the use of the Monte Carlo method, it is highly recommended to adopt data with assuming their uniform distribution. Let it be illustrated by the example where the pedestrian velocity is defined by a witness e.g. by the words "he walked at a normal pace". Although the research on pedestrian movement gives results distributed in accordance with the Gaussian curve (i.e. normal distribution curve), nobody knows how accurate the witness's assessment was in a specific individual case; therefore, an assumption should be made that every value from within the range of "normal walking pace" as specified in relevant tables is equally likely to occur.

3. Pre-impact velocities: a model used for accident reconstruction calculations

At the accident reconstruction calculations of collision parameters, the post-impact motion of the vehicles involved is examined at first; only then, the pre-impact velocities are calculated based on the momentum and angular momentum theorems (i.e. in accordance with the "classical collision model") or on the momentum and energy conservation laws. For the analysis to be simplified, an example has been chosen where the momentum conservation law would be sufficient for solving the problem. In the vector notation, this law has the form:

$$m_1 \mathbf{v}'_{IS_1,C} + m_2 \mathbf{v}'_{IS_2,C} = m_1 \mathbf{v}_{IS_1,C} + m_2 \mathbf{v}_{IS_2,C} \, , \qquad (5)$$

where: i=1.2 – index representing the number of a specific vehicle; m_i – mass of the vehicle; $v'_{IS_i,C}$ – velocity vector of the centre of vehicle mass S_i at the instant of vehicle separation; $v_{IS_i,C}$ – velocity vector of the centre of vehicle mass S_i at the instant just before the impact.

This law is a special case of the momentum theorem and may be applied to the central oblique collision (illustrated in Fig. 2a), where the pre-impact velocity directions should be far from being parallel for the dividing by zero in formulas (8) and (9) to be avoided.

In the case of strongly eccentric collisions, the failure of considering the angular momentum theorem may lead to major errors (see sub-item 3.2). In such a situation, the full classical collision model must be used, inclusive of the conditions related to the coefficient of restitution and friction in the tangential plane of the vehicles involved in the collision.

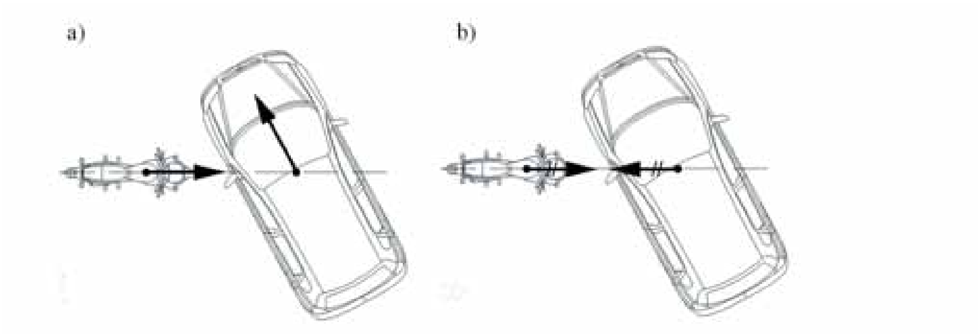

Fig. 2. Types of central collisions: a) – oblique collision; b) – head-on collision

The motor vehicles in their impact and final (rest) positions have been shown in Fig. 3.

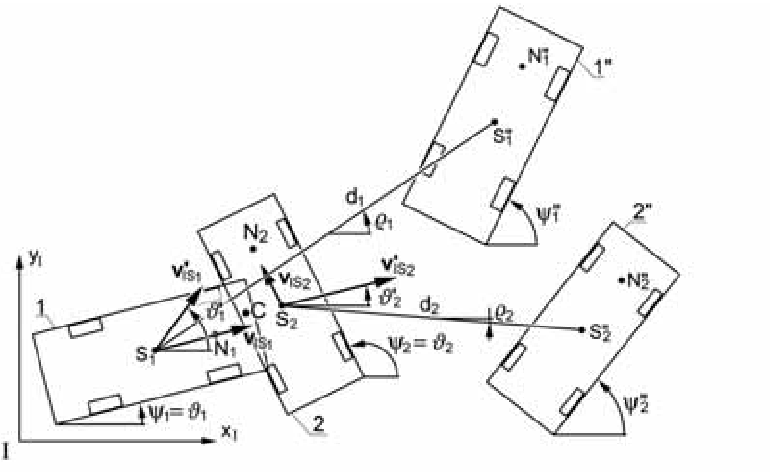

Fig. 3. Schematic drawing for the analysis of a two-vehicle collision

In the analysis, the notation as shown below has been adopted. Firstly, a symbol without an additional superscript (e.g. v_i) relates to the pre-impact state; the sign "prime" (as in e.g. v_i') has the meaning that the symbol relates to the instant of separation; and the sign "double prime" (as in e.g. v_i'') shows that the symbol relates to the final position. Moreover:

$\{I\}$ – global inertial coordinate system with its origin being placed at point I;

$\{N_i\}$ – local coordinate system with its origin being placed in the middle of the front wheel axis of the i^{th} vehicle N_i, parallel to the $\{S_i\}$ coordinate system with its origin being placed at the centre of vehicle mass S_i;

I_{z_i} – vehicle's moment of inertia in relation to the vertical axis going through the centre of mass S_i;

L_i, – vehicle wheelbase;

$\boldsymbol{r}_{N_i S_i, N_i}$ – position of the centre of vehicle mass in relation to the $\{N_i\}$ system, i.e. the vector extending from N_i to S_i expressed in the $\{N_i\}$ system;

$\boldsymbol{r}''_{IN_i, N_i}$ – vector extending from I to N_i in the final position of the vehicle, in relation to the $\{N_i\}$ system;,

ψ_i – instantaneous yaw angle in relation to the $\{I\}$ system;

$\Delta\psi_i$ – total vehicle yaw angle in the post-impact motion;

$\omega_i = \dot{\psi}_i$ – angular velocity of vehicle yaw;

ϑ_i – angle of the velocity vector of the centre of vehicle mass, measured in relation to the $\{I\}$ system as shown in Fig. 3;

δ_i – angle of vehicle 2 in relation to vehicle 1 just before the impact;

vi_{min} – minimum pre-impact velocity;

vi_{max} – maximum pre-impact velocity;

μ_i – tyre-to-road friction oefficient;

f_i – coefficient of resistance to post-impact motion from 0 for all the wheels freely rolling to 1 for all the wheels locked without rolling

EES_i – Energy-Equivalent Speed, i.e. the vehicle velocity equivalent to the energy of plastic deformation of the vehicle;

ϱ_i – angle of vector $\boldsymbol{r}_{S'_i S''_i, I}$, i.e. of the vector extending from point S'_i to point S''_i measured from the x_I axis of the $\{I\}$ system.

Assumptions:

- The impact velocity vectors lay on the longitudinal vehicle centrelines and their sense was such that they pointed towards the vehicle front ($\vartheta_i = \psi_i$).
- The model is applicable to passenger cars or similar vehicles whose motion can be classified as planar.
- Pre-impact angular velocities of the vehicles were $\omega_1 = \omega_2 = 0$.
- The road surface reactions acting on vehicle tyres were ignored.
- The collision had the nature of a rough collision.
- The pre-impact and separation positions of the vehicles were identical.
- The resistance to post-impact motion and the coefficient of friction were constant.

Pre-impact velocities

For the assumptions made, the separation velocities v'_i and ω'_i can be calculated with employing the kinetic energy and friction work equivalency principle. Taking this as a basis and considering the bicycle vehicle model, Marquard [13] developed a method where he expressed the rotational resistance by means of correction factors. Below, the McHenry modification (known under the name of SPIN 2 [6]) was used, where the diversified resistance at wheels has been taken into account with the use of factor f_i (for more details see [19, 22, 26]). In this modification, the following formulas have been adopted:

$$\omega_i^{'} = \text{sign}\, \Delta\psi_i \sqrt{\frac{\mu_i\, g\, \Delta\psi_i^2}{\frac{I_{z\,i}}{m_i L_i}(1-f_i)|\Delta\psi_i| + \frac{d_i}{1{,}7}}}, \tag{6}$$

$$v_i^{'} = 1{,}7\left[\frac{\mu_i\, g\, \Delta\psi_i}{\omega_i^{'}} - \frac{I_{z\,i}}{m_i L_i}(1-f_i)\omega_i^{'}\right]. \tag{7}$$

The vector equation (5) may be expressed by two algebraic equations in the $\{I\}$ system. After transformations, the pre-impact velocities are given by formulas:

$$v_1 = \frac{m_1 v_1^{'} \sin(\vartheta_2 - \vartheta_1^{'}) + m_2 v_2^{'} \sin(\vartheta_2 - \vartheta_2^{'})}{m_1 \sin(\vartheta_2 - \vartheta_1)}, \tag{8}$$

$$v_2 = -\frac{m_1 v_1^{'} \sin(\vartheta_1 - \vartheta_1^{'}) + m_2 v_2^{'} \sin(\vartheta_1 - \vartheta_2^{'})}{m_2 \sin(\vartheta_2 - \vartheta_1)}. \tag{9}$$

Algorithm of the Monte Carlo method

The final (rest) vehicle positions are precisely known; therefore, the matrixes of rotation from the local coordinate systems at the final positions $\{N_i\}$ to the inertial system $\{I\}$ (see (10)) may be calculated only once before starting the Monte Carlo simulation procedure.

$$\mathbf{A}_{IN_i}^{''} = \begin{bmatrix} \cos\psi_i^{''} & -\sin\psi_i^{''} \\ \sin\psi_i^{''} & \cos\psi_i^{''} \end{bmatrix}, i = 1{,}2 \tag{10}$$

A single cycle of the calculation procedure is carried out pursuant to the algorithm as specified below.

1. Input data: picking out a value of each parameter on a pseudo-random basis from within the predefined ranges of uncertainty.
2. Angle of the velocity vector of vehicle 2 at the instant of impact:

$$\vartheta_2 = \delta_2 - \vartheta_1. \tag{11}$$

3. Positions of the centre of mass in the coordinate systems related to the vehicles:

$$\mathbf{r}_{N_i S_i, N_i} = [-c_i, 0]^T, i = 1{,}2. \tag{12}$$

4. Matrixes of rotation from the local coordinate systems at the pre-impact positions $\{N_i\}$ to the inertial system $\{I\}$

$$\mathbf{A}_{IN_i} = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}, i = 1{,}2. \tag{13}$$

5. Position of vehicle 2 at the instant of impact in the inertial system $\{I\}$:

$$\mathbf{r}_{IN_2,I} = \mathbf{r}_{IC,I} - \mathbf{A}_{IN_2}\mathbf{r}_{N_2C_2,N_2}. \tag{14}$$

6. Separation velocities:
- centres of mass in the final positions

$$\mathbf{r}''_{IS_i,I} = \mathbf{r}''_{IN_i,I} + \mathbf{A}''_{IN_i}\mathbf{r}_{N_iS_i,N_i}, i = 1{,}2, \tag{15}$$

- centres of mass in the pre-impact positions

$$\mathbf{r}_{IS_i,I} = \mathbf{r}_{IN_i,I} + \mathbf{A}_{IN_i}\mathbf{r}_{N_iS_i,N_i}, i = 1{,}2, \tag{16}$$

- distance between the centres of mass in the pre-impact position and the final position

$$d_i = \left|\mathbf{r}''_{IS_i,I} - \mathbf{r}_{IS_i,I}\right|, i = 1{,}2, \tag{17}$$

- separation velocities according to McHenry formulas (6) and (7).

7. Impact velocities, based on the momentum conservation law, according to formulas (8) and (9).

The cycle consisting of steps from 1 to 7 is repeated n times. The results that do not meet the criteria of "conditional sampling" are rejected (cf. [20]). Thus, by introducing e.g. the energy conservation law to the equations, it is made possible to reduce the influence of the problem of parallelism of the longitudinal vehicles' centrelines at the instant just before the impact, inherent in the momentum conservation law (see the said denominator in formulas (8) and (9)). The conditional sampling usually results in asymmetry of the probability density function.

3.1. Example 2: Collision of two motorcars

In this example, data obtained from crash test No. 7 documented in [4] were used; hence, the results will simultaneously be usable for verification purposes.

Input data

A sketch of the accident scene has been shown in Fig. 4 and the numerical data inclusive of the uncertainty ranges for the parameters whose values are difficult for being ascertained in the accident reconstruction practice have been specified in Table 1. At the collision under consideration, the eccentricity was very small; therefore, the use of the momentum conservation principle alone may be considered acceptable.

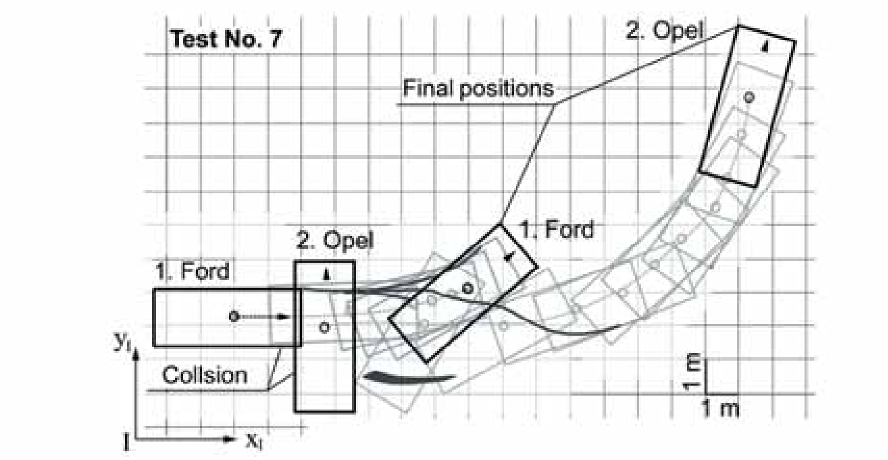

Fig. 4. Accident scene [4]

Table 1. Data for test No. 7 in [4]

Parameter	1. Ford Taunus 1300	2. Opel Commodore GS
Distance between centre of mass and front axle [m]	1.14	1.24
Wheelbase L_i [m]	2.58	2.672
Coefficient of friction μ_i	0.7±0.1	0.7±0.1
Coefficient of resistance to the post-impact motion f_i	0.7±0.2	0.6±0.2
Mass m_i [kg]	990±30	1220±30
Moment of inertia I_{zi} [kgm^2]	1387±200	1877±200
Total vehicle yaw angle in the post-impact motion $\Delta\psi_i$ [°]	42°	–13°
Final velocity v_i'' [km/h]	0	0
Angle of the pre-impact velocity vector ϑ_i in $\{I\}$ [°]	0.0±2.0	90.0±2.0
Angle of the separation velocity vector ϑ_i' in $\{I\}$ [°]	1.7±3.0	0.6±3.0

Calculations

In result of the Monte Carlo simulation, probability density functions of velocities $f(v_i), i$ = 1,2 were obtained; they have been represented in Fig. 5 by solid lines. The expected (nominal) values are v_1 = 86.6 km/h and v_2 = 1.2 km/h. For comparison, these values actually were $v_1^{(r)}$ =87.7 km/h i $v_2^{(r)}$ = 0 km/h.

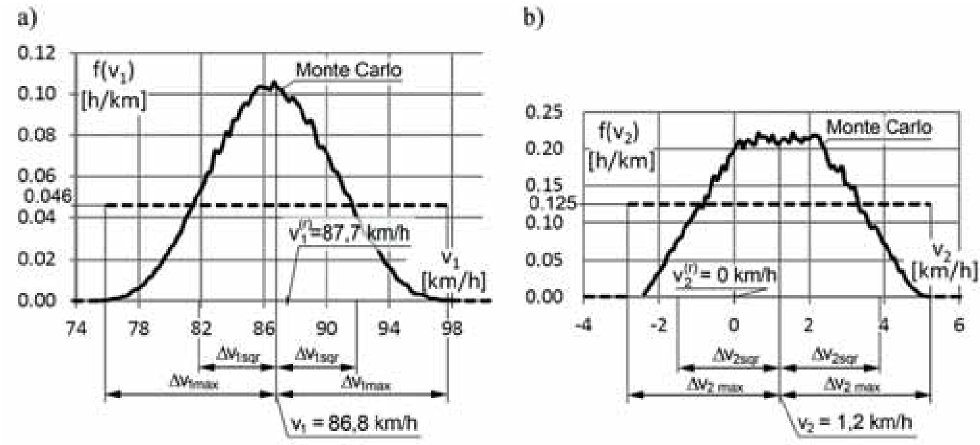

Fig. 5. Probability density function of the pre-impact velocity: a) for vehicle 1; b) for vehicle 2

The presentation of results in the form of univariate graphs entails a risk of selective and uncorrelated picking out of any of the values e.g. to convince the court of the rightness of one's position during legal proceedings. Therefore, a better solution from the formal point of view is the presentation in the form of bivariate probability density functions $f(v_1, v_2)$ or cumulative distribution functions $d(v_1, v_2)$, where the velocities v_1 and v_2 depend on each other (as in Fig. 6). The probability that the impact velocities fall within ranges $v_1 \in <a,b>$ and $v_2 \in <c,d>$ can be calculated from a formula

$$P_{v_1,v_2} = P(v_1 \in \langle a,b \rangle \cap v_2 \in \langle c,d \rangle) = \int_a^b \int_c^d f(v_1,v_2)\, dv_2 dv_1, \tag{18}$$

but it can be directly read from Fig. 6b that e.g.

$$P(v_1 \leq 88 \text{ km/h} \cap v_2 \leq 3 \text{ km/h}) = 0.62.$$

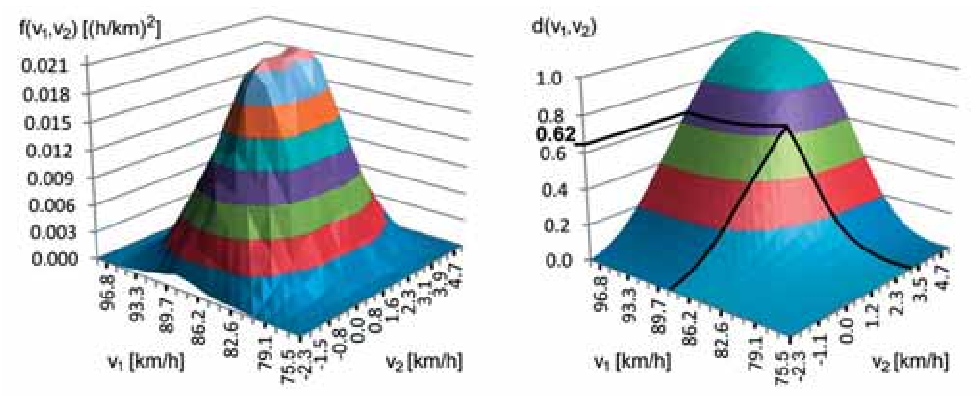

Fig. 6. Results of the calculation of pre-impact velocities v_1 and v_2 by the Monte Carlo method as: a) bivariate probability density function, b) bivariate cumulative distribution function

3.2. Example 3: Collision of a motorcycle with a passenger car

Now let us look into the problem of a strongly eccentric impact of a motorcycle against a motorcar, taking as an example the crash test No. 3 described in [16], with limiting the range of the input data to those usually available to a forensic expert when undertaking an accident reconstruction task. The accident scene has been schematically presented in Fig. 7 and the other data have been given in Table 2. The directions of the separation velocities together with the ranges of uncertainty have been shown in Fig. 8.

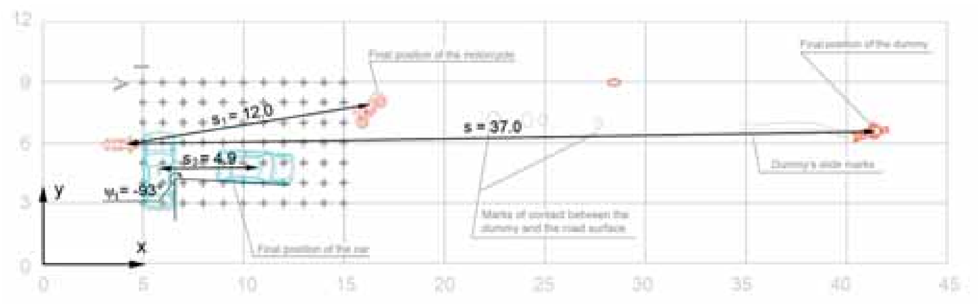

Fig. 7. Accident scene (test No. 3 [16])

Table 2. Data for test No. 3 in [16]

Parameter	1. Motorcycle Yamaha XS 400	2. Motorcar Mazda 323 LX
Wheelbase L_i [m]	1.38	2.5
Coefficient of friction in the post-impact motion μ_i	0.40±0.10	0.75±0.10
Coefficient of resistance to the post-impact motion f_i	0.75±0.25	0.80±0.20
Mass m_i [kg]	mY = 182±30 *	996±30
Moment of inertia I_{zi} [kgm^2]	88±10	1273±200
Total vehicle yaw angle in the post-impact motion $\Delta\psi_i$ [°]	180°	–93°
Final velocity v_i'' [km/h]	0	0
Angle of the pre-impact velocity vector ϑ_i in {I} [°]	0.0±0.5	90.0±0.5
Angle of the separation velocity vector ϑ_i' in {I} [°]	8.0±3.0	5.0±4.0

*m_Y – mass of the motorcycle without the dummy

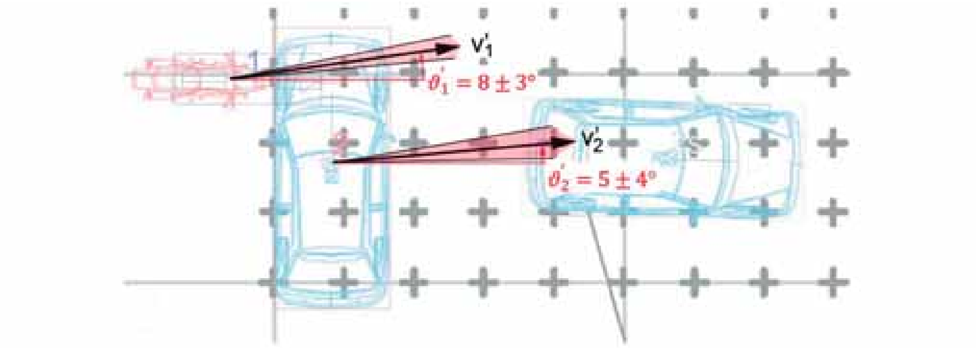

Fig. 8. Directions of the separation velocities adopted at the calculations

The mass of the test dummy representing the motorcyclist was m_0 = 82 kg. After the motorcycle impact against the car, the dummy flew over the car bonnet without touching it; therefore, based on Priester's suggestions [18], a single combined value of the reduced mass of the motorcycle together with the motorcyclist, equal to $m_1 \approx m_Y + 0.3 \cdot m_0$ = 207 kg ($\pm\Delta m_1$ = 30 kg) was assumed for the calculations where the momentum conservation principle was employed.

In the Monte Carlo simulation carried out, the algorithm of reconstruction calculations was identical to that used in example 1. The results were obtained in the form of a bivariate probability density function represented in Fig. 9. The pre-impact velocities fell within ranges v_1 = **131÷183 km/h** and v_2 = **1.3÷7.1 km/h**; their expected (nominal) values were v_1 = **162 km/h** and v_2 = **4 km/h** (note: the probability distribution is asymmetrical; therefore, the expected values are not the mean values from the whole ranges of uncertainty of the results). On the other hand, should the calculations be made only once for the nominal values then the results together with the rms uncertainty values would be v_1 = 162±24 km/h and v_2 = 4±2 km/h. Hence, a typical conclusion arises that "the Monte Carlo simulation has made it possible to narrow the range of real results by rejecting the extreme areas of low probability". This time, however, such a conclusion would be exceedingly hasty, because the measured actual velocities were $v_1^{(r)}$ = **122 km/h** and $v_2^{(r)}$ = **0 km/h,** i.e. the error in the expected value $\Delta v_1 = 100 \cdot (v_1 - v_1^{(r)})/v_1^{(r)}$ was as great as 33%.

In spite of the very suggestive form of the probability density function, the result of the Monte Carlo simulation is far from reality. The main reason for this fact lies in inadequacy of the collision model adopted, i.e. in the application of the momentum conservation rule to a strongly eccentric impact. As it can be seen, the Monte Carlo method taken alone cannot improve the imperfections of a deterministic model; instead, it may only help in obtaining a lucid form of the result for all combinations of data from among the uncertainty ranges adopted.

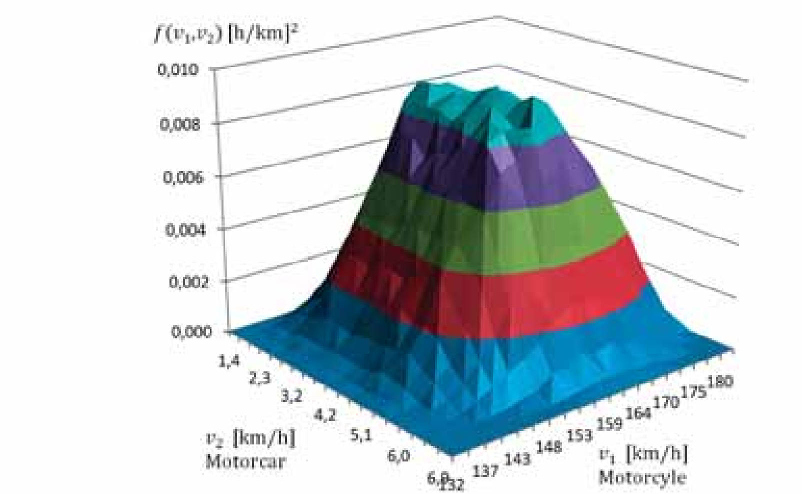

Fig. 9. Probability density function of the pre-impact velocity values obtained from the Monte Carlo simulation (according to the momentum conservation principle)

In this situation, the complete classical collision theory (e.g. the Kudlich-Slibar model, see [26]) must be employed. The velocity values obtained from the calculations corrected as appropriate would be v_1 = 123±17 km/h and v_2 = 9±3 km/h. Thus, the v_1 velocity value would fall to the realistic range. Note: for low velocity values (such as v_2) confronted with very high velocities (such as v_1), no great importance should be attached to the numerical values ofv_2; instead, an assumption should be made that vehicle 2 moved with an unspecified low velocity close to 0 or it was at a standstill.

4. Example 4: Searching for the collision point on the road

Let us address the intriguing problem of analytical determining of the point of collision between two vehicles in the case that any marks that would unequivocally indicate the location of this point were not found on the road surface. In this analysis, the impact test No. 7 [4], dealt with in sub-item 3.2 of this article, will be used again as an example, but now this case will be examined as if only the vehicle deformations, final (rest) positions of the vehicles, and the fact that the vehicles were perpendicular to each other at the moment of impact, were known. The task is to determine analytically the impact position of the vehicles in relation to the road surface. A sketch of the accident scene has been shown in Fig. 10 (cf. Fig. 4).

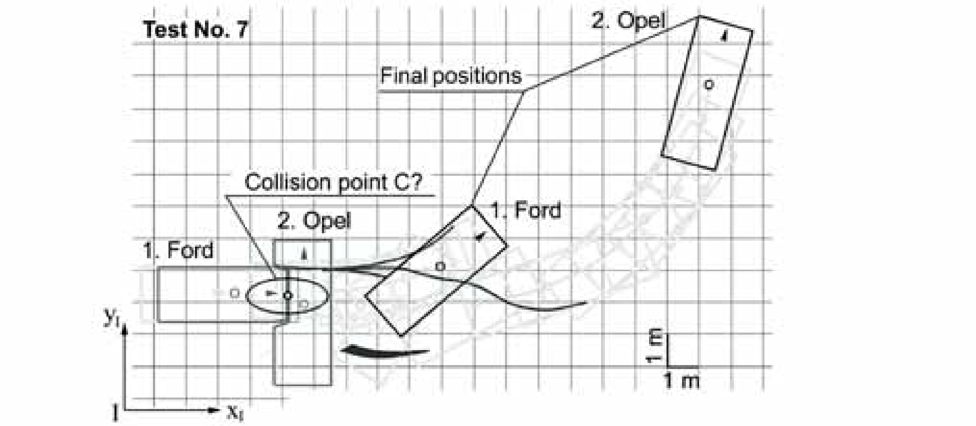

Fig. 10. Accident scene [4]; the location of collision point C, defining the collision position of vehicles relative to the road, is to be found

Let us use a mathematical model similar to that applied to the work described in publication [24]. The collision point is a selected point C situated on the tangential plane of the vehicles brought to contact with each other in accordance with the vehicle deformations. Since the position of this point in relation to both of the vehicles is constant, the calculation of its coordinates will in effect be the determining of the collision point.

The shortened algorithm of the Monte Carlo simulation will include the following steps repeated n times:

1. Pick out a value of each input parameter on a pseudo-random basis from within the appropriate predefined range of uncertainty. Note: for the separation velocities v_i' to be calculated in step 2, the pre-impact velocities v_i must be known; therefore, let us assume that they have been recorded by an EDR (Event Data Recorder) as v_i = 87.8±0.5 km/h and v_2 = 0.0±0.5 km/h (without these data, further calculations would not make any sense).
2. Calculate the separation velocities v_i' of both vehicles with the use of the McHenry model.
3. Calculate the distances travelled by the centres of mass in the post-impact motion.
4. Calculate the coordinates of the centres of mass at the instant of impact.
5. Calculate the coordinates of the collision point C.
6. Archive the results.
7. Jump to step 1.

A block diagram of the calculation procedure has been shown in Fig. 11.

The results obtained for n = 20 000 have been presented in the form of a bivariate probability density function in Fig. 12. In part a) of this figure, this function has been shown as a horizontal projection against the background of the actual configuration of the vehicles at the instant of impact (the vehicle contours have been marked by solid lines) and the actual collision point. The highest concentration of results was obtained for point C with coordinates

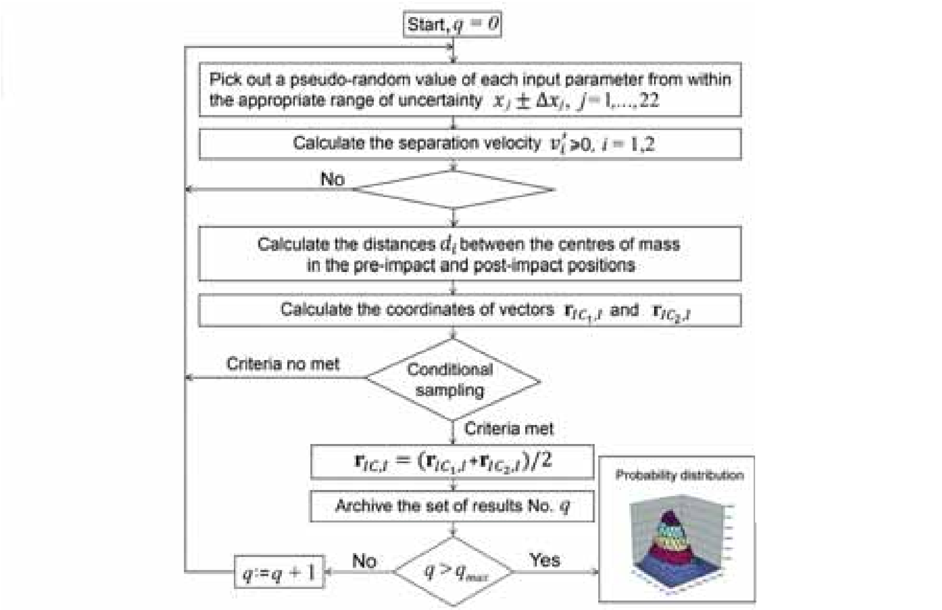

Fig. 11. The algorithm of the Monte Carlo simulation

C = [5.51 m; 3.77 m]; the vehicle contours corresponding to this point have been marked by dashed lines. The total span of the distribution was δ_x = 3.31 m and δ_y = 0.96 m.

The calculations made in accordance with the complete classical collision theory (i.e. the Kudlich-Slibar model) yielded results with a similar uncertainty class; therefore, they do not have to be quoted here.

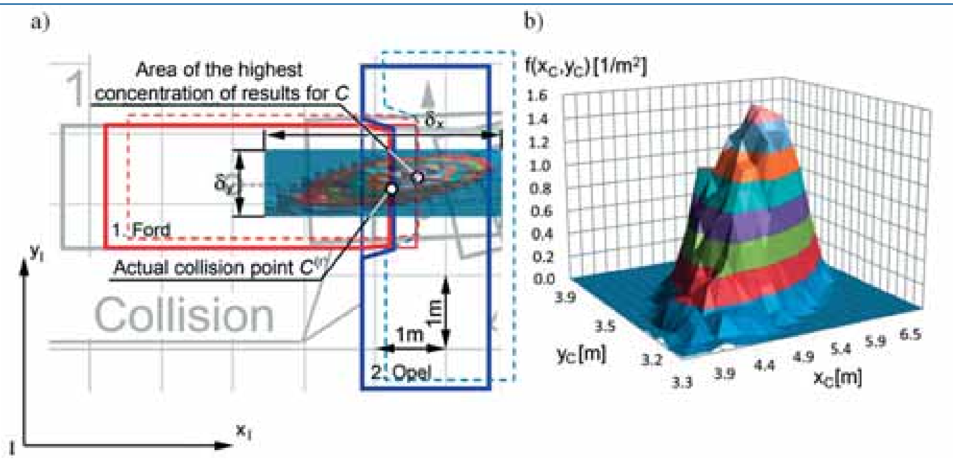

Fig. 12. Probability density function of the collision point location on the road, obtained from the Monte Carlo simulation (according to the momentum conservation principle and the McHenry method): a) horizontal projection; b) 3D view

The area with the highest concentration of results is shifted by about 0.39 m to the right from the actual collision point $C^{(r)}$. On the other hand, the probability distributions make it possible to narrow the area of the most likely results to a considerable degree. For comparison, in the case of the calculations being made only once, the maximum coordinate uncertainty values would be $\Delta x_C \approx \delta_x/2$ and $\Delta y_C \approx \delta_y/2$ and any result falling within the ranges $(x_C - \Delta x_C) < x_C < (x_C + \Delta x_C)$ and $(y_C - \Delta y_C) < y_C < (y_C + \Delta y_C)$ should be considered as being equally likely to occur.

The coordinate values to be found are very susceptible to the pre-impact velocities and the directions of the pre-impact and post-impact velocity vectors, i.e. to the data that are most difficult to be determined. Therefore, results of the Monte Carlo simulations should be treated with great caution, with remembering that the area of the highest concentration of results is only the region of most frequently occurring results of calculations carried out for the specific input data assumed and for the mathematical models adopted. If an accuracy of 0.10–0.50 m is desired (as it is in the case of a problem of crossing the carriageway centreline by any of the vehicles involved), then the uncertainty of calculations of the order of several meters completely disqualifies the method.

When a real case is examined, it will be recommendable to make comparisons between results obtained with the use of different collision models and post-impact motion models (cf. [21]) as well as to carry out verifying dynamic simulations, although fully reliable solutions should not be expected even in spite of such precautions having been taken.

5. Example 5: Simulation of a collision and the post-impact motion

Let us get back to the problem of calculating pre-impact velocities, raised in sub-item 3.1. This time, the task will be solved with the use of the PC-Crash program, by calculating the separation parameters based on pre-impact parameters and by simulation of the post-impact motion of the vehicles. The set of pre-impact input data will be identified by means of the Monte Carlo simulation.

At this analysis, the Kudlich-Slibar collision model and a model of vehicle motion dynamics with 10 degrees of freedom were used. The solution searching process was automated, with the Monte Carlo method and an optimizing tool [15] being used. The goal was to find such a set of input data for which the quality function, i.e. the weighted relative error defined by formula (19), would reach its minimum value Q_{min}

$$Q = \sqrt{\frac{\sum_{i=1}^{n}(w_i q_i)^2}{\sum_{i=1}^{n} w_i^2}} 100\% \tag{19}$$

where: q_i – relative difference between the actual value of the i^{th} parameter and the corresponding value obtained from the simulation; w_i – weight of the i^{th} parameter.

The set of the basic input data used was identical to that presented in sub-item 3.1, but it was extended by adding some data that supplemented the parametrization of the model of motion dynamics. The calculation results in the form of a set of points defined by errors Q

and corresponding impact velocities, obtained from 4 000 steps, have been presented in Fig. 13. At an assumption that the error Q should not exceed 10 %, the result obtained from 48 steps of 4 000 steps in total was v_1 = 74–86 km/h and v_2 = 0–8 km/h. The best fit between the virtual and actual final positions (Q = 2 %) was achieved for v_1 = 79 km/h and v_2 = 0 km/h.

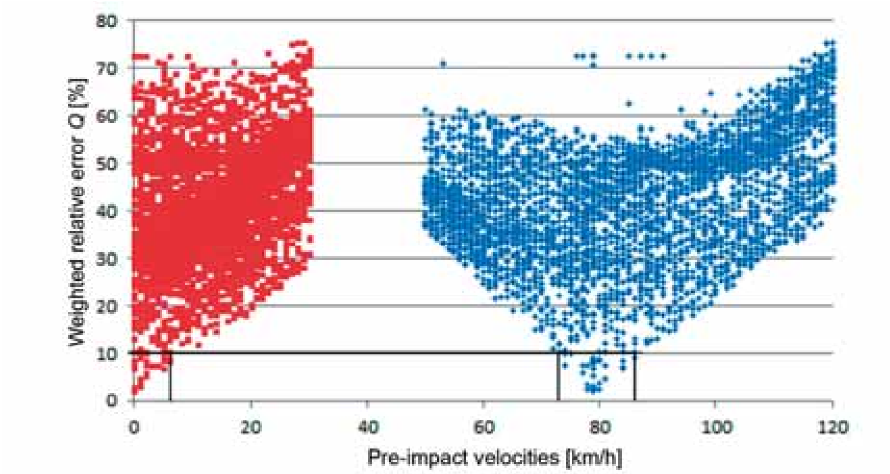

Fig. 13. Quality function values vs. pre-impact velocities obtained at the Monte Carlo simulation

The fact that the expected value of the velocity of vehicle 1 determined from classical accident reconstruction calculations (86.8 km/h, see sub-item 3.1) was closer to the actual value (87.7 km/h) than that calculated with the use of the PC-Crash program (79.0 km/h) was pure chance. The simulations, which are virtual experiments, offer an incomparably more comprehensive view of the influence of different parameters than the simple momentum equations and McHenry formulas do.

6. Example 6: Pedestrian accident

In this example, calculations will be carried out to analyse the kinematics of the pre-collision phase of an accident where a pedestrian was struck by a motor vehicle. The main goal is to answer the question whether driver's reaction to the beginning of the hazardous situation, i.e. to the fact that the pedestrian entered the carriageway, was delayed or not. In mathematical terms, this problem will be expressed as follows:

The reaction delay τ is the time interval between the beginning of a hazardous situation (i.e. the moment when an average, sufficiently good, and fit driver should have begun to react – e.g. the fact that a pedestrian entered the roadway) and the beginning of the reaction of the real driver involved in the accident. In other words, this is the time of driver's reaction delay in relation to the required beginning of the reaction. This parameter has been defined by a formula

$$\tau = t_p - (t_r + t_n + t_{h1}), \qquad (20)$$

where: t_p – emergency duration time (sometimes referred to in English as "time-to-collision" or "TTC"), i.e. the time that elapsed from objective beginning of the hazardous situation from the driver's point of view (e.g. the instant when a pedestrian entered the carriageway) and the collision; t_r – time of reaction of an average, sufficiently good, and fit driver; t_n – braking deceleration rise time; t_{h1} – braking time before the collision.

One of the key parameters that are decisive for the uncertainty of analysis results is the driver's reaction time, which depends on many factors, such as e.g. degree of complexity of the road situation, emergency duration time (TTC), time of the day (daytime or night-time), element of surprise, or "information noise" in the road environment [8, 9].

The notation "$\tau>0$" means that the driver's decision about applying brakes was delayed should be interpreted as a statement that the driver reacted immediately.

When appropriate transformations are made (see [25]), the following formula may be derived:

$$\tau = \frac{\sqrt{2a_h s_{h2}}\left(1+\frac{m_p}{m_s}\right) - \sqrt{2a_h\left[s_{h1}+s_{h2}\left(1+\frac{m_p}{m_s}\right)^2\right]}}{a_h} - t_n - t_r + \frac{d_p}{v_p}, \qquad (21)$$

where: d_p – distance walked by the pedestrian during the emergency duration time, i.e. from entering the carriageway to the collision; v_p – pedestrian velocity; s_{h1} – braking distance before the collision; s_{h2} – braking distance after the collision; a_h – vehicle deceleration during the braking; m_p – pedestrian mass; m_s – vehicle mass.

Table 3. Data adopted for the calculations

Parameter	Value
a_h [m/s²]	(0.75±0.04) g
d_p [m]	3.8±0.2
$a_n = a\_h/s$ [m/s²]	(0.38±0.02) g
t_n [s]	0.20±0.02
s_{h1} [m]	12.00±0.10
s_{h2} [m]	4.00±0.05
t_r [s]	0.85±0.15
m_p [kg]	68±2
m_s [kg]	1200±20
v_{p1} [m/s] <u>according to pedestrian's version</u> (the velocity of a pedestrian, a male 29 years old, walking at a slow pace, based on data taken from the whole range determined by the compilation of research results published by various authors, see [26])	1.13±0.16
v_{p2} [m/s] <u>according to driver's version</u> <u>(the velocity of a pedestrian running, according to the same sources)</u>	3.22±0.78

The data adopted for the calculations inclusive of their uncertainty ranges have been brought together in Table 3 and presented in Fig. 14. A serious difficulty arose from the fact that two completely different versions existed that described the velocity of motion of the pedestrian (a male 29 years old): the pedestrian claimed to "be walking at a slow pace" while the driver reported that the pedestrian "was running across the road".

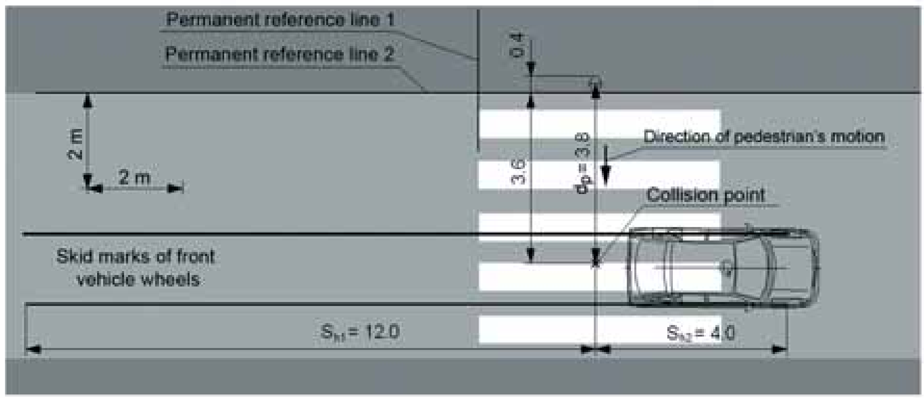

Fig. 14. Accident scene (all dimensions in [m])

Instead of making separate calculations for version 1 and version 2 of the pedestrian velocity, one common velocity range, appropriately widened, was adopted, which covered all the possible velocity values, from the minimum for the pedestrian walking at a slow pace to the maximum for the pedestrian running (cf. [5, 26]), in order to simplify the interpretation of analysis results and to reduce the labour input required. In numerical terms, the velocity range adopted was as specified below:

$$v_p = 2.49 \pm 1.52 \text{ m/s}.$$

The reaction delay τ was calculated in three different ways, i.e. analytically, graphically, and with the use of the Monte Carlo method.

Analytical method

According to the analytical method, the nominal value τ with the maximum uncertainty value $\Delta\tau_{max}$ and the root mean square value $\Delta\tau_{sqr}$ were:

$$\tau \pm \Delta\tau_{max} = -0.53 \pm 1.22 \text{ s},$$

$$\tau \pm \Delta\tau_{sqr} = -0.53 \pm 0.95 \text{ s}.$$

Since the results covered both positive and negative values ("delayed driver's reaction" and "immediate driver's reaction", respectively), the issue of delay in driver's reaction remained unsolved. In addition to this, any value of driver's reaction delay within the ranges $\tau \pm \Delta\tau_{max}$ or $\tau \pm \Delta\tau_{sqr}$ is equally likely to occur.

Graphical method

The results obtained from the graphical method have been presented in Fig. 15. In this case, too, any value of driver's reaction delay within the range $-1.0 < \tau$ [s] < 3.7 is equally likely to occur.

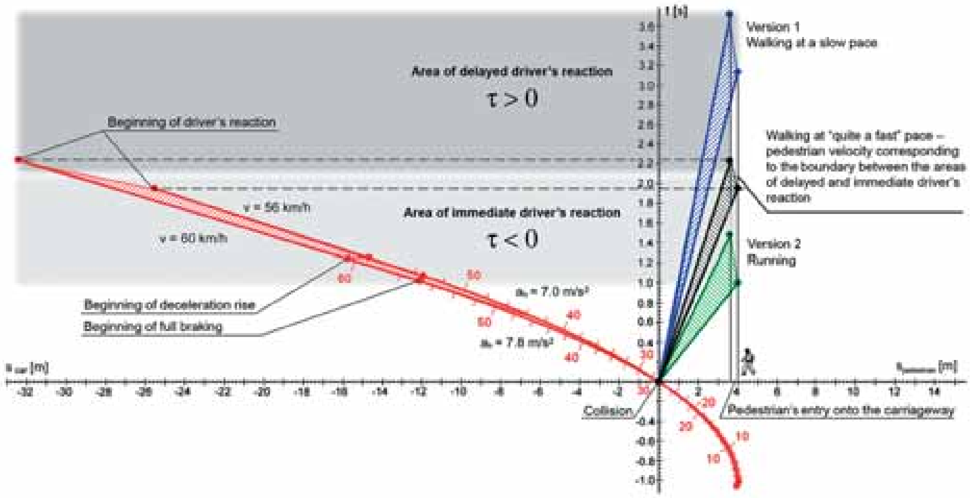

Fig. 15. Time-distance relationship with separation of two versions of pedestrian's velocity

Monte Carlo method

A Monte Carlo simulation was carried out with calculations to formula (21) being repeated 40 000 times, but with modifying every time the values of each of the input parameters on a pseudo-random basis within the predefined ranges of uncertainty. Uniform distributions of all the input data were adopted to avoid the blame for arbitrary rejection of marginal areas.

The calculation results in the form of a probability density function represented by a strongly asymmetrical bell-shaped curve have been presented in Fig. 16. The expected value of the reaction delay (i.e. the one for which the highest concentration of results was recorded) is identical to that obtained from the analytical method, i.e. $\tau = -0.53$ s; however, when the distribution is taken into account as a whole, the results can be seen to fall within the range $-1.3 < \tau$ [s] < 2.1, which means that they still do not offer a solution for this issue.

Comparison and discussion of the results

For comparison, the uniform probability density functions determined at the analytical and graphical methods (according to formula (4), with substituting τ for v) have been additionally plotted in Fig. 16.

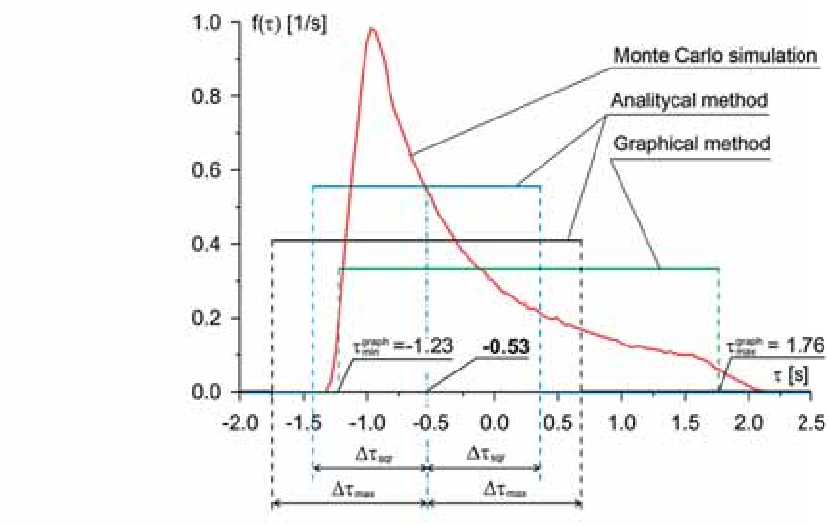

Fig. 16. Comparison of probability density functions of reaction delay τ obtained when the analytic, graphic, and Monte Carlo methods were used

The uncertainty ranges determined with the use of the graphical and Monte Carlo methods practically coincide with each other and, in the nature of things, cover the whole span of the results possible. However, a good point of the Monte Carlo method is the fact that it additionally makes it possible to prepare the results in a form suitable for statistical processing.

In the case of the analytical method, the uncertainty range is somewhat narrower and shifted towards the peak of the probability density curve. This effect is caused by the linearization of nonlinear equation (21) around the nominal values during the calculation of susceptibility coefficients (i.e. the partial derivatives necessary for calculating the values of uncertainty $\Delta\tau_{max}$ and $\Delta\tau_{sqr}$ of results) and by the use of high values of uncertainty of the data, especially the pedestrian velocity.

The symmetry of the probability density function at the Monte Carlo method will improve with narrowing the range of the possible pedestrian velocity values; therefore, if in a specific case it is ascertained that the pedestrian walked at e.g. a fast pace then this effect will disappear and the interpretation troubles will be avoided.

It is easy to notice that the analytical calculations, covering merely the ranges $\tau\pm\Delta\tau_{max}$ or $\tau\pm\Delta\tau_{sqr}$, will omit some solutions (on the right side of the graphs in Fig. 16) while allowing others, although being redundant (on the left side of the graphs). The other two methods are free from this drawback.

In the case of wide scatter of the input data, the introduction of a well-thought-out mathematical criterion will make it possible to formulate lucid conclusions, which will be identical in qualitative terms regardless of the calculation method used. Such a criterion will reduce the necessity of repeating the calculations for various, hastily created, "sub-versions" of the course of the accident. An example of such a criterion has been presented in publication [25].

7. Conclusions

1. The Monte Carlo method lies in making repeated calculations with the use of the same deterministic mathematical model, but with picking out every time the values of specific parameters on a pseudo-random basis from within predefined ranges of uncertainty, with these input data being configured in almost all the possible combinations. Thanks to this, the results are obtained in the form of a probability density function similar, in terms of its graphical representation, to a bell-shaped curve.

2. The Monte Carlo method cannot improve the imperfections of a deterministic model. Instead, it is a tool making it possible to obtain results in a form facilitating the statistical interpretation of data and the uncertainty analysis. Thanks to this, it is possible to narrow the range of results to realistic values by rejecting the extreme areas of low probability.

3. **When interpreting results of a Monte Carlo simulation, one should remember that the area of the highest concentration of results is actually the region of most frequently occurring results of calculations carried out for the specific input data assumed and for the deterministic models adopted, but not necessarily a reflection of the truth. An issue of fundamental importance is the developing of an adequate mathematical model, because an incorrect model will produce erroneous results but presented in a suggestive form.**

4. Calculations according to simple formulas can be easily made with the use of spreadsheet programs (e.g. MS Excel).

5. At the reconstruction of a specific road accident for forensic investigation purposes, it is recommended to adopt uniform distributions of most of the data, because nobody can assure that a specific parameter in a specific case did not happen to assume its extreme value. As an example: although results of field measurements of distances have, in statistical terms, a normal distribution [2], nobody knows how carefully a police officer measured a specific distance. On the other hand, the use of actual distributions of input data in statistical and generalized applications is strongly advisable.

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