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A screw dislocation located outside, inside or on the interface of a parabolic elastic inhomogeneity

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Using conformal mapping techniques, superposition and analytic continuation, we derive analytic solutions to the problem of a screw dislocation interacting with a parabolic elastic inhomogeneity. The screw dislocation can be located anywhere either in the surrounding matrix or in the parabolic inhomogeneity or simply on the parabolic interface itself. We obtain explicit expressions for the two analytic functions in the image plane characterizing the elastic fields describing displacement and stresses in the two-phase composite. Using the Peach-Koehler formula, we also obtain the image force acting on the screw dislocation. The analytic function defined in the parabolic inhomogeneity in the physical plane can be interpreted in terms of real and image screw dislocations for any location of the real screw dislocation.

Key words: screw dislocation, parabolic elastic inhomogeneity, superposition, conformal mapping, analytic continuation.

1. Introduction

The performance and strength of composite materials used in the design and manufacture of a number of devices and structures rely significantly on the understanding of the overall influence of material defects present, often naturally, in the constituent components of the material. One such defect known to play a critical role (in e.g. failure analysis) is the dislocation and its elastic interaction with its material surroundings. For example, the study of dislocations interacting with inhomogeneities is fundamental to a better understanding of the strengthening and hardening mechanism of composite materials [1]. Investigations into the dislocation-inhomogeneity interaction problem are abundant in the literature (see Zhou *et al.* [2] for an updated review). Previous investigations on this topic have been predominately confined to the scenario in which the inhomogeneities are bounded by closed curvilinear interfaces (e.g., a circular inhomogeneity in [1]; or an elliptical inhomogeneity in [3, 4]; or an inhomogeneity of arbitrary shape in [5]). Although some analytical solutions exist for a line dislocation or a line force in anisotropic or isotropic elastic materials with a parabolic open boundary, these are confined to the cases when the parabolic boundary can be either only traction-free or with fixed displacements (see, for example, [6–8]).

In this paper, our objective is to derive analytic solutions to the problem associated with a screw dislocation interacting with a parabolic elastic inhomogeneity. The screw dislocation can be located anywhere either in the matrix or in the parabolic inhomogeneity or solely on the parabolic interface itself. The original boundary value problem is first decomposed into two separate sub-problems. The two analytic functions in the image plane for each sub-problem can be expressed in terms of a single analytic function and its analytic continuations. Once this single analytic function is determined, the two analytic functions for each sub-problem are known. Consequently, using superposition, the two analytic functions for the original boundary value problem can be conveniently obtained. Using the Peach–Koehler formula [1], the image force acting on the screw dislocation is presented when the screw dislocation is located either in the matrix or in the parabolic inhomogeneity. The analytic function defined in the parabolic inhomogeneity can be expediently interpreted in terms of real and image screw dislocations in an infinite plane. It is found that all the image screw dislocations for the parabolic inhomogeneity are either located on a parabola or on a semi-infinite line.

2. Complex variable formulation

The Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$ is established. In the anti-plane shear deformations of an isotropic elastic material, the two anti-plane shear stress components σ_{31} and σ_{32}, the out-of-plane displacement $w = u_3$ and the single stress function ϕ can be expressed in terms of a single analytic function $f(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [4]

$$\sigma_{32} + \mathrm{i}\sigma_{31} = \mu f'(z), \tag{2.1}$$

$$\phi + \mathrm{i}\mu w = \mu f(z), \tag{2.2}$$

where μ is the shear modulus. The two stress components can be expressed in terms of the single stress function as follows [4]

$$\sigma_{32} = \phi_{,1}, \qquad \sigma_{31} = -\phi_{,2}. \tag{2.3}$$

3. General solution

As shown in Fig. 1, we consider a domain in $\Re^2$, infinite in extent, containing a parabolic elastic inhomogeneity with elastic properties distinct from those of the surrounding matrix. Let

$$S_1:\ x_1 \leq H - \frac{x_2^2}{4H} \quad \text{with } H > 0 \quad \text{and} \quad S_2:\ x_1 \geq H - \frac{x_2^2}{4H}$$

denote the inhomogeneity and the matrix, which are perfectly bonded across the interface L: $x_1 = H - \frac{x_2^2}{4H}$. The linearly elastic materials occupying the parabolic inhomogeneity and the matrix are assumed to be homogeneous and isotropic with associated shear moduli μ_1 (> 0) and μ_2 (> 0), respectively. In addition, a screw dislocation with the Burgers vector b is located at $z = z_0 = x_0 + \mathrm{i}y_0$ with x_0 and y_0 being the real and imaginary parts of z_0 either in the matrix or in the inhomogeneity or solely on the parabolic interface. In what follows, the subscripts 1 and 2 are used to identify the respective quantities in S_1 and S_2.

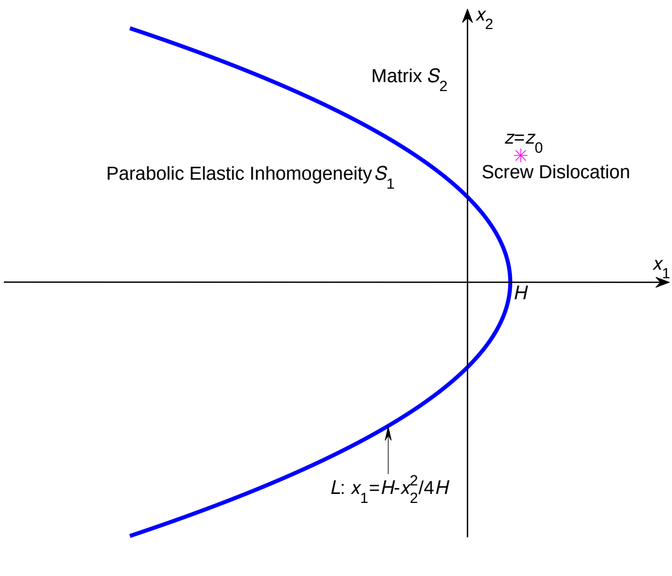

FIG. 1. A screw dislocation interacting with a parabolic elastic inhomogeneity.

We introduce the following conformal mapping function

$$z = \omega(\xi) = \xi^2, \qquad \xi = \omega^{-1}(z) = \sqrt{z}, \qquad \mathrm{Re}\{\xi\} \geq 0, \tag{3.1}$$

which maps the negative x_1-axis onto the straight vertical line $\{\mathrm{Re}\{\xi\} = 0, -\infty < \mathrm{Im}\{\xi\} < +\infty\}$; and the interface L onto another straight vertical line $\{\mathrm{Re}\{\xi\} = h, -\infty < \mathrm{Im}\{\xi\} < +\infty\}$ with $h = \sqrt{H}$.

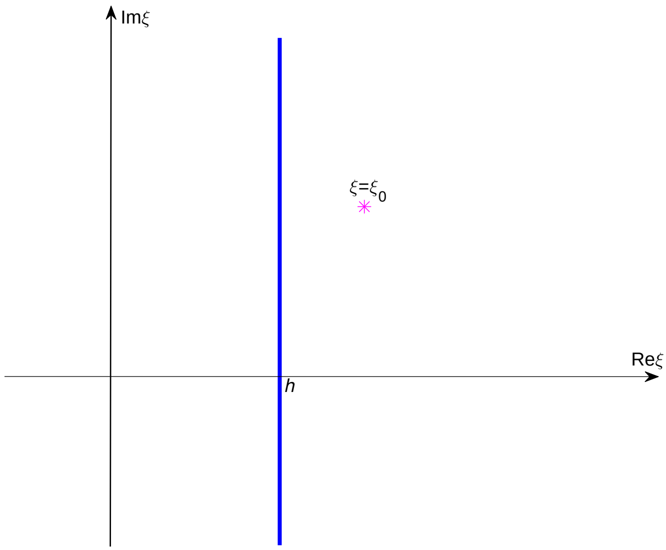

FIG. 2. The image ξ-plane.

Thus, as shown in Fig. 2, by using the mapping function in Eq. (3.1), S_1 and S_2 are mapped onto $0 \le \mathrm{Re}\{\xi\} \le h$ and $\mathrm{Re}\{\xi\} \ge h$, respectively; the location of the screw dislocation at $z = z_0$ is mapped onto the point $\xi = \xi_0$ with $\xi_0 = \sqrt{z_0}$. When the screw dislocation is located in the matrix, we have $\mathrm{Re}\{\xi_0\} > h$; when the screw dislocation lies in the parabolic inhomogeneity, we have $\mathrm{Re}\{\xi_0\} < h$; when the screw dislocation is located on the parabolic interface, we have $\mathrm{Re}\{\xi_0\} = h$. For convenience, we write $f_1(\xi) = f_1(\omega(\xi))$ and $f_2(\xi) = f_2(\omega(\xi))$.

Traction and displacement should be continuous across the negative x_1-axis. Such continuity conditions can be expressed as follows

$$
\begin{aligned}
&f_1(\xi) + \overline{f_1(\xi)} = f_1(-\xi) + \overline{f_1(-\xi)},\\
&f_1(\xi) - \overline{f_1(\xi)} = f_1(-\xi) - \overline{f_1(-\xi)}, \quad \mathrm{Re}\{\xi\} = 0,
\end{aligned}
\tag{3.2}
$$

which are equivalent to

$$
f_1(\xi) = f_1(-\xi), \qquad \mathrm{Re}\{\xi\} = 0. \tag{3.3}
$$

The function $f_1(\xi)$ obtained should always satisfy the condition in Eq. (3.3), which represents the analyticity condition for $f_1(z)$ across the negative x_1-axis. Recall that for an elliptical inhomogeneity with a closed surface, a condition similar to Eq. (3.3) also exists (see Eq. (9) in [9]). The original boundary value problem can be decomposed into the following two sub-problems:

(i) A screw dislocation with the Burgers vector $b/2$ located at $z = z_0$ while another screw dislocation with the Burgers vector $b/2$ is located at $z = \bar{z}_0$;

(ii) A screw dislocation with the Burgers vector $b/2$ located at $z = z_0$ while another screw dislocation with the Burgers vector $-b/2$ is located at $z = \bar{z}_0$.

The solutions to the original problem can be obtained through superposition of the solutions to the above two sub-problems. The superscript '$*$' is attached to the analytic functions for the first sub-problem whilst the superscript '$**$' is attached to the analytic functions for the second sub-problem.

The two analytic functions $f_1^*(\xi) = f_1^*(\omega(\xi))$ and $f_2^*(\xi) = f_2^*(\omega(\xi))$ for the first sub-problem can be expressed in terms of a single analytic function $\Omega(\xi)$ and its analytic continuations as follows:

$$f_1^*(\xi) = \Omega(\xi)+\bar{\Omega}(-\xi), \qquad 0 \le \mathrm{Re}\{\xi\} \le h, \tag{3.4}$$

$$\frac{2}{\Gamma+1} f_2^*(\xi) = \Omega(\xi)-M\Omega(\xi-2h)+\bar{\Omega}(-\xi)-M\bar{\Omega}(-\xi+2h), \qquad \mathrm{Re}\{\xi\} \ge h, \tag{3.5}$$

where

$$\Gamma = \frac{\mu_1}{\mu_2}, \qquad M = \frac{1-\Gamma}{1+\Gamma}. \tag{3.6}$$

The two analytic functions $f_1^{**}(\xi) = f_1^{**}(\omega(\xi))$ and $f_2^{**}(\xi) = f_2^{**}(\omega(\xi))$ for the second sub-problem can be expressed in terms of a single analytic function $\Phi(\xi)$ and its analytic continuations as follows:

$$f_1^{**}(\xi) = \Phi(\xi)-\bar{\Phi}(-\xi), \qquad 0 \le \mathrm{Re}\{\xi\} \le h, \tag{3.7}$$

$$\frac{2}{\Gamma+1} f_2^{**}(\xi) = \Phi(\xi)+M\Phi(\xi-2h)-\bar{\Phi}(-\xi)-M\bar{\Phi}(-\xi+2h), \qquad \mathrm{Re}\{\xi\} \ge h. \tag{3.8}$$

Once the specific expressions for the two analytic functions $\Omega(\xi)$ and $\Phi(\xi)$ are given, the functions $f_1^*(\xi)$, $f_2^*(\xi)$ and $f_1^{**}(\xi)$, $f_2^{**}(\xi)$ follow readily. Consequently, through superposition, we obtain

$$\begin{aligned} f_1(\xi) &= f_1^*(\xi) + f_1^{**}(\xi), \\ f_2(\xi) &= f_2^*(\xi) + f_2^{**}(\xi). \end{aligned} \tag{3.9}$$

In the following three sections, we derive analytic solutions to the three cases: (i) a screw dislocation in the matrix ($\mathrm{Re}\{\xi_0\} > h$); (ii) a screw dislocation in the parabolic inhomogeneity ($\mathrm{Re}\{\xi_0\} < h$); (iii) a screw dislocation on the parabolic interface ($\mathrm{Re}\{\xi_0\} = h$).

4. A screw dislocation in the matrix ($\mathrm{Re}\{\xi_0\} > h$)

When $\mathrm{Re}\{\xi_0\} > h$, $\Omega(\xi)$ and $\Phi(\xi)$ take the specific forms

$$\Omega(\xi) = \frac{b}{2\pi(\Gamma+1)} \sum_{n=0}^{+\infty} M^n \ln[(\xi - \xi_0 - 2nh)(\xi - \bar{\xi}_0 - 2nh)], \tag{4.1}$$

$$\Phi(\xi) = \frac{b}{2\pi(\Gamma+1)} \sum_{n=0}^{+\infty} (-M)^n \ln \frac{\xi - \xi_0 - 2nh}{\xi - \bar{\xi}_0 - 2nh}. \tag{4.2}$$

The function $\Omega(\xi)$ in Eq. (4.1) is obtained by requiring that the principal part of $f_2^*(\xi)$, denoted by $f_{2s}^*(\xi)$, takes the form $f_{2s}^*(\xi) = \frac{b}{4\pi} \ln[(\xi - \xi_0)(\xi - \bar{\xi}_0)]$; similarly, the function $\Phi(\xi)$ in Eq. (4.2) is obtained by requiring that the principal part of $f_2^{**}(\xi)$, denoted here by$f_{2s}^{**}(\xi)$, is given by $f_{2s}^{**}(\xi) = \frac{b}{4\pi} \ln \frac{\xi-\xi_0}{\xi-\bar{\xi}_0}$.

By substituting Eq. (4.1) into Eqs. (3.4) and (3.5), we obtain the following expressions for $f_1^*(\xi)$ and $f_2^*(\xi)$

$$\begin{aligned} f_1^*(\xi) &= \frac{b}{2\pi(\Gamma+1)} \sum_{n=0}^{+\infty} M^n \ln[(\xi-\xi_0-2nh)(\xi-\bar{\xi}_0-2nh)] \\ &\quad + \frac{b}{2\pi(\Gamma+1)} \sum_{n=0}^{+\infty} M^n \ln[(\xi+\xi_0+2nh)(\xi+\bar{\xi}_0+2nh)], \quad 0 \le \mathrm{Re}\{\xi\} \le h, \end{aligned} \tag{4.3}$$

$$\begin{aligned} f_2^*(\xi) &= \frac{b}{4\pi} \ln[(\xi-\xi_0)(\xi-\bar{\xi}_0)] + \frac{b}{4\pi} \sum_{n=0}^{+\infty} M^n \ln[(\xi+\xi_0+2nh)(\xi+\bar{\xi}_0+2nh)] \\ &\quad - \frac{b}{4\pi} \sum_{n=0}^{+\infty} M^{n+1} \ln[(\xi+\xi_0+2h(n-1))(\xi+\bar{\xi}_0+2h(n-1))], \quad \mathrm{Re}\{\xi\} \ge h. \end{aligned} \tag{4.4}$$

By substituting Eq. (4.2) into Eqs. (3.7) and (3.8), we obtain the following expressions for $f_1^{**}(\xi)$ and $f_2^{**}(\xi)$

$$\begin{aligned} f_1^{**}(\xi) &= \frac{b}{2\pi(\Gamma+1)} \sum_{n=0}^{+\infty} (-M)^n \ln \frac{\xi - \xi_0 - 2nh}{\xi - \bar{\xi}_0 - 2nh} - \frac{b}{2\pi(\Gamma+1)} \\ &\quad \times \sum_{n=0}^{+\infty} (-M)^n \ln \frac{\xi + \bar{\xi}_0 + 2nh}{\xi + \xi_0 + 2nh}, \quad 0 \le \mathrm{Re}\{\xi\} \le h, \end{aligned} \tag{4.5}$$

$$\begin{aligned} f_2^{**}(\xi) &= \frac{b}{4\pi} \ln \frac{\xi - \xi_0}{\xi - \bar{\xi}_0} - \frac{b}{4\pi} \sum_{n=0}^{+\infty} (-M)^n \ln \frac{\xi + \bar{\xi}_0 + 2nh}{\xi + \xi_0 + 2nh} + \frac{b}{4\pi} \\ &\quad \times \sum_{n=0}^{+\infty} (-M)^{n+1} \ln \frac{\xi + \bar{\xi}_0 + 2h(n-1)}{\xi + \xi_0 + 2h(n-1)}, \quad \mathrm{Re}\{\xi\} \ge h. \end{aligned} \tag{4.6}$$

Consequently, it follows from Eqs. (3.9) and (4.3)–(4.6) that

$$(4.7)\quad f_1(\xi)=\frac{b}{2\pi(\Gamma+1)}\times\sum_{n=0}^{+\infty}M^n\ln[(\xi-\xi_0-2nh)(\xi+\xi_0+2nh)(\xi-\bar{\xi}_0-2nh)(\xi+\bar{\xi}_0+2nh)]+\frac{b}{2\pi(\Gamma+1)}\sum_{n=0}^{+\infty}(-M)^n\ln\frac{(\xi-\xi_0-2nh)(\xi+\xi_0+2nh)}{(\xi-\bar{\xi}_0-2nh)(\xi+\bar{\xi}_0+2nh)},\quad 0\le\mathrm{Re}\{\xi\}\le h,$$

$$(4.8)\quad f_2(\xi)=\frac{b}{2\pi}\ln(\xi-\xi_0)+\frac{b(1-M^2)}{2\pi}\sum_{n=0}^{+\infty}M^{2n}\ln(\xi+\xi_0+4nh)+\frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n+1}\ln[\xi+\bar{\xi}_0+2h(2n+1)]-\frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n+1}\ln[\xi+\bar{\xi}_0+2h(2n-1)],\quad \mathrm{Re}\{\xi\}\ge h.$$

It is easily verified that $f_1(\xi)$ in Eq. (4.7) satisfies the analyticity condition in Eq. (3.3). In other words, the right hand side of Eq. (4.7) is indeed an analytic function of the complex variable z. The stresses and displacement in the inhomogeneity and in the matrix can be obtained by substituting Eqs. (4.7) and (4.8) into Eqs. (2.1) and (2.2). Using Eq. (4.8) and the Peach–Koehler formula [1], the image force acting on the screw dislocation can eventually be derived as

$$(4.9)\quad F_1-\mathrm{i}F_2=\frac{\mu_2 b^2}{8\pi\xi_0}\left[(1-M^2)\sum_{n=1}^{+\infty}\frac{M^{2n}}{\xi_0+2nh}-\frac{M^2}{\xi_0}\right]-\frac{\mu_2 hb^2}{\pi\xi_0}\sum_{n=0}^{+\infty}\frac{M^{2n+1}}{[\xi_0+\bar{\xi}_0+2h(2n+1)][\xi_0+\bar{\xi}_0+2h(2n-1)]},$$

where F_1 and F_2 are, respectively, the force components along the x_1 and x_2 directions. We can see from Eq. (4.9) that the normalized force $H(F_1-\mathrm{i}F_2)/(\mu_2 b^2)$ can be completely determined once the two dimensionless parameters M and ξ_0/h are given.

REMARK. The Peach-Koehler formula in the current setting can be expressed as: $F_1-\mathrm{i}F_2=b(\sigma_{32}^d+\mathrm{i}\sigma_{31}^d)$, where σ_{31}^d and σ_{32}^d are the bounded stresses at the position of the screw dislocation after excluding those due to the screw dislocation itself.

In the remainder of this section, we endeavor to present an interpretation of the analytic function defined in the parabolic inhomogeneity in terms of image screw dislocations when the screw dislocation is located in the matrix. Equation (4.7) can be re-written into the following equivalent form

$$f_1(z) = \frac{b(1+M)}{2\pi} \sum_{n=0}^{+\infty} M^n \ln(z - z^{(n)}), \qquad z \in S_1, \tag{4.10}$$

where

$$z^{(0)} = z_0, \qquad z^{(n)} = \begin{cases} (2nH^{1/2} + \bar{z}_0^{1/2})^2, & n = 1,3,5,\dots, \\ (2nH^{1/2} + z_0^{1/2})^2, & n = 2,4,6,\dots. \end{cases} \tag{4.11}$$

The expression of $f_1(z)$ in Eq. (4.10) can be simply interpreted as the contribution from the image screw dislocations at $z^{(n)}$, $n = 0, 1, 2, \dots, +\infty$ with Burgers vectors $bM^n(1+M)$ in an infinite plane. When the real screw dislocation in the matrix is not located on the x_1-axis, all the image screw dislocations are located on the parabola described by

$$x_1 = -\frac{|z_0| - x_0}{2} + \frac{x_2^2}{2(|z_0| - x_0)}, \qquad |z_0| - x_0 > 0. \tag{4.12}$$

When the real screw dislocation is located on the x_1-axis in the matrix, all the image screw dislocations are located on the semi-infinite line $\{x_0 \le x_1 < +\infty,$

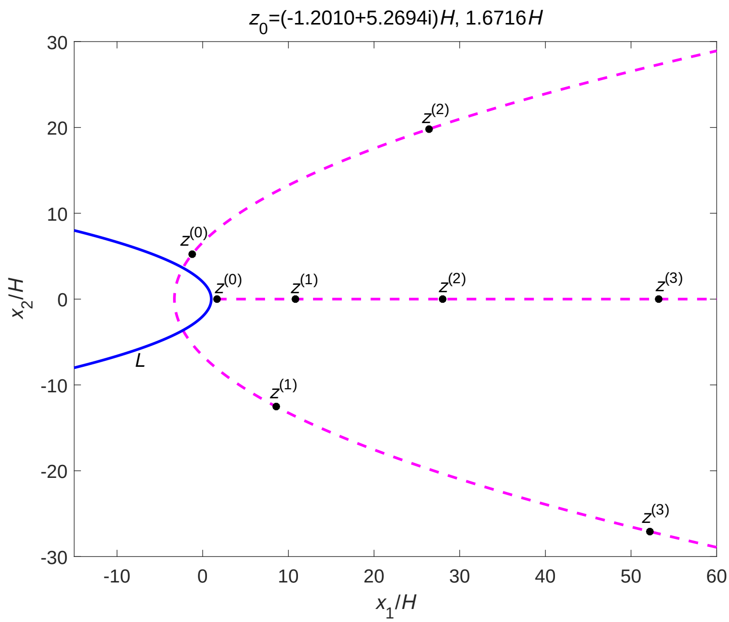

Fig. 3. The locations of the image dislocations for the parabolic inhomogeneity when $z_0 = (-1.2010 + 5.2694\mathrm{i})H$, $1.6716H$ located in the matrix.

$x_2 = 0\}$. The locations of the image dislocations for the parabolic inhomogeneity are illustrated in Fig. 3 for the two typical cases of $z_0 = (-1.2010 + 5.2694\mathrm{i})H$, $1.6716H$ located in the matrix.

5. A screw dislocation inside the parabolic inhomogeneity ($\mathrm{Re}\{\xi_0\} < h$)

When $\mathrm{Re}\{\xi_0\} < h$, $\Omega(\xi)$ and $\Phi(\xi)$ can be specifically given by

$$\begin{aligned}(5.1)\qquad \Omega(\xi) &= \frac{b}{4\pi}\sum_{n=0}^{+\infty} M^n \ln[(\xi-\xi_0-2nh)(\xi-\bar{\xi}_0-2nh)] \\ &+ \frac{b}{4\pi}\sum_{n=0}^{+\infty} M^{n+1} \ln[[\xi+\bar{\xi}_0-2h(n+1)][\xi+\xi_0-2h(n+1)]],\end{aligned}$$

$$\begin{aligned}(5.2)\qquad \Phi(\xi) &= \frac{b}{4\pi}\sum_{n=0}^{+\infty} (-M)^n \ln\frac{\xi-\xi_0-2nh}{\xi-\bar{\xi}_0-2nh} \\ &- \frac{b}{4\pi}\sum_{n=0}^{+\infty} (-M)^{n+1} \ln\frac{\xi+\bar{\xi}_0-2h(n+1)}{\xi+\xi_0-2h(n+1)}.\end{aligned}$$

By substituting Eq. (5.1) into Eqs. (3.4) and (3.5), we obtain the following expressions for $f_1^*(\xi)$ and $f_2^*(\xi)$

$$\begin{aligned}(5.3)\quad f_1^*(\xi) &= \frac{b}{4\pi}\sum_{n=0}^{+\infty} M^n \ln\big[(\xi-\xi_0-2nh)(\xi+\xi_0+2nh)(\xi-\bar{\xi}_0-2nh)(\xi+\bar{\xi}_0+2nh)\big] \\ &+\frac{b}{4\pi}\sum_{n=0}^{+\infty} M^{n+1} \ln\big[[\xi-\xi_0+2h(n+1)] \\ &\times[\xi+\xi_0-2h(n+1)][\xi-\bar{\xi}_0+2h(n+1)][\xi+\bar{\xi}_0-2h(n+1)]\big], \\ &\qquad 0 \le \mathrm{Re}\{\xi\} \le h,\end{aligned}$$

$$\begin{aligned}(5.4)\quad f_2^*(\xi) &= \frac{b(1-M)}{4\pi} \\ &\times\sum_{n=0}^{+\infty} M^n \ln\big[(\xi-\xi_0+2nh)(\xi+\xi_0+2nh)(\xi-\bar{\xi}_0+2nh)(\xi+\bar{\xi}_0+2nh)\big], \\ &\qquad \mathrm{Re}\{\xi\} \ge h.\end{aligned}$$

By substituting Eq. (5.2) into Eqs. (3.7) and (3.8), we obtain the following expressions for $f_1^{**}(\xi)$ and $f_2^{**}(\xi)$

$$(5.5)\quad f_1^{**}(\xi) = \frac{b}{4\pi}\sum_{n=0}^{+\infty}(-M)^n \ln\frac{(\xi-\xi_0-2nh)(\xi+\xi_0+2nh)}{(\xi-\bar{\xi}_0-2nh)(\xi+\bar{\xi}_0+2nh)}$$
$$-\frac{b}{4\pi}\sum_{n=0}^{+\infty}(-M)^{n+1}\ln\frac{[\xi+\bar{\xi}_0-2h(n+1)][\xi-\bar{\xi}_0+2h(n+1)]}{[\xi+\xi_0-2h(n+1)][\xi-\xi_0+2h(n+1)]},$$
$$0\le \operatorname{Re}\{\xi\}\le h,$$

$$(5.6)\quad f_2^{**}(\xi) = \frac{b(1-M)}{4\pi}\sum_{n=0}^{+\infty}(-M)^n \ln\frac{(\xi-\xi_0+2nh)(\xi+\xi_0+2nh)}{(\xi+\bar{\xi}_0+2nh)(\xi-\bar{\xi}_0+2nh)},$$
$$\operatorname{Re}\{\xi\}\ge h.$$

Consequently, it follows from Eqs. (3.9) and (5.3)–(5.6) that

$$(5.7)\quad f_1(\xi) = \frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n}\ln\big[(\xi-\xi_0-4nh)(\xi+\xi_0+4nh)\big]$$
$$+\frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n+1}\ln\big[[\xi-\bar{\xi}_0-2h(2n+1)][\xi+\bar{\xi}_0+2h(2n+1)]\big]$$
$$+\frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n+2}\ln\big[[\xi-\xi_0+4h(n+1)][\xi+\xi_0-4h(n+1)]\big]$$
$$+\frac{b}{2\pi}\sum_{n=0}^{+\infty}M^{2n+1}\ln\big[[\xi-\bar{\xi}_0+2h(2n+1)][\xi+\bar{\xi}_0-2h(2n+1)]\big],$$
$$0\le \operatorname{Re}\{\xi\}\le h,$$

$$(5.8)\quad f_2(\xi) = \frac{b(1-M)}{4\pi}\sum_{n=0}^{+\infty}M^n\ln\big[(\xi-\xi_0+2nh)(\xi+\xi_0+2nh)$$
$$\times(\xi-\bar{\xi}_0+2nh)(\xi+\bar{\xi}_0+2nh)\big]$$
$$+\frac{b(1-M)}{4\pi}\sum_{n=0}^{+\infty}(-M)^n\ln\frac{(\xi-\xi_0+2nh)(\xi+\xi_0+2nh)}{(\xi+\bar{\xi}_0+2nh)(\xi-\bar{\xi}_0+2nh)},$$
$$\operatorname{Re}\{\xi\}\ge h.$$

It is easily verified that $f_1(\xi)$ in Eq. (5.7) satisfies the analyticity condition in Eq. (3.3). In other words, the right-hand side of Eq. (5.7) is indeed an analytic function of the complex variable z. The stresses and displacement in the inhomogeneity and in the matrix can be obtained by substituting Eqs. (5.7) and (5.8) into Eqs. (2.1) and (2.2). Again, using Eq. (5.7) and the Peach–Koehler formula [1], the image force acting on the screw dislocation can be finally obtained as

$$F_1 - \mathrm{i}F_2 = \frac{\mu_1 b^2}{4\pi}\sum_{n=1}^{+\infty}\frac{M^{2n}}{\xi_0^2 - 4n^2h^2} + \frac{\mu_1 b^2}{2\pi}\sum_{n=0}^{+\infty} M^{2n+1}\left[\frac{1}{\xi_0^2 - [\bar{\xi}_0 + 2h(2n+1)]^2} + \frac{1}{\xi_0^2 - [\bar{\xi}_0 - 2h(2n+1)]^2}\right]. \tag{5.9}$$

When the screw dislocation is located at the parabola focus $z = 0$, the image force in Eq. (5.9) reduces to

$$F_1 = -\frac{\mu_1 b^2}{4\pi H}\sum_{n=1}^{+\infty}\frac{M^n}{n^2}, \qquad F_2 = 0. \tag{5.10}$$

That Eq. (5.10) is indeed correct can be understood from a careful examination of OBNOSOV [10]. Furthermore, when the inhomogeneity is much softer than the surrounding matrix ($M = 1$), Eq. (5.10) becomes

$$F_1 = -\frac{\mu_1 b^2}{4\pi H}\sum_{n=1}^{+\infty}\frac{1}{n^2} = -\frac{\pi\mu_1 b^2}{24H} < 0. \tag{5.11}$$

At the other extreme, when the inhomogeneity is much harder than the surrounding matrix ($M = -1$), Eq. (5.10) becomes

$$F_1 = -\frac{\mu_1 b^2}{4\pi H}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2} = -\frac{\mu_1 b^2}{4\pi H}\left(\frac{\pi^2}{24} - \frac{\pi^2}{8}\right) = \frac{\pi\mu_1 b^2}{48H} > 0. \tag{5.12}$$

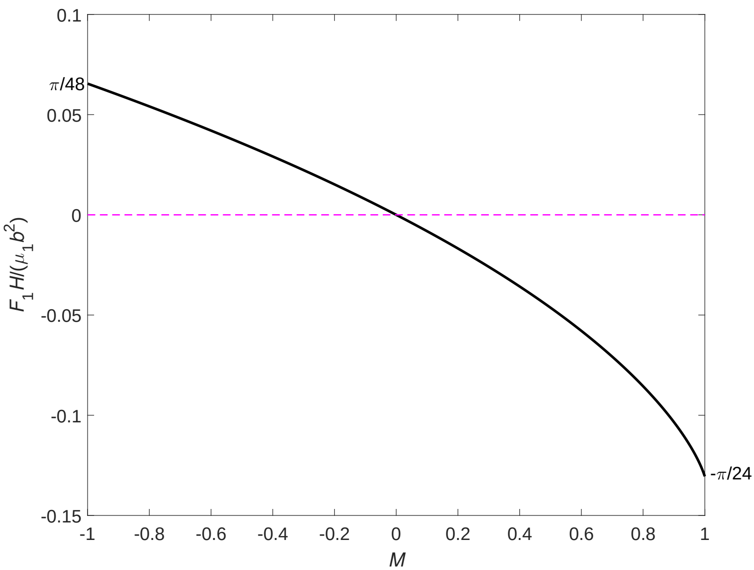

FIG. 4. The image force on a screw dislocation at the parabola focus in Eq. (5.10) as a function of the mismatch parameter M.

The accuracy of the analytic results in Eqs. (5.11) and (5.12) is verified numerically in Fig. 4 for the calculated image force in Eq. (5.10) as a function of the mismatch parameter M. It is seen from Fig. 4 that: $F_1 > 0$ when $-1 \leq M < 0$; $F_1 < 0$ when $0 < M \leq 1$. In other words, the screw dislocation at the origin will be attracted to the parabolic interface when the inhomogeneity is harder than the matrix and will be repelled from the parabolic interface when the inhomogeneity is softer than the matrix.

When the screw dislocation is located on the x_1-axis inside the inhomogeneity with $z_0 = \bar{z}_0 = x_0$ $(-\infty < x_0 \leq H)$, the image force in Eq. (5.9) becomes

$$F_1 = \frac{\mu_1 b^2}{4\pi} \sum_{n=1}^{+\infty} \frac{M^n}{x_0 - n^2 H}, \qquad F_2 = 0, \tag{5.13}$$

which is illustrated in Fig. 5 for various values of x_0 and M. It is observed from Fig. 5 that: (i) the screw dislocation is attracted to the parabolic interface $(F_1 > 0)$ when the inhomogeneity is harder than the matrix and is repelled from the parabolic interface $(F_1 < 0)$ when the inhomogeneity is softer than the matrix; (ii) the magnitude of the image force increases with x_0.

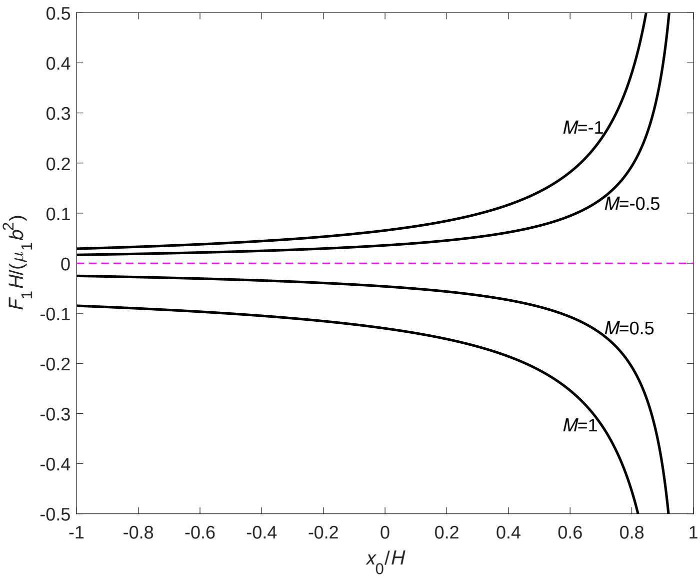

Fig. 5. The image force on a screw dislocation located on the x_1-axis in the inhomogeneity for different values of the mismatch parameter M.

In the remainder of this section, we present an interpretation of the analytic function defined in the parabolic inhomogeneity in terms of real and image screw dislocations when the screw dislocation is located inside the inhomogeneity.

Equation (5.7) can be re-written in the following equivalent form

$$f_1(z) = \frac{b}{2\pi}\ln(z-z_0) + \frac{b}{2\pi}\sum_{n=1}^{+\infty} M^n[\ln(z-z_1^{(n)}) + \ln(z-z_2^{(n)})], \quad z \in S_1, \tag{5.14}$$

where

$$z_1^{(n)}, z_2^{(n)} = \begin{cases} (2nH^{1/2} \pm \bar{z}_0^{1/2})^2 & \text{for } n = 1, 3, 5, \ldots, \\ (2nH^{1/2} \pm z_0^{1/2})^2 & \text{for } n = 2, 4, 6, \ldots. \end{cases} \tag{5.15}$$

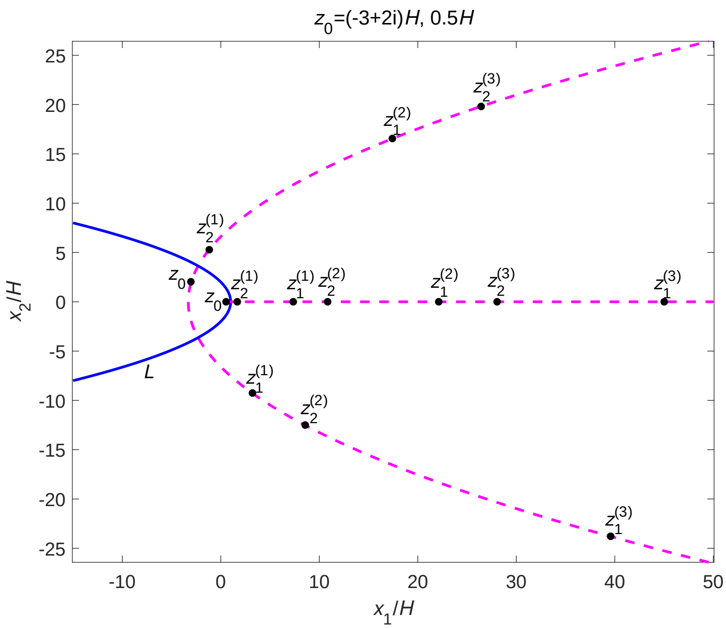

FIG. 6. The locations of the image dislocations for the parabolic inhomogeneity when $z_0 = (-3+2\mathrm{i})H$, $0.5H$ located in the parabolic inhomogeneity.

The form of $f_1(z)$ in Eq. (5.14) can be interpreted as the contribution from the real screw dislocation at $z = z_0$ with the Burger vector b in an infinite plane and that from the image screw dislocations at $z_1^{(n)}$, $z_2^{(n)}$, $n = 1, 2, \ldots, +\infty$ with Burgers vectors $M^n b$ in an infinite plane. When the real screw dislocation is not located on the segment $[0, H]$ on the x_1-axis, all of the real and image screw dislocations are located on the parabola described by Eq. (4.12). When the real screw dislocation is located on the segment $[0, H]$ on the x_1-axis, all of the real and image screw dislocations are located on the semi-infinite line $\{x_0 \leq x_1 < +\infty, x_2 = 0\}$. The locations of the image dislocations for the parabolic inhomogeneity are illustrated in Fig. 6 for the two typical cases of $z_0 = (-3+2\mathrm{i})H, 0.5H$ located inside the parabolic inhomogeneity.

It is deduced from Eqs. (4.10) and (5.14) that if we place a screw dislocation with the Burgers vector b at $z = z_0$ inside the parabolic inhomogeneity and meanwhile locate two screw dislocations with identical Burgers vectors $-\frac{Mb}{1+M}$ at $z = (2H^{1/2} \pm \bar{z}_0^{1/2})^2$ in the matrix, the analytic function defined in the inhomogeneity takes the simple form $f_1(z) = \frac{b}{2\pi}\ln(z - z_0)$, $z \in S_1$, which is just that for a screw dislocation in a homogeneous plane. In this case, the image force on the screw dislocation inside the inhomogeneity is zero.

6. A screw dislocation on the parabolic interface ($\mathrm{Re}\{\xi_0\} = h$)

The solution to the case when $\mathrm{Re}\{\xi_0\} = h$ can be obtained quite simply via a limiting procedure from the solution obtained in Section 4 for a screw dislocation in the matrix. In this case, we have the following identity:

$$\bar{\xi}_0 = 2h - \xi_0. \tag{6.1}$$

Substitution of the above identity into Eqs. (4.7) and (4.8) leads to the following expressions for $f_1(\xi)$ and $f_2(\xi)$:

$$\begin{aligned} f_1(\xi) = \frac{b}{\pi(\Gamma+1)}\Bigg\{ & \sum_{n=0}^{+\infty} M^{2n}\ln[(\xi - \xi_0 - 4nh)(\xi + \xi_0 + 4nh)] \\ & + \sum_{n=1}^{+\infty} M^{2n-1}\ln[(\xi + \xi_0 - 4nh)(\xi - \xi_0 + 4nh)]\Bigg\}, \quad 0 \le \mathrm{Re}\{\xi\} \le h, \end{aligned} \tag{6.2}$$

$$\begin{aligned} f_2(\xi) = {} & \frac{b(1-M)}{2\pi}\ln(\xi - \xi_0) \\ & + \frac{b(1-M^2)}{2\pi}\left[\sum_{n=0}^{+\infty} M^{2n}\ln(\xi + \xi_0 + 4nh) + \sum_{n=1}^{+\infty} M^{2n-1}\ln(\xi - \xi_0 + 4nh)\right], \\ & \mathrm{Re}\{\xi\} \ge h. \end{aligned} \tag{6.3}$$

We can again easily verify that $f_1(\xi)$ in Eq. (6.2) satisfies the analyticity condition in Eq. (3.3). In other words, the right-hand side of Eq. (6.2) is indeed an analytic function of the complex variable z. Equations (6.2) and (6.3) can also be established by substituting the identity in Eq. (6.1) into Eqs. (5.7) and (5.8) for a screw dislocation in the parabolic inhomogeneity. The stresses and displacement in the inhomogeneity and in the matrix can be obtained by substituting Eqs. (6.2) and (6.3) into Eqs. (2.1) and (2.2). As a check of our results, we note that when the inhomogeneity and the matrix are identical ($\Gamma = 1$ or $M = 0$), we deduce from Eqs. (6.2) and (6.3) that

$$f_1(z) = f_2(z) = \frac{b}{2\pi}\ln(z - z_0), \tag{6.4}$$

which is simply the result for a screw dislocation in a homogeneous elastic plane.

In the remainder of this section, we present an interpretation of the analytic function defined in the parabolic inhomogeneity in terms of real and image screw dislocations when the screw dislocation is located only on the parabolic interface. Equation (6.2) can be written in the equivalent form:

$$f_1(z) = \frac{b(1+M)}{2\pi}\ln(z-z_0) + \frac{b(1+M)}{2\pi}\sum_{n=1}^{+\infty} M^n \ln(z-z_1^{(n)}), \quad z \in S_1, \tag{6.5}$$

where

$$z_1^{(n)} = z^{(n)} = \left[[2n+1-(-1)^n]H^{1/2} + (-1)^n z_0^{1/2}\right]^2. \tag{6.6}$$

Once again, that Eq. (6.5) is indeed correct in the case of $z_0 = H$ can also be verified from a careful reading of the result in Obnosov [10]. The expression for $f_1(z)$ in Eq. (6.5) allows for an interesting interpretation: as the sum of contributions from the addition of the real and image screw dislocations both at $z = z_0$ with the total Burgers vector $b(1+M)$ in an infinite plane and that from the image screw dislocations at $z_1^{(n)}$, $n = 1, 2, \ldots, +\infty$ with Burgers vectors $bM^n(1+M)$ in an infinite plane. The locations of the image dislocations for the parabolic inhomogeneity are illustrated in Fig. 7 for the two typical cases of $z_0 = (-3+4\mathrm{i})H$, H located on the parabolic interface.

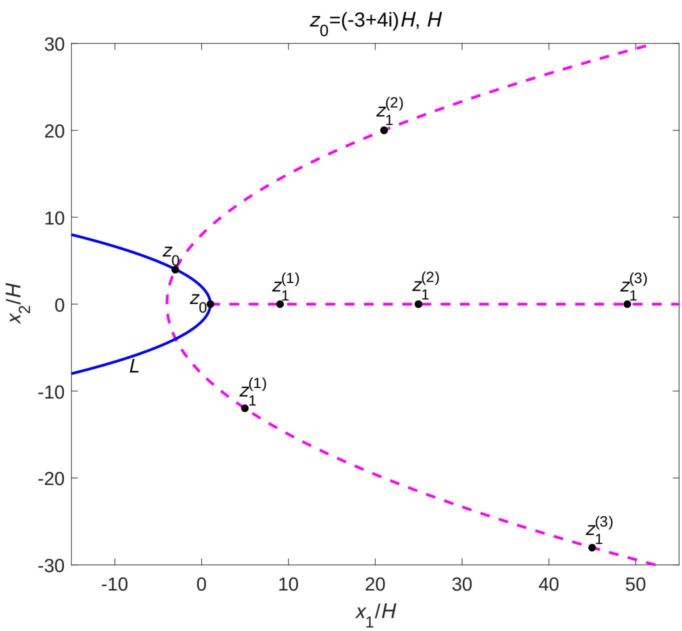

Fig. 7. The locations of the image dislocations for the parabolic inhomogeneity when $z_0 = (-3+4\mathrm{i})H$, H just located on the parabolic interface.

It is seen from Eqs. (4.10) and (6.5) that if we place a screw dislocation with the Burgers vector b at $z = z_0$ on the parabolic interface itself while another with the Burgers vector $-Mb$ is located at $z = (2H^{1/2} + \bar{z}_0^{1/2})^2$ in the matrix, the analytic function in the inhomogeneity takes the simple form

$$f_1(z) = \frac{b(1+M)}{2\pi} \ln(z - z_0), \qquad z \in S_1,$$

which corresponds to the result for a screw dislocation on the interface of a bimaterial composed of two bonded half-planes [4].

7. Conclusions

We have derived analytic solutions to the problem of a screw dislocation near a parabolic elastic inhomogeneity. When the screw dislocation is located in the matrix, the two analytic functions $f_1(\xi)$ and $f_2(\xi)$ are obtained in Eqs. (4.7) and (4.8), and the image force is given by Eq. (4.9). When the screw dislocation is located in the parabolic inhomogeneity, the two analytic functions $f_1(\xi)$ and $f_2(\xi)$ are obtained in Eqs. (5.7) and (5.8), and the image force is given by Eq. (5.9). When the screw dislocation is located solely on the parabolic interface, the two analytic functions $f_1(\xi)$ and $f_2(\xi)$ are obtained in Eqs. (6.2) and (6.3). The solutions obtained in Sections 4 and 5 can be further employed as Green's functions to investigate the interaction of a finite crack with the parabolic elastic inhomogeneity when the surrounding matrix is subjected to uniform remote anti-plane stresses. The solution derived in Section 6 can be used quite conveniently as the Green's function to study a partially debonded parabolic elastic inhomogeneity. It is more advantageous to construct the Cauchy singular integral equation for a partially debonded parabolic elastic inhomogeneity in the image ξ-plane. Note that the stresses inside a perfectly bonded parabolic inhomogeneity are uniform when the surrounding matrix is subjected to uniform remote anti-plane stresses [11].

We conclude by noting that the method of solution presented here can be easily adapted to accommodate the problem of a screw dislocation located outside, inside or precisely on the interface of a parabolic *piezoelectric* inhomogeneity. Previous studies on dislocations in piezoelectric solids are abundant in the literature and can be found in, for example, [12, 13].

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