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ANALYTICAL AND NUMERICAL ASPECTS OF DAMPING OSCILLATIONS OF COLUMNS UNDER CIRCULATORY LOADING CONDITIONS

Abstract

In this paper we study the influence of damping on the stability of columns under variable loads ofcirculatory type. As opposed to the case of static loading by potential forces, energy dissipation by damping does – in general – not ensure a stabilizing effect. For purely elastic columns, the stability limit can be improved by shape optimization. Here we show that by optimizing the distribution of damping effects along a column it is also possible to increase the critical load of a given column.

INTRODUCTION

We consider the Bernoulli-Euler beam equation with an additional damping term as a model for a column under a compressing longitudinal force. At the bottom, clamped boundary conditions are assumed. For the tip, Beck's or Reut's conditions are imposed, exemplary for a circulatory load. As a consequence, we obtain a non-self-adjoint boundary value problem for the lateral displacement.

In Figure 1, models of Beck, Reut and Euler columns are presented. Only the last case turns out to be conservative, i.e., the total energy is preserved. In the two other cases, during an oscillation around the trivial equilibrium position energy may be gained from the external force – which means that a *flatter*-type instability may occur, see e.g. [2-4] and [7].

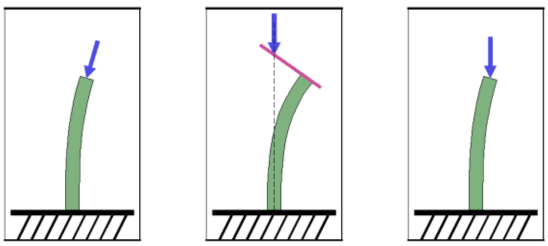

Fig. 1.*Academic column models: Beck, Reut and Euler*

In the Euler case, the instability is of divergent type, at the critical load there is a bifurcation of the equilibrium solution path.

Fig. 2.*Real-world construction under follower force*

For real-world applications, cf. Figure 2, the static critical dead load is well known, it can be calculated by a singularity analysis of the stiffness matrix describing the tower. It is much more complex to consider the dynamical behaviour, most important are the eigen-frequencies and eigenforms of the construction under the real loading conditions.

For this paper, we perform our analysis on academic examples, preferably on the Beck's column, bearing in mind, however, applications as shown in Figure 2. For other applications see [5,6] and [9].

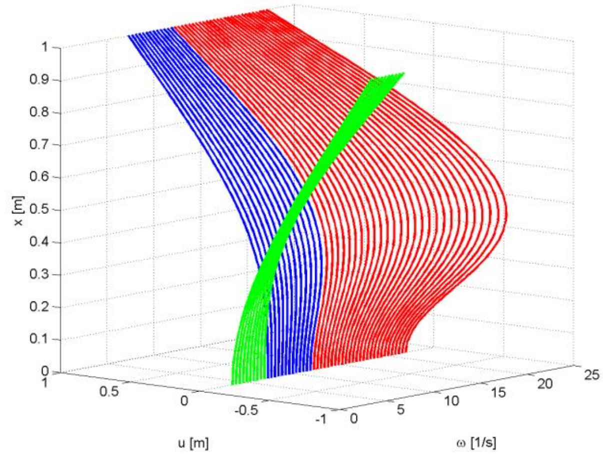

Fig. 3.*Change of critical eigenform with increasing load at tip*

When the load parameter is increased, on the one hand a change in the eigenfrequency is observed, which finally ceases to be real-valued, so that instability occurs. On the other hand the form of the free oscillations changes. This is shown in Figure 3. At a certain load, the number of nodes of a chosen eigenformmay jump e.g. from one to two (green to blue), later instability happens – the amplitude begins to increase with time (red).

Our particular interest is in the first eigenforms of the construction under a compressive load. Usually one of them – but not necessarily that corresponding to the lowest frequency – leads to the loss of stability. This is the case, when for the first time,with growing load,a negative imaginary part of an eigenfrequency appears.

Figure 3, compare also Figure 6, illustrates the fact that the mode – in the sense of number of nodes – of the first eigenform may be higher than one. Knowledge of the eigenform that turns critical is very helpful for determining a suitable damping distribution.

It should be mentioned that similar considerations concern the stability of plates under follower forces applied along the edges, [1].

1. MODEL EQUATIONS

The partial differential equation governing lateral displacements in an Euler beam is the following:

$$(Su_{xx} + Pu)_{xx} + \rho u_{tt} - b = 0\,. \tag{1}$$

The boundary conditions are:

$$u(0,t) = u_x(0,t) = 0, \tag{2}$$

$$u_{xx}(1,t) = (Su_{xx}(1,x))_x = 0\,. \tag{3}$$

Here $u = u(x,t)$ denotes the lateral displacement of the middle line of the considered column, position x and time $t$are the independent variables. Partial derivatives we indicate by a subscript. Equation (1) expresses the balance of momentum in the lateral direction, (2) represents clamped conditions at the bottom, (3) demands the moment at the tip of the column to vanish and the lateral force to be proportional to the inclination, so that the force is always tangential to the center line, the modulus of the compression force is kept constant P. By S the bending stiffness is abbreviated. Notice that the linear mass density ρ as well as the stiffness may be dependent on the location – but not on time. On the other hand, P is a concentrated force, hence it does not depend on space position x. While it might be a function of time t, for this paper we assume it to be constant. Of course, P is just the modulus – the direction of the force is variable, thus the system (1-3) is a special case of a problem with a follower force. Analogous problems may be considered for other choices of (3), e.g. the Reut case is very similar. The Euler case, on the other side, is distinguished by a constant force, acting at always the same particle, so it is a so-called *dead load*.

For better comparison, the variables in (1-3) are assumed to be dimensionless, e.g. $\in [0,1]$, and the load is measured in multiples of $S^{-1}L^{-2}$, where L denotes the true physical length of the column.

2. SOLUTION METHODS

In the homogeneous case, i.e. for vanishing body forces b, there are homogenous solutions to (1-3) in the form:

$$u(x,t) = u(x)e^{i\omega t}\,, \tag{4}$$

respectively

$$u(x,t) = u(x)\sin(\omega t)\,. \tag{5}$$

All scalar multiples and linear combinations of such oscillating solutions fulfil also the equation of motion (1) and both boundary conditions (2) and (3).

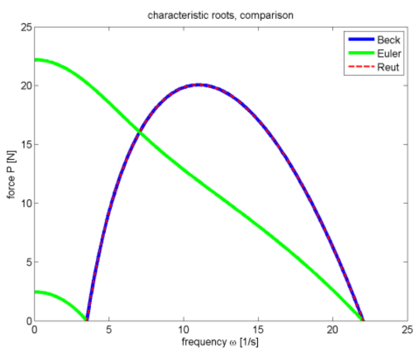

Fig. 4.*Real branches of characteristic root curves for elastic Beck's column, together with Reut and Euler cases for comparison*

For a prismatic column of a rectangular profile with unit mass density and stiffness, below the critical value of $P_{cr} \cong 20.05$ each frequency ω compatible with (1-4) is a real number. Above the critical load P_{cr}, there appear solutions to the homogeneous version of (1), i.e. for $b \equiv 0$, with conjugate complex, not real, parameters ω and $\overline{\omega}$.

Of course, one of the two corresponding solutions exhibits unbounded growth, so that the zero solution becomes unstable. In fact, a value ω with positive imaginary part results in an amplitude of the form $\exp(-Im(\omega)t)$, which is decaying, but then the conjugate complex has a negative imaginary part, which causes exponential growth with time. This fact motivates the term *critical* load P_{cr} – it marks the onset of instability of solutions to the system (1-3).

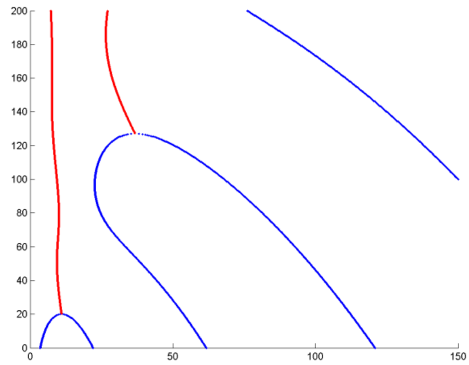

Fig. 5.*Characteristic root curves for elastic Beck's column, non-real parts projected on real plane, in red*

Classically, the root curves are obtained by a study of the determinant of the transfer matrix. By such an approach plots like that in Figure 3 are obtained. For our present purposes, however, it is essential to find also the non-real branches of the root curves, and the number, order and positions of nodes of the forms of oscillations. Eigenforms are shown in Figure 3, nodes are counted in Figure 6.

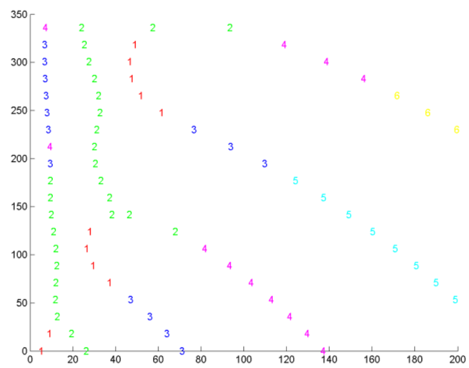

Fig. 6.*Node numbers along characteristic root curves for elastic Beck's column, based oncount of zeros of eigenforms*

3. OPTIMIZATION OF THE CRITICAL LOAD

The phenomenon of *flatter* type instability was widely discussed in the literature, e.g. in [2]. Attempts were made to increase the load carrying capacity P_{cr}, for instance by redistributing material along its length, [10-14]. Here it is sound to assume that the stiffness S is related to the mass density ρ by a law of the type

$$S = S_0\rho^3 \tag{6}$$

in the case of a column with constant depth. When it is assumed that all cross sections are similar, the exponent changes to 4.By similar is meant that they have identical width to depth ratio.

Now, for a given amount of material

$$m = \int_0^1 \rho(x)dx \tag{7}$$

it is possible to obtain higher values of P_{cr} by making the upper part of the construction more slender, while the lower part is made wider.

In fact, the end segment is now less stiff by virtue of (6), but it has also less inertia, so that the overall behaviour is more stable.

Figure 7 shows the critical load that can be supported by a modified Beck's column. The modification uses exactly the same amount of material, no lateral supports are introduced. However, a considerably higher critical load is achieved.

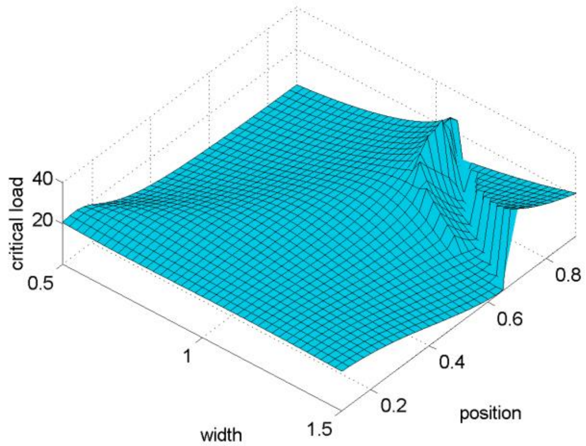

Fig. 7.*Optimamizing a two-segment version of an elastic Beck's column*

It turns out that the limit load can be pushed further up, when more than two segments of constant width are allowed, or if a continuously variable profile, and correspondingly stiffness and density distributions, are taken into account as a design variables. By means of shape optimizations the value of P_{cr} can reach up to seven times its basic value. Even critical loads of almost 200 are reported, such profiles turn out, unfortunately, to be very sensitive to small perturbations, so that they have no practical use, [14].

4. SUPPORTS

Further options for gaining higher load carrying capacities are the introduction of lateral supports, either elastic, viscous or visco-elastic elements may serve this purpose, [7]. Springs and/or dampers, attached to a lateral frame, or a Winkler-type bedding may be studied in this context. In the former case, concentrated forces, multiples of a Dirac distribution, appear as an additional term b in (1), e.g.:

$$b(x,t) = (-ku(x,t) - du_t(x,t))\delta(x - x_s)\ , \qquad (8)$$

where x_s is the position of the support, $k \geq 0$ is the elasticity modulus, $d \geq 0$ is the damper constant.

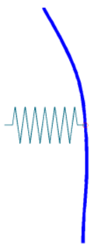

Fig. 8.*Beck's column with single elastic lateral support*

Figure 8 shows the special case that the support is positioned exactly in the midpoint of the column. This is the perfect placement in the limit case of an infinite modulus of the spring. It turns out that for smaller, more realistic, spring constants, a higher position gives a larger increment of the critical load. In [4] it was shown that in certain regions of the parameter space unexpected drops of the objective function may occur. These are typically correlated to changes of eigenmodesprovoked by the support. The change in the root curves is shown in Figure 9. Notice the increase of $P_{ct}$by almost 200% and the branch along the imaginary axis, showing that divergent instability may be observed at supercritical load.

In the distributed bedding case, instead of (8) we have:

$$b(x,t) = -k(x)u(x,t) - d(x)u_t(x,t) \qquad (9)$$

with classical, i.e. not Dirac distribution type, material functions k and d.

Obviously, (8) can be considered a singular limit case of (9), and analogously, cases with several different supports can be derived.

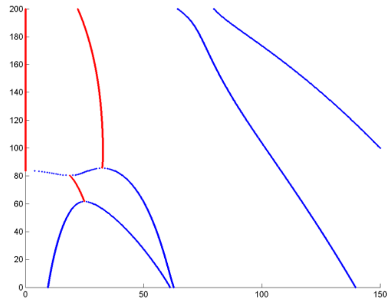

Fig. 9.*Characteristic root curves for Beck's column with single elastic lateral support*

A study of the dynamics of an unstable Beck's column shows that the upper part gains quickly kinetic energy. It seems more promising to substitute the spring from Figure 8 by a spring/damper or a single damper.

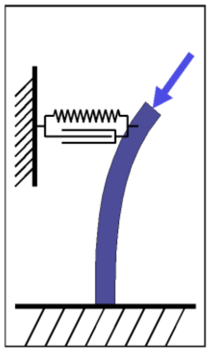

Fig. 10.*Beck's column with single viscoelastic lateral support*

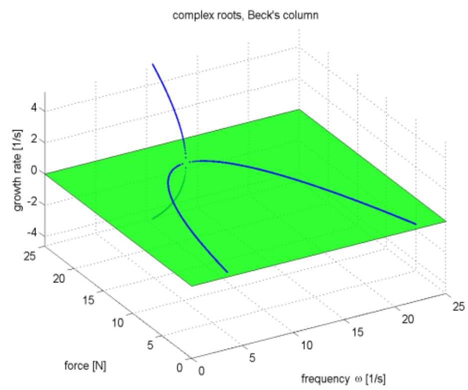

Fig. 11.*Shift of characteristic root curves in damped case*

Figure 11 shows a result obtained for the case of a damping according to (9) with a constant function d and vanishing k. Calculations are based of the exponential form of solutions (4), i.e., we split the wanted solution into a product of anoscillation form function anda harmonic time dependence.Discretization in space leads to a matrix eigenvalue problem:

$$\begin{bmatrix} 0 & I \\ \rho^{-1}K(P) & \rho^{-1}D \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = i\omega \begin{bmatrix} U \\ V \end{bmatrix}. \quad (10)$$

The dimension of the stiffnessmatrix K equals the number of nodes we chose in $[0,1]$ for the space discretization. The identity matrix I and the damping $D$are of the same size. Notice that the stiffness matrix Kcomprises the term Pu_{xx}, so it depends on the external load$P$.Consequently, for each load level an eigenvalue problem of dimension $2n$ has to be solved. The solutions form sets of branches in the $\omega - P$-space, which is three-dimensional due to the fact that the frequency may be complex, not real. We speak about complex root curves, see Figure 11 as opposed to the real projections in Figures 4 and 9.

The corresponding first branch of the complex root curve, parametrized by the compressive load, is shown in Fig. 11. Comparison with the elastic case shows that a shift along the imaginary axis causes a slight increase of the critical force.

Application of an evolutionary strategy allows enhancing this effect. On the other hand, a bad distribution again may cause a drop of the load carrying capacity.

CONCLUSION

The stability analysis of elastic columns with damping leads to complex eigenvalue problems. At the same level of discretization, the matrix dimension doubles.Further, circulatory loads causethe lack of symmetry in the matrix formulation. In the effect, the evaluation of the critical load, i.e. the onset of instability, has a much higher numerical cost. Nonetheless, an increase of the limit load can be achieved by application of non-uniform damping distributed along a Beck’s column.

So far we have studied here the case of damping depending on the lateral velocity of the construction, i.e. a drag force due to interaction with an external medium or caused by a lateral support. Damping may be caused as well by internal dissipation. This means that terms featuring the same spatial operators on u_t as so far applied on u enter the equation. After discretization, a similar problem as in (10) is to be solved, where D no is no longer a diagonal matrix. Consequently, the computational cost for an optimal distribution of material damping – which means for instance the distribution of damping material, compare [8], – may be expected to be comparable with that of shape optimization.

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ANALITYCZNE I NUMERYCZNE ASPEKTY TŁUMIENIA KOLUMN PODDANYCH OBCIĄŻENIOM CYRKULACYJNYM

Streszczenie

W pracy badany jest wpływ tłumienia na stateczność kolumn w przypadku działania obciążeń zmiennych typu cyrkulacyjnego. Wykazono możliwość zwiększenia siły krytycznej przez dobór rozkładu tłumienia wzdłuż konstrukcji analogicznie do optymalizacji kształtu kolumn czysto sprężystych.

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