Control and Cybernetics

vol. **47** (2018) No. 1

Optimality conditions and duality for generalized fractional minimax programming involving locally Lipschitz (b, Ψ, Φ, ρ) -univex functions*

by

Tadeusz Antczak[1], Shashi K. Mishra[2] and Balendu B. Upadhyay[3]

[1]Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland
antczak@math.uni.lodz.pl
[2]Department of Mathematics, Institute of Science, Banaras Hindu University Varanasi-221005, India
bhu.skmishra@gmail.com
[3]Department of Mathematics, Indian Institute of Technology Patna Bihta-801103, Bihar, India
bhooshan@iitp.ac.in

Abstract: In this paper, we are concerned with optimality conditions and duality results of generalized fractional minimax programming problems. Sufficient optimality conditions are established for a class of nondifferentiable generalized fractional minimax programming problems, in which the involved functions are locally Lipschitz (b, Ψ, Φ, ρ)-univex. Subsequently, these optimality conditions are utilized as a basis for constructing various parametric and nonparametric duality models for this type of fractional programming problems and proving appropriate duality theorems.

Keywords: generalized fractional minimax programming, locally Lipschitz (b, Ψ, Φ, ρ)-univex function, optimality conditions, duality

1. Introduction

In this paper, we consider the following generalized fractional minimax programming problem:

$$\begin{array}{ll} \text{Minimize } \Phi(x) = \sup\limits_{y\in Y} \frac{f(x,y)}{g(x,y)} & \\ \text{subject to} \quad h_j(x) \leq 0, \quad j = 1, ..., m, & \qquad (P) \\ \qquad x \in R^n, & \end{array}$$

*Submitted: December 2015; Accepted: August 2018

where Y is a specified subset of R^m, $f : R^n \times Y \to R$, $g : R^n \times Y \to R$, $h_j : R^n \to R$, $j \in J = \{1, ..., m\}$, are locally Lipschitz functions on $R^n \times Y$.

Let $D := \{x \in R^n : h_j(x) \leq 0, \;\; j \in J\}$ (assumed to be nonempty) denote the set of all feasible solutions in the considered nonsmooth generalized fractional programming problem (P). Further, we assume that $f(x, y) \geq 0$ and $g(x, y) > 0$ for all $(x, y) \in D \times Y$.

Throughout this paper, we also assume that Y is a compact set. This means that, for every $\widetilde{x} \in D$, there exists $\widetilde{y} \in Y$ with the following property:

$$\frac{f(\widetilde{x}, \widetilde{y})}{g(\widetilde{x}, \widetilde{y})} = \sup_{y \in Y} \frac{f(\widetilde{x}, y)}{g(\widetilde{x}, y)}.$$

We denote by $J(x)$ the set of active constraints at $x \in D$, that is, $J(x) := \{j \in J : h_j(x) = 0\}$. Further, let us denote

$$Y(x) := \left\{ y \in Y : \frac{f(x, y)}{g(x, y)} = \sup_{z \in Y} \frac{f(x, z)}{g(x, z)} \right\}.$$

Then note that $Y(x)$ is a compact subset of Y in R^m. Therefore, for each $x \in D$, the continuous function $f(x, \cdot)$ on the compact set Y atains its maxima for at most finite number of points, say α points.

Optimization problems, in which both a minimization and a maximization process of fractional objectives are performed, are usually referred in the optimization literature as generalized fractional minimax programming problems. Recently, there has been an increasing interest in studying generalized convexity for both differentiable and nondifferentiable generalized fractional minimax programming (see, for example, Ahmad, 2003; Ahmad and Husain, 2006; Ahmed, 2004; Antczak, 2008; Chandra et al., 1986; Ho and Lai, 2014; Liang and Shi, 2003; Liu and Wu, 1998a, 1998b; Liu et al., 1997; Mishra, 1997; Mishra et al., 2003; Mishra and Upadhyay, 2014; Upadhyay and Mishra, 2015; Yang and Hou, 2005; Zalmai, 1995, and others). For a bibliography of fractional programming, see Stancu-Minasian (1999).

In this paper, we shall establish both parametric and non-parametric sufficient optimality conditions and construct several parametric and non-parameter duality models for a new class of nonconvex generalized fractional minimax problems involving locally Lipschitz functions. In order to prove the main results in the paper, we define a new concept of generalized convexity. Namely, we introduce the definition of a locally Lipschitz (b, Ψ, Φ, ρ)-univex function, which generalizes both the definition of univex functions, Bector et al. (1994), and the definition of a class of locally Lipschitz (Φ, ρ)-invex functions, see Antczak and Stasiak (2011). Then, under nondifferentiable (b, Ψ, Φ, ρ)-univexity, we establish sufficient optimality conditions for the considered generalized fractional minimax problem involving locally Lipschitz functions. Further, motivated by Mond and Weir (1981), Bector et al. (1989), Schaible (1976), we define dual models for the primal generalized fractional minimax problem (P). We prove several duality results under various (b, Ψ, Φ, ρ)-univexity hypotheses imposed on the functions constituting the considered generalized fractional minimax programming problem (P). In particular, it does not seem that the optimality and duality results have been established previously in the literature for such a large class of nonconvex nondifferentiable generalized fractional minimax problems.

2. Preliminaries

In this section, we introduce a new concept of generalized convexity, namely, a nondifferentiable (b, Ψ, Φ, ρ)-univexity notion.

DEFINITION 1 *Let $f : R^n \to R$ be a locally Lipschitz function and $u \in R^n$. If there exist functions $\Psi : R \to R$, $b : X \times X \to R_+ \backslash \{0\}$, $\Phi : R^n \times R^n \times R^n \times R \to R$ and a real number ρ such that, for all $x \in R^n$, $\Phi(x, u; (\cdot, \cdot))$ is convex, $\Phi(x, u; (0, a)) \geq 0$ for all $a \in R_+$, such that the inequality*

$$b(x, u)\Psi(F(x) - F(u)) \geq \Phi(x, u; (\xi, \rho)) \quad (>) \tag{1}$$

holds for each $\xi \in \partial f(u)$ and all $x \in R^n$ ($x \neq u$), then f is said to be locally Lipschitz (strictly) (b, Ψ, Φ, ρ)-univex at u on R^n. If the inequality (1) is satisfied at any $u \in R^n$, then f is said to be locally Lipschitz (strictly) (b, Ψ, Φ, ρ)-univex on R^n. If the inequality (1) is satisfied for any $x \in X$, where X is a nonempty subset of R^n, then f is said to be locally Lipschitz (strictly) (b, Ψ, Φ, ρ)-univex on X.

REMARK 1 *In order to define an analogous class of (strictly) (b, Ψ, Φ, ρ)-unicave functions, the direction of the inequality (1) should be reversed.*

REMARK 2 *Note that the definition of a locally Lipschitz (b, Ψ, Φ, ρ)-univex function generalizes and extends many other generalized convexity notions. Indeed, from Definition 1, there are the following special cases:*

i) If $\Phi(x, u, (\xi, \rho)) = \xi^T(x - u)$, $\Psi(a) \equiv a$ and $b(x, u) \equiv 1$ for all $x, u \in R^n$, then we obtain the definition of a (nondifferentiable) convex function.

ii) If $\Phi(x, u, (\xi, \rho)) = \xi^T(x - u)$ and $\Psi(a) \equiv a$, then we obtain the definition of a (nondifferentiable) b-convex function.

iii) If $\Phi(x, u, (\xi, \rho)) = \xi^T \eta(x, u)$ for a certain mapping $\eta : R^n \times R^n \to R^n$, $\Psi(a) = a$ and $b(x, u) \equiv 1$ for all $x, u \in R^n$, then we obtain the definition of a (locally Lipschitz) invex function (with respect to the function η); see Reiland (1990) for the nonsmooth scalar case and, for the vectorial case, Kim and Schaible (2004), Lee (1994).

iv) If $\Phi(x, u, (\xi, \rho)) = \xi^T \eta(x, u)$ for a certain mapping $\eta : R^n \times R^n \to R^n$, then we obtain the definition of a (locally Lipschitz) univex function (with respect to the function η); see Bector et al. (1994) for the differentiable scalar case.

v) If $\Phi(x,u,(\xi,\rho)) = \frac{1}{b(x,u)}\xi^T\eta(x,u)$, $\Psi(a) \equiv a$, and $\eta : R^n \times R^n \to R^n$, then we obtain the definition of a nondifferentiable b-invex function (with respect to the function η); see Li et al. (1997).

vi) If $\Phi(x,u,(\xi,\rho)) = \xi^T(x-u) + \rho\|x-u\|^2$, $\Psi(a) \equiv a$ and $b(x,u) \equiv 1$ for all $x,u \in R^n$, then (b,Ψ,Φ,ρ)-univexity reduces to the definition of a nonsmooth ρ-convex function defined by Vial (1983) in the scalar case; see also Zalmai (1995) for the nondifferentiable case.

vii) If $\Phi(x,u,(\xi,\rho)) = \xi^T\eta(x,u) + \rho\|\theta(x,u)\|^2$, $\Psi(a) \equiv a$ and $b(x,u) \equiv 1$ for all $x,u \in R^n$, $\eta : R^n \times R^n \to R^n$, $\theta : R^n \times R^n \to R^n$, $\theta(x,u) \neq 0$, whenever $x \neq u$, then (b,Ψ,Φ,ρ)-univexity reduces to the definition of a nonsmooth ρ-invex function (with respect to η and θ) introduced by Jeyakumar (1988) in the scalar case; see also Craven (2010) and Suneja and Lalitha (1993) for the vectorial case.

viii) If $\Phi(x,u,(\xi,\rho)) = F(x,u,\xi)$, where $F(x,u,\cdot)$ is a sublinear functional on R^n, $\Psi(a) \equiv a$ and $b(x,u) \equiv 1$ for all $x,u \in R^n$, then the definition of a (b,Ψ,Φ,ρ)-univex function reduces to the definition of F-convexity introduced by Hanson and Mond (1982) in the scalar case.

ix) If $\Phi(x,u,(\xi,\rho)) = F(x,u,\xi) + \rho d^2(x,u)$, where $F(x,u,\cdot)$ is a sublinear functional on R^n, $\Psi(a) \equiv a$ and $b(x,u) \equiv 1$ for all $x,u \in R^n$, then the definition of a (b,Ψ,Φ,ρ)-univex function reduces to the definition of (F,ρ)-convexity, considered by Mukherjee and Rao (1996) in the scalar case, and by Bhatia and Jahn (1994), Craven (2010) in the vectorial case.

x) If $\Phi(x,u,(\xi,\rho)) = \alpha(x,u)\xi^T\eta(x,u)$, where $\eta : R^n \times R^n \to R^n$, $\alpha : R^n \times R^n \to R_+\backslash\{0\}$, $\alpha(x,u) = \frac{1}{b(x,u)}$, then (b,Ψ,Φ,ρ)-univexity reduces to the definition of a nonsmooth α-invex function (with respect to η), introduced by Mishra et al. (2008), see also Jayswal et al. (2013).

xi) If $\Psi(a) \equiv a$ and $b(x,u) \equiv 1$ for all $x,u \in R^n$, then we obtain the definition of a locally Lipschitz (Φ,ρ)-invex function; see Antczak and Stasiak (2011) for the scalar case, and Antczak (2014) for the nondifferentiable vectorial case.

3. Optimality

In this section, for the considered nonsmooth generalized fractional minimax programming problem, we prove sufficient optimality conditions under a variety of (b,Ψ,Φ,ρ)-univexity hypotheses.

In the sequel, we shall use the following parametric necessary optimality conditions, established by Ho and Lai (2012).

First, we introduce the generalized Slater constraint qualification for a nonsmooth generalized fractional minimax programming problem, in which the inequality constraint functions are $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex.

The generalized Slater constraint qualification: It is said that the generalized Slater constraint qualification is satisfied at $\overline{x} \in D$ for problem (P) if there exists another feasible solution $\widetilde{x}$ such that $h_j(\widetilde{x}) < 0$, $j = 1, ..., m$, and, moreover, $h_j(\cdot)$, $j \in J(\overline{x})$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{x}$ on D, where Ψ_j^h is increasing and $\Psi_j^h(0) = 0$.

We now give the parametric necessary optimality conditions under the above introduced generalized Slater constraint qualification.

THEOREM 1 *(Parametric necessary optimality conditions): Let $\overline{x} \in D$ be an optimal solution of the considered nonsmooth generalized fractional minimax programming problem (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then, there exist a positive integer $\overline{\alpha}$ such that $1 \leq \overline{\alpha} \leq n + 1$, scalars $\overline{\lambda}_i$, $i = 1, ..., \overline{\alpha}$, scalars $\overline{\mu}_j$, $j = 1, ..., m$, vectors $\overline{y}^i$, $i = 1, ..., \overline{\alpha}$, and scalar $\overline{v}$, such that*

$$0 \in \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\partial f\left(\overline{x}, \overline{y}^i\right) - \overline{v} \partial g\left(\overline{x}, \overline{y}^i\right) \right) + \sum_{j=1}^{m} \overline{\mu}_j \partial h_j\left(\overline{x}\right), \tag{2}$$

$$f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right) = 0, \; i = 1, ..., \overline{\alpha}, \tag{3}$$

$$\overline{\mu}_j h_j\left(\overline{x}\right) = 0, \; j = 1, ..., m, \tag{4}$$

$$\overline{y} = \left(y^1, ..., y^{\overline{\alpha}}\right), \; y^i \in Y\left(\overline{x}\right), \; i = 1, ..., \overline{\alpha}, \; \overline{\lambda}_i \geq 0, \; \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i = 1, \; \overline{\mu}_j \geq 0, \; j = 1, ..., m. \tag{5}$$

Now, we prove the following parametric sufficient optimality conditions under various (b, Ψ, Φ, ρ)-univexity hypotheses, imposed on the functions, constituting the considered generalized minimax fractional optimization problem (P).

THEOREM 2 *Let $\overline{x} \in D$ be a feasible solution of the considered nonsmooth generalized fractional minimax programming problem (P) and, moreover, there exist an integer $\overline{\alpha}$, $1 \leq \overline{\alpha} \leq n+1$, scalars $\overline{\lambda}_i$, $i = 1, ..., \overline{\alpha}$, scalars $\overline{\mu}_j$, $j = 1, ..., m$, and vectors $\overline{y}^i \in Y(\overline{x})$, $i = 1, ..., \overline{\alpha}$, such that the necessary optimality conditions (2)-(5) are satisfied at $\overline{x}$. Further, assume that either one of the following three sets of hypotheses is satisfied:*

a) *(i)* *for each $i = 1, ..., \overline{\alpha}$, $f\left(\cdot, \overline{y}^i\right) - \overline{v} g\left(\cdot, \overline{y}^i\right)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{x}$ on D and $a < 0 \Longrightarrow \Psi_i(a) < 0$;*

(ii) *for each $j \in J\left(\overline{x}\right)$, $h_j(\cdot)$ is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{x}$ on D, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$;*

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j} \geq 0.$

b) for each $i = 1, ..., \overline{\alpha}$, $f\left(\cdot, \overline{y}^i\right) - \overline{v} g\left(\cdot, \overline{y}^i\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$ *is* $(b_i, \Psi_i, \Phi, \rho_i)$*-univex at* $\overline{x}$ *on* D *and* $a < 0 \Longrightarrow \Psi_i(a) < 0$*;*

c) the so-called $\overline{\alpha}$*-reduced Lagrange function for problem (P), that is, the function* $z \to L_{\overline{\alpha}}\left(z, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)$*, where* $L_{\overline{\alpha}}\left(z, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) := \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i (f(z, y^i) - \overline{v} g(z, y^i)) + \sum_{j=1}^{m} \overline{\mu}_j h_j(z)$*, is* (b, Ψ, Φ, ρ)*-univex at* $\overline{x}$ *on* D*, where* $\rho \geq 0$ *and* $\Psi(a) \geq 0 \Longrightarrow a \geq 0$*.*

Then $\overline{x}$ *is an optimal solution of the problem (P).*

Proof. Let $\overline{x}$ be an arbitrary feasible solution of the considered nonsmooth generalized fractional minimax programming problem (P). Furthermore, we assume that there exist an integer number $\overline{\alpha}$, $1 \leq \overline{\alpha} \leq n+1$, scalars $\overline{\lambda}_i$, vectors $\overline{y}^i \in Y(\overline{x})$, $i = 1, ..., \overline{\alpha}$, and scalars $\overline{\mu}_j$, $j = 1, ..., m$, such that the necessary optimality conditions (2)-(5) are fulfilled at $\overline{x}$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not optimal for (P). Then, there exists a feasible solution $\widetilde{x}$ of the problem (P) such that

$$\overline{v} = \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)} > \sup_{y \in Y} \frac{f(\widetilde{x}, y)}{g(\widetilde{x}, y)}. \tag{6}$$

Hence, (6) gives

$$f\left(\widetilde{x}, \overline{y}^i\right) - \overline{v} g\left(\widetilde{x}, \overline{y}^i\right) < 0, \; i = 1, ..., \overline{\alpha}. \tag{7}$$

Thus, by the necessary optimality conditions (3) and (5), (7) yields

$$f\left(\widetilde{x}, \overline{y}^i\right) - \overline{v} g\left(\widetilde{x}, \overline{y}^i\right) < f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right), \; i = 1, ..., \overline{\alpha}. \tag{8}$$

Proof of this theorem under hypothesis a).

By hypothesis a) (i) - (ii), $f\left(\cdot, \overline{y}^i\right) - \overline{v} g\left(\cdot, \overline{y}^i\right)$, $i \in I$, is $(b_i, \Psi_i, \Phi_i, \rho_i)$-univex at $\overline{x}$ on D and $h_j(\cdot)$, $j \in J(\overline{x})$, is $(b_{h_j}, \Psi, \Phi_j^h, \rho_{h_j})$-univex at $\overline{x}$ on D. Then, by Definition 1, the following inequalities

$$\begin{aligned} b_i(x, \overline{x}) \Psi_i \left(f\left(x, \overline{y}^i\right) - \overline{v} g\left(x, \overline{y}^i\right) - \left[f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right)\right]\right) \geq \\ \Phi\left(x, \overline{x}, (\xi_i, \rho_i)\right), \; i = 1, ..., \overline{\alpha}, \end{aligned} \tag{9}$$

and

$$b_{h_j}(x, \overline{x}) \Psi_j^h \left(h_j(x) - h_j(\overline{x})\right) \geq \Phi_j^h \left(x, \overline{x}, \left(\zeta_j, \rho_{h_j}\right)\right), \; j \in J(\overline{x}) \tag{10}$$

hold for all $x \in D$ and for each $\xi_i \in \partial\left(f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right)\right)$, $i \in I$, and for each $\zeta_j \in \partial\left(h_j(\overline{x})\right)$, $j \in J(\overline{x})$, respectively. Therefore, so for $x = \widetilde{x} \in D$. Thus, (9) and (10) yield

$$\begin{aligned} b_i(\widetilde{x}, \overline{x}) \Psi_i \left(f\left(\widetilde{x}, \overline{y}^i\right) - \overline{v} g\left(\widetilde{x}, \overline{y}^i\right) - \left[f\left(\overline{x}, \overline{y}^i\right) - \overline{v} g\left(\overline{x}, \overline{y}^i\right)\right]\right) \\ \geq \Phi\left(\widetilde{x}, \overline{x}, (\xi_i, \rho_i)\right), \; i = 1, ..., \overline{\alpha}, \end{aligned} \tag{11}$$

$$b_{h_j}(\widetilde{x},\overline{x})\Psi_j^h\left(h_j(\widetilde{x})-h_j(\overline{x})\right)\geq\Phi\left(\widetilde{x},\overline{x},\left(\zeta_j,\rho_{h_j}\right)\right),\ j=1,...,m. \tag{12}$$

Using hypothesis $a<0\Longrightarrow\Psi_i(a)<0$, $i=1,...,\overline{\alpha}$, together with $b_i(\widetilde{x},\overline{x})>0$, $i=1,...,\overline{\alpha}$, we deduce that inequalities (8) imply

$$b_i(\widetilde{x},\overline{x})\Psi_i\left(f\left(\widetilde{x},\overline{y}^i\right)-\overline{v}g\left(\widetilde{x},\overline{y}^i\right)-\left[f\left(\overline{x},\overline{y}^i\right)-\overline{v}g\left(\overline{x},\overline{y}^i\right)\right]\right)<0,\ i=1,...,\overline{\alpha}. \tag{13}$$

Combining (11) and (13), we obtain the inequalities

$$\Phi\left(\widetilde{x},\overline{x},(\xi_i,\rho_i)\right)<0,\ i=1,...,\overline{\alpha} \tag{14}$$

hold for all $x\in D$ and for each $\xi_i\in\partial\left(f\left(\overline{x},\overline{y}^i\right)-\overline{v}g\left(\overline{x},\overline{y}^i\right)\right)$, $i=1,...,\overline{\alpha}$. Since $\overline{\lambda}_i\geq 0$, $\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i=1$, (14) gives

$$\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\Phi\left(\widetilde{x},\overline{x},(\xi_i,\rho_i)\right)<0. \tag{15}$$

By $\widetilde{x}\in D$, $\overline{x}\in D$ and by the definition of $J\left(\overline{x}\right)$, we have

$$h_j\left(\widetilde{x}\right)-h_j(\overline{x})\leq 0,\ j\in J\left(\overline{x}\right). \tag{16}$$

Since Ψ_j^h is an increasing functional with $\Psi_j^h(0)=0$ and $b_{h_j}(\widetilde{x},\overline{x})>0$, $j\in J\left(\overline{x}\right)$, (13) yields

$$b_{h_j}(\widetilde{x},\overline{x})\Psi_j^h\left(h_j(\widetilde{x})-h_j(\overline{x})\right)\leq 0,\ j\in J\left(\overline{x}\right). \tag{17}$$

By combining (16) and (17), we get

$$\Phi\left(\widetilde{x},\overline{x},\left(\zeta_j,\rho_{h_j}\right)\right)\leq 0,\ j\in J\left(\overline{x}\right).$$

Thus,

$$\sum_{j\in J(\overline{x})}\overline{\mu}_j\Phi\left(\widetilde{x},\overline{x},\left(\zeta_j,\rho_{h_j}\right)\right)\leq 0. \tag{18}$$

Let us denote

$$\overline{\vartheta}_i=\frac{\overline{\lambda}_i}{\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i+\sum_{j=1}^m\overline{\mu}_j},\ i=1,...,\overline{\alpha}, \tag{19}$$

$$\overline{\delta}_j=\frac{\overline{\mu}_j}{\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i+\sum_{j=1}^m\overline{\mu}_j},\ j\in J\left(\overline{x}\right). \tag{20}$$

By (19) and (20), it follows that $0\leq\overline{\vartheta}_i\leq 1$, $i=1,...,\overline{\alpha}$, but, for at least one $i\in\{1,...,\overline{\alpha}\}$, $\overline{\vartheta}_i>0$, $0\leq\overline{\delta}_j\leq 1$, $j\in J\left(\overline{x}\right)$ and, moreover,

$$\sum_{i=1}^{\overline{\alpha}}\overline{\vartheta}_i+\sum_{j\in J(\overline{x})}\overline{\delta}_j=1. \tag{21}$$

Taking into account (19) and (20) in (15) and (18), and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{\overline{\alpha}} \overline{\vartheta}_i \Phi\left(\widetilde{x}, \overline{x}, (\xi_i, \rho_i)\right) + \sum_{j \in J(\overline{x})} \overline{\delta}_j \Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{h_j}\right)\right) < 0. \tag{22}$$

By Definition 1, $\Phi(\widetilde{x}, \overline{x}; (\cdot, \cdot))$ is convex. Since (21) holds, by the definition of convexity, (22) yields

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{\overline{\alpha}} \overline{\vartheta}_i \left(\xi_i, \rho_i\right) + \sum_{j \in J(\overline{x})} \overline{\delta}_j \left(\zeta_j, \rho_{h_j}\right)\right) < 0. \tag{23}$$

Taking into account the Lagrange multipliers as being equal to 0, we have

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{\overline{\alpha}} \overline{\vartheta}_i \xi_i + \sum_{j=1}^{m} \overline{\delta}_j \zeta_j \ , \ \sum_{i=1}^{\overline{\alpha}} \overline{\vartheta}_i \rho_i + \sum_{j=1}^{m} \overline{\delta}_j \rho_{h_j}\right) < 0. \tag{24}$$

Using (19) and (20) in (24), we obtain

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\mu}_j}\left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \xi_i + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j, \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j}\right)\right) < 0. \tag{25}$$

By the necessary optimality condition (2), it follows that

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\mu}_j}\left(0 \ , \ \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j}\right)\right) < 0. \tag{26}$$

By Definition 1, $\Phi(\widetilde{x}, \overline{x}; (0, a)) \geq 0$ for all $a \in R_+$. Then, by hypothesis (iii), the following inequality

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\mu}_j}\left(0 \ , \ \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j}\right)\right) \geq 0$$

holds, contradicting (26). This completes the proof of this theorem under hypothesis (a).

Proof of the theorem under hypothesis b) is similar to the one under hypothesis a) and, therefore, it has been omitted in the paper.

Proof of theorem under hypothesis c).

By assumption, the $\overline{\alpha}$-reduced Lagrangian $z \to L_{\overline{\alpha}}\left(z, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)$ is (b, Ψ, Φ, ρ)-univex at $\overline{x}$ on D. Hence, by Definition 1, the following inequality

$$b(x, \overline{x})\Psi\left(L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) - L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)\right) \geq \Phi\left(x, \overline{x}, (\xi, \rho)\right) \tag{27}$$

holds for all $x \in D$ and for each $\xi \in \partial L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)$. Since (27) is satisfied for each $\xi \in \partial L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)$, by the definition of the $\overline{\alpha}$-reduced Lagrangian, it is fulfilled for each $\xi \in \partial\left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(\overline{x}, y^i) - \overline{v}g(\overline{x}, y^i)\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\overline{x})\right)$. By Corollary 2, for Proposition 2.3.3 (see, Clarke, 1983), it follows that

$$\xi \in \left(\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(\partial f(\overline{x}, y^i) - \overline{v}\partial g(\overline{x}, y^i)\right) + \sum_{j=1}^{m} \overline{\mu}_j \partial h_j(\overline{x})\right).$$

Hence, by the necessary optimality condition (2), (27) gives

$$b(x, \overline{x})\Psi\left(L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) - L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)\right) \geq \Phi\left(x, \overline{x}, (0, \rho)\right). \tag{28}$$

Since $\Phi\left(x, \overline{x}, (0, a)\right) \geq 0$ for any $a \geq 0$, by hypothesis $\rho \geq 0$, (28) implies

$$b(x, \overline{x})\Psi\left(L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) - L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)\right) \geq 0. \tag{29}$$

By definition, $b(x, \overline{x}) > 0$ for all $x \in D$. Then, (29) yields

$$\Psi\left(L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) - L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right)\right) \geq 0.$$

Using hypothesis $\Psi(a) \geq 0 \Longrightarrow a \geq 0$, we get

$$L_{\overline{\alpha}}\left(x, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right) \geq L_{\overline{\alpha}}\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu}, \overline{v}\right).$$

Hence, by the definition of the $\overline{\alpha}$-reduced Lagrange function, we have

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(x, y^i) - \overline{v}g(x, y^i)\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(x) \geq$$
$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(\overline{x}, y^i) - \overline{v}g(\overline{x}, y^i)\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\overline{x}). \tag{30}$$

By the necessary optimality conditions (3) and (4), (30) yields

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(x, y^i) - \overline{v}g(x, y^i)\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(x) \geq 0.$$

From the feasibility of x in (P) and $\overline{\mu}_j \geq 0$, $j = 1, ..., m$, we obtain that the inequality

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\left(f(x, y^i) - \overline{v}g(x, y^i)\right) \geq 0$$

holds for all $x \in D$. Since $\overline{\lambda}_i \geq 0$ for $i = 1, ..., \overline{\alpha}$, then there exists i^* such that the inequality

$$f(x, \overline{y}^{i^*}) - \overline{v} g(x, \overline{y}^{i^*}) \geq 0$$

holds for all $x \in D$. Thus,

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \overline{v}.$$

Since $\sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)} = \overline{v}$, the following inequality

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)}$$

holds for all $x \in D$. This means that $\overline{x}$ is optimal of the problem (P). ■

Remark 3 *Note that in the proof of this theorem under hypothesis c), we have not used the assumption that the functional $\Phi(x, \overline{x}; (\cdot, \cdot))$ is convex for all $x \in D$.*

In order to discuss various nonparametric dual models for the considered generalized fractional minimax programming problem (P), we state another version of the necessary optimality conditions, formulated in Theorem 1. This can be accomplished by simply replacing the parameter $\overline{v}$ with $f(\overline{x}, \overline{y}^i)/g(\overline{x}, \overline{y}^i)$ and rewriting the multiplier functions, associated with the inequality constraints. Hence, the nonparametric necessary optimality conditions can be formulated as follows:

Theorem 3 *(Nonparametric necessary conditions): Let $\overline{x}$ be an optimal solution in (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then there exist a positive integer $\overline{\alpha}$, scalars $\overline{\lambda}_i \geq 0$, $i = 1, ..., \overline{\alpha}$, scalars $\overline{\mu}_j \geq 0$, $j = 1, ..., m$, and vectors $\overline{y}^i$, $i = 1, ..., \overline{\alpha}$, such that*

$$0 \in \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \left(\partial f \left(\overline{x}, \overline{y}^i \right) g \left(\overline{x}, \overline{y}^i \right) - f \left(\overline{x}, \overline{y}^i \right) \partial g \left(\overline{x}, \overline{y}^i \right) \right) + \sum_{j=1}^{m} \overline{\mu}_j \partial h_j \left(\overline{x} \right) = 0, \quad (31)$$

$$\overline{\mu}_j h_j \left(\overline{x} \right) = 0, \quad j = 1, ..., m, \quad (32)$$

$$\overline{\lambda}_i \geq 0, \ \sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i = 1, \ \overline{y}^i \in Y \left(\overline{x} \right), \ i = 1, ..., \overline{\alpha}, \ \overline{\mu}_j \geq 0, \ j = 1, ..., m. \quad (33)$$

4. Duality

In this section, let Q denote the set of triples (α, λ, y), where α ranges over the integers such that $1 \leq \alpha \leq n + 1$, $\lambda \in R_+^{\alpha}$, $\sum_{i=1}^{\alpha} \lambda_i = 1$, and $\overline{y} = \left(y^1, ..., y^{\alpha} \right)$ is an $m\alpha$-dimensional vector with $y^i \in Y(x)$ for all $i = 1, ..., \alpha$ and for some $x \in R^n$.

4.1. Schaible type dual

Now, for the nonsmooth generalized fractional minimax problem (P), we consider a dual problem (SD) in the sense of Schaible as follows:

$$\max_{(\alpha,\lambda,\overline{y})\in Q} \quad \sup_{(u,\mu,v)\in W_1(\alpha,\lambda,\overline{y})} \quad v \tag{34}$$

where $W_1(\alpha,\lambda,\overline{y})$ is the set of all triples $(u,\mu,v)\in X\times R^m_+\times R_+$, satisfying the following conditions:

$$0\in\sum_{i=1}^{\alpha}\lambda_i\left(\partial f_i(u)-v\partial g_i(u)\right)+\sum_{j=1}^{m}\mu_j\partial h_j(u), \qquad \text{(SD)} \tag{35}$$

$$f\left(u,y^i\right)-vg\left(u,y^i\right)\geq 0,\ i=1,...,\alpha, \tag{36}$$

$$\mu_j h_j(u)\geq 0,\ \ j=1,...,m, \tag{37}$$

$$\lambda_i\geq 0,\ \sum_{i=1}^{\alpha}\lambda_i=1,\ \overline{y}=\left(y^1,...,y^{\alpha}\right),\ y^i\in Y\left(u\right),\ i=1,...,\alpha,\ \mu_j\geq 0,$$
$$j=1,...,k. \tag{38}$$

If, for a triplet $(\alpha,\lambda,\overline{y})\in Q$, the set $W_1(\alpha,\lambda,\overline{y})$ is empty, then we define the supremum over it to be $-\infty$.

Let Ω_{SD} denote the set of all feasible solutions for the dual problem (SD), that is, the set of $(\alpha,\lambda,\overline{y},u,\mu,v)$ satisfying the constraints (35)-(38). Further, we denote by S_{SD} the set $S_{SD}=\{u\in X:(\alpha,\lambda,\overline{y},u,\mu,v)\in\Omega_{SD}\}$ and, for $u\in S_{SD}$, $J(u)=\{j\in J:h_j(u)\geq 0\}$.

Theorem 4 *(Weak Duality). Let x and $(\alpha,\lambda,\overline{y},u,\mu,v)$ be feasible solutions for problems (P) and (SD), respectively. Further, assume that one of the following two sets of hypotheses is satisfied:*

(a) *(i)* $f\left(\cdot,y^i\right)-vg\left(\cdot,y^i\right)$, $i=1,...,\alpha$, *is* (b_i,Ψ_i,Φ,ρ_i)*-univex at* u *on* $D\cup S_{SD}$ *and* $a<0\Longrightarrow\Psi_i(a)<0$*;*

(ii) $h_j(\cdot)$, $j\in J(u)$, *is* $(b_{h_j},\Psi^h_j,\Phi,\rho_{h_j})$*-univex at* u *on* $D\cup S_{SD}$, Ψ^h_j *is increasing and* $\Psi^h_j(0)=0$*;*

(iii) $\sum_{i=1}^{\alpha}\lambda_i\rho_i+\sum_{j=1}^{m}\mu_j\rho_{h_j}\geq 0$.

(b) $f\left(\cdot,y^i\right)-vg\left(\cdot,y^i\right)+\sum_{j=1}^{m}\mu_j h_j(\cdot)$, $i=1,...,\alpha$, *is* (b_i,Ψ_i,Φ,ρ_i)*-univex at* u *on* $D\cup S_{SD}$ *and* $a<0\Longrightarrow\Psi_i(a)<0$.

Then

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)}\geq v.$$

Proof. **We prove this theorem under hypothesis (a).**
Suppose, contrary to the result, that there exist $x \in D$ and $(\alpha, \lambda, y, u, \mu, v) \in \Omega_{SD}$ such that

$$\sup_{y \in Y} \frac{f(x,y)}{g(x,y)} < v. \tag{39}$$

Hence, (39) gives

$$f(x, y^i) - vg(x, y^i) < 0,\ i = 1, ..., \alpha.$$

By the constraint (36), it follows that

$$f(x, y^i) - vg(x, y^i) < f\left(u, y^i\right) - vg\left(u, y^i\right),\ i = 1, ..., \alpha. \tag{40}$$

By hypothesis (i), $f\left(\cdot, y^i\right) - vg\left(\cdot, y^i\right)$, $i = 1, ..., \alpha$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at u on $D \cup S_{SD}$. Using hypothesis (i) again, by (40), we conclude that

$$\Psi_i\left(f(x, y^i) - vg(x, y^i) - \left[f\left(u, y^i\right) - vg\left(u, y^i\right)\right]\right) < 0,\ i = 1, ..., \alpha. \tag{41}$$

Since $b_i\left(x, u\right) > 0$, $i = 1, ..., \alpha$, the inequalities (41) yield

$$b_i\left(x, u\right) \Psi_i\left(f(x, y^i) - vg(x, y^i) - \left[f\left(u, y^i\right) - vg\left(u, y^i\right)\right]\right) < 0,\ i = 1, ..., \alpha. \tag{42}$$

Hence, by Definition 1, inequalities (42) imply that the following inequalities

$$\Phi\left(x, u, (\xi_i, \rho_i)\right) < 0,\ i = 1, ..., \alpha \tag{43}$$

hold for each $\xi_i \in \partial\left(f\left(u, y^i\right) - vg\left(u, y^i\right)\right)$, $i = 1, ..., \alpha$.
Since $\lambda_i \geq 0$, $\sum_{i=1}^{\alpha} \lambda_i = 1$, inequalities (43) yield

$$\sum_{i=1}^{\alpha} \lambda_i \Phi\left(x, u, (\xi_i, \rho_i)\right) < 0. \tag{44}$$

By $x \in D$ and $(\alpha, \lambda, \overline{y}, u, \mu, v) \in \Omega_{SD}$, it follows that $h_j(x) \leq h_j\left(u\right)$, $j \in J\left(u\right)$. By hypothesis (ii), it follows that

$$\Psi_j^h\left(h_j(x) - h_j\left(u\right)\right) \leq 0,\ j \in J\left(u\right). \tag{45}$$

By Definition 1, we have that $b_{h_j}\left(x, u\right) > 0$, $j \in J\left(u\right)$. Thus, (45) gives

$$b_{h_j}\left(x, u\right) \Psi_j^h\left(h_j(x) - h_j\left(u\right)\right) \leq 0,\ j \in J\left(u\right). \tag{46}$$

Using hypothesis (ii) again, by Definition 1, we get that the inequalities

$$\Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0,\ j \in J\left(u\right)$$

hold for each $\zeta_j \in \partial\left(h_j\left(u\right)\right)$, $j \in J\left(u\right)$. Since $\mu_j \geq 0$, $j \in J$, the above inequalities yield

$$\sum_{j \in J(u)} \mu_j \Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0. \tag{47}$$

Combining (44) and (47), we get

$$\sum_{i=1}^{\alpha}\lambda_i\Phi\left(x,u,(\xi_i,\rho_i)\right)+\sum_{j\in J(u)}\mu_j\Phi\left(x,u,\left(\zeta_j,\rho_{h_j}\right)\right)<0. \tag{48}$$

Let us denote

$$\vartheta_i=\frac{\lambda_i}{\sum_{i=1}^{\alpha}\lambda_i+\sum_{j=1}^{m}\mu_j},\ i=1,...,\alpha, \tag{49}$$

$$\delta_j=\frac{\mu_j}{\sum_{i=1}^{\alpha}\lambda_i+\sum_{j=1}^{m}\mu_j},\ j\in J\left(u\right). \tag{50}$$

As it follows from (49) and (50), $0\leq\vartheta_i\leq 1$, $i=1,...,\alpha$, but for at least one $i\in\{1,...,\alpha\}$, $\vartheta_i>0$, $0\leq\delta_j\leq 1$, $j\in J\left(u\right)$, and, moreover,

$$\sum_{i=1}^{\alpha}\vartheta_i+\sum_{j\in J(u)}\delta_j=1. \tag{51}$$

Using (49) and (50) together with (48), we obtain

$$\sum_{i=1}^{\alpha}\vartheta_i\Phi\left(x,u,(\xi_i,\rho_i)\right)+\sum_{j\in J(u)}\delta_j\Phi\left(x,u,\left(\zeta_j,\rho_{h_j}\right)\right)<0. \tag{52}$$

By Definition 1, $\Phi(x,u;(\cdot,\cdot))$ is convex. Since (51) is satisfied, then, by the definition of convexity, (52) implies

$$\Phi\left(x,u,\sum_{i=1}^{\alpha}\vartheta_i\left(\xi_i,\rho_i\right)+\sum_{j\in J(u)}\delta_j\left(\zeta_j,\rho_{h_j}\right)\right)<0.$$

Taking into account the Lagrange multipliers as being equal to 0, by (49) and (50), we have that the inequality

$$\Phi\left(x,u,\frac{1}{\sum_{i=1}^{\alpha}\lambda_i+\sum_{j=1}^{m}\mu_j}\left(\sum_{i=1}^{\alpha}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j,\ \sum_{i=1}^{\alpha}\lambda_i\rho_i+\sum_{j\in J(u)}\mu_j\rho_{h_j}\right)\right)<0 \tag{53}$$

holds for each $\xi_i\in\partial\left(f\left(u,y^i\right)-vg\left(u,y^i\right)\right)$, $i=1,...,\alpha$, and for each $\zeta_j\in\partial h_j\left(u\right)$, $j\in J$. By the constraint (35), inequality (53) yields

$$\Phi\left(x,u,\frac{1}{\sum_{i=1}^{\alpha}\lambda_i+\sum_{j=1}^{m}\mu_j}\left(0\ ,\ \sum_{i=1}^{\alpha}\lambda_i\rho_i+\sum_{j\in J(u)}\mu_j\rho_{h_j}\right)\right)<0. \tag{54}$$

By Definition 1, $\Phi(x,u,(0,a)) \geq 0$ for all $a \in R_+$. Due to hypothesis (iii), the following inequality

$$\Phi\left(x, u, \frac{1}{\sum_{i=1}^{\alpha}\lambda_i + \sum_{j=1}^{m}\mu_j}\left(0\ ,\ \sum_{i=1}^{\alpha}\lambda_i\rho_i + \sum_{j\in J(u)}\mu_j\rho_{h_j}\right)\right) \geq 0$$

holds, contradicting (54). This completes the proof of this theorem under hypothesis (a).

Proof of this theorem under hypothesis (b) is similar to the one under hypothesis (a) and, therefore, it has been omitted in the paper. ■

Theorem 5 *(Strong Duality). Let $\overline{x} \in D$ be an optimal point of the considered generalized fractional minimax problem (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then, there exist $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right) \in Q$ and $(\overline{x}, \overline{\mu}, \overline{v}) \in W_1\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right)$ such that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}, \overline{v}\right)$ is optimal for (SD). If also all hypotheses of Theorem 4 are fulfilled, then the corresponding optimal values of (P) and (SD) are the same.*

Proof. By assumption, $\overline{x} \in D$ is an optimal point of (P) and the generalized Slater constraint qualification is satisfied at $\overline{x}$. Hence, by Theorem 1, there exist a positive integer $\overline{\alpha}$, scalars $\overline{\lambda}_i \geq 0$, $i = 1, ..., \overline{\alpha}$, scalars $\overline{\xi}_j \geq 0$, $j = 1, ..., k$, and vectors $\overline{y}^i \in Y(\overline{x})$, $i = 1, ..., \overline{\alpha}$, such that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}, \overline{v}\right)$ is feasible for (SD). Since

$$\overline{v} = \frac{f(\overline{x}, \overline{y})}{g(\overline{x}, \overline{y})},$$

using the weak duality theorem (Theorem 4), we conclude that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\xi}, \overline{v}\right)$ is optimal for (SD). Hence, the corresponding optimal values of (P) and (SD) are the same. ■

Theorem 6 *(Converse Duality). Let $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}, \overline{v}\right)$ be an optimal point of (SD) such that $\overline{u} \in D$. Further, assume that one of the following two sets of hypotheses is satisfied:*

(a) (i) $f\left(\cdot, y^i\right) - \overline{v}g\left(\cdot, y^i\right)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{SD}$ and $a < 0 \Longrightarrow \Psi_i(a) < \Psi_i(0) = 0$;

(ii) $h_j(\cdot)$, $j \in J(\overline{u})$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{u}$ on $D \cup S_{SD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$;

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i\rho_i + \sum_{j=1}^{m} \overline{\mu}_j\rho_{h_j} \geq 0$.

(b) $f\left(\cdot, y^i\right) - \overline{v}g\left(\cdot, y^i\right) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{SD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;

Then $\overline{u}$ is optimal for the generalized fractional minimax problem (P).

Proof. The proof of this theorem follows directly from weak duality (Theorem 4). ■

THEOREM 7 *(Strict Converse Duality). Let $\overline{x}$ and $\left(\overline{\alpha},\overline{\lambda},\overline{y},\overline{u},\overline{\mu},\overline{v}\right)$ be optimal points of the problems (P) and (SD), respectively, and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Assume, furthermore, that one of the following two sets of hypotheses is fulfilled:*

(a) (i) $f\left(\cdot,y^i\right)-\overline{v}g\left(\cdot,y^i\right)$, $i=1,...,\overline{\alpha}$, is strictly (b_i,Ψ_i,Φ,ρ_i)-univex at $\overline{u}$ on $D\cup S_{SD}$ and Ψ_i is strictly increasing and $\Psi_i(0)=0$;

(ii) $h_j\left(\cdot\right)$, $j\in J\left(\overline{u}\right)$, is $(b_{h_j},\Psi_j^h,\Phi,\rho_{h_j})$-univex at $\overline{u}$ on $D\cup S_{SD}$, Ψ_j^h is increasing and $\Psi_j^h(0)=0$;

(iii) $\sum_{i=1}^{\overline{\alpha}}\overline{\lambda}_i\rho_i+\sum_{j=1}^{m}\overline{\mu}_j\rho_{h_j}\geq 0$.

(b) $f\left(\cdot,y^i\right)-\overline{v}g\left(\cdot,y^i\right)+\sum_{j=1}^{m}\overline{\mu}_jh_j\left(\cdot\right)$, $i=1,...,\overline{\alpha}$, is strictly (b_i,Ψ_i,Φ,ρ_i)-univex at $\overline{u}$ on $D\cup S_{SD}$ and $a<0\Longrightarrow\Psi_i(a)<0$.

Then $\overline{x}=\overline{u}$ and $\overline{v}=\frac{f(\overline{u},\overline{y})}{g(\overline{u},\overline{y})}$.

PROOF. **We prove this theorem under hypothesis a).**
Suppose, contrary to the result, that $\overline{x}\neq\overline{u}$. Hence,

$$\overline{v}\neq\frac{f(\overline{u},\overline{y})}{g(\overline{u},\overline{y})}.$$

From the strong duality theorem (Theorem 5), we have

$$\overline{v}=\frac{f(\overline{x},\overline{y})}{g(\overline{x},\overline{y})}. \tag{55}$$

Thus, (55) gives

$$f(\overline{x},y^i)-\overline{v}g(\overline{x},y^i)=0,\ i=1,...,\overline{\alpha}. \tag{56}$$

Since $\left(\overline{\alpha},\overline{\lambda},\overline{y},\overline{u},\overline{\mu},\overline{v}\right)\in\Omega_{SD}$, using the constraint (36) together with (56) and

$$\overline{v}\neq\frac{f(\overline{u},\overline{y})}{g(\overline{u},\overline{y})},$$

we get

$$f(\overline{x},y^i)-\overline{v}g(\overline{x},y^i)\leq f\left(\overline{u},y^i\right)-\overline{v}g\left(\overline{u},y^i\right),\ i=1,...,\overline{\alpha}, \tag{57}$$

$$f(\overline{x},y^i)-\overline{v}g(\overline{x},y^i)<f\left(\overline{u},y^i\right)-\overline{v}g\left(\overline{u},y^i\right)\ \text{ for at least one } i\in\{1,...,\overline{\alpha}\}. \tag{58}$$

By hypothesis (i), $f\left(\cdot,y^i\right)-vg\left(\cdot,y^i\right)$, $i=1,...,\overline{\alpha}$, is (b_i,Ψ_i,Φ,ρ_i)-univex at $\overline{u}$ on $D\cup S_{SD}$. Since $a<0\Longrightarrow\Psi_i(a)<\Psi_i(0)=0$, by Definition 1, inequalities (57) and (58) imply

$$\Psi_i\left(f(\overline{x},y^i)-\overline{v}g(\overline{x},y^i)-\left[f\left(\overline{u},y^i\right)-\overline{v}g\left(\overline{u},y^i\right)\right]\right)<0,\ i=1,...,\overline{\alpha}. \tag{59}$$

Since $b_i\left(\overline{x},\overline{u}\right)>0$, $i=1,...,\alpha$, inequalities (41) yield

$$b_i\left(\overline{x},\overline{u}\right)\Psi_i\left(f(\overline{x},y^i)-\overline{v}g(\overline{x},y^i)-\left[f\left(\overline{u},y^i\right)-\overline{v}g\left(\overline{u},y^i\right)\right]\right)<0,\ i=1,...,\overline{\alpha}. \tag{60}$$

Hence, by Definition 1, inequalities (42) imply that the following inequalities

$$\Phi\left(\overline{x}, \overline{u}, (\xi_i, \rho_i)\right) < 0, \ i = 1, ..., \alpha \tag{61}$$

hold for each $\xi_i \in \partial\left(f\left(\overline{u}, y^i\right) - \overline{v}g\left(\overline{u}, y^i\right)\right), \ i = 1, ..., \overline{\alpha}$. From the feasibility of $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}, \overline{v}\right)$ in the Schaible dual problem (SD), we have that $\overline{\lambda}_i \geq 0$, $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i = 1$. Hence, inequalities (61) yield

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \Phi\left(\overline{x}, \overline{u}, (\xi_i, \rho_i)\right) < 0. \tag{62}$$

Using $\overline{x} \in D$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\xi}, \overline{v}\right) \in \Omega_{SD}$, together with the assumptions that Ψ_j^h is increasing and $\Psi_j^h(0) = 0$, we obtain

$$\Psi_j^h\left(h_j(\overline{x}) - h_j\left(\overline{u}\right)\right) \leq 0, \ j \in J\left(\overline{u}\right). \tag{63}$$

Since $b_{h_j}\left(\overline{x}, \overline{u}\right) > 0$, $j \in J\left(\overline{u}\right)$, by Definition 1, we get that the inequalities

$$\Phi\left(\overline{x}, \overline{u}, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0, \ j \in J\left(\overline{u}\right)$$

hold for each $\zeta_j \in \partial\left(h_j\left(\overline{u}\right)\right)$, $j \in J\left(\overline{u}\right)$. Therefore, by $\overline{\mu}_j \geq 0$, $j \in J$, the above inequalities yield

$$\sum_{j \in J(\overline{u})} \overline{\mu}_j \Phi\left(\overline{x}, \overline{u}, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0. \tag{64}$$

By (62) and (64), it follows that

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \Phi\left(\overline{x}, \overline{u}, (\xi_i, \rho_i)\right) + \sum_{j \in J(\overline{u})} \overline{\mu}_j \Phi\left(\overline{x}, \overline{u}, \left(\zeta_j, \rho_{h_j}\right)\right) < 0. \tag{65}$$

The rest of this proof is similar to the proof of Theorem 4. ■

4.2. Weir duality

Now, following the lines of Weir (1986), we consider a dual problem (WD) to (P) as follows:

$$\max_{(\alpha,\lambda,\overline{y}) \in Q} \ \sup_{(u,\xi) \in W_2(\alpha,\lambda,\overline{y})} H(u) = \frac{f(u,y)}{g(u,y)},$$

where the set $W_2\left(\alpha, \lambda, y\right)$ is the set of all $(u, \mu) \in X \times R_+^m$, satisfying the following conditions:

$$0 \in \sum_{i=1}^{\alpha} \lambda_i \left(\partial f(u, y^i) g(u, y^i) - f(u, y^i) \partial g(u, y^i)\right) + \sum_{j=1}^{m} \mu_j \partial h_j(u), \text{ (WD)} \tag{66}$$

$$\mu_j h_j(u) \geq 0, \; j = 1, ..., m, \tag{67}$$

$$\lambda_i \geq 0, \; \sum_{i=1}^{\alpha} \lambda_i = 1, \; \overline{y} = \left(y^1, ..., y^\alpha\right), \; y^i \in Y\left(u\right), \; i = 1, ..., \alpha, \; \mu_j \geq 0, \; j = 1, ..., k. \tag{68}$$

If, for a triplet $(\alpha, \lambda, \overline{y}) \in Q$, the set $W_2\left(\alpha, \lambda, \overline{y}\right)$ is empty, then we define the supremum over it to be $-\infty$.

Let Ω_{WD} denote the set of all feasible solutions for problem (WD), and

$$S_{WD} := \{u \in X : (\alpha, \lambda, \overline{y}, u, \mu) \in \Omega_{WD}\}$$

and, for $u \in S_{WD}$,

$$J(u) := \{j \in J : h_j(u) \geq 0\}.$$

Theorem 8 *(Weak Duality). Let x and $(\alpha, \lambda, \overline{y}, u, \mu)$ be feasible solutions of the problems (P) and (WD), respectively. Assume, furthermore, that one of the following two sets of hypotheses is satisfied:*

(a) *(i)* *$f(\cdot, y^i)g(u, y^i) - f(u, y^i)g(\cdot, y^i)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at u on $D \cup S_{WD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;*

(ii) *$h_j\left(\cdot\right)$, $j \in J\left(u\right)$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at u on $D \cup S_{WD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$;*

(iii) $\sum_{i=1}^{\alpha} \lambda_i \rho_i + \sum_{j=1}^{m} \mu_j \rho_{h_j} \geq 0.$

(b) *$f(\cdot, y^i)g(u, y^i) - f(u, y^i)g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j\left(\cdot\right)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at u on $D \cup S_{WD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;*

Then

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq H(u). \tag{69}$$

Proof. **We prove theorem under hypothesis (a).**
We proceed by contradiction. Suppose, contrary to the result, that there exist $x \in D$ and $(\alpha, \lambda, \overline{y}, u, \mu) \in \Omega_{WD}$ such that

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} < H(u).$$

Hence, we have

$$\frac{f(x, \overline{y})}{g(x, \overline{y})} < \frac{f(u, \overline{y})}{g(u, \overline{y})}.$$

Thus,

$$f(x, y^i)g(u, y^i) - f(u, y^i)g(x, y^i) < 0, \; i = 1, .., \alpha. \tag{70}$$

By hypothesis (i), we have that $a < 0 \Longrightarrow \Psi_i(a) < 0$. By (70), this implies that

$$\Psi_i(f(x,y^i)g(u,y^i) - f(u,y^i)g(x,y^i) - \left[f(u,y^i)g(u,y^i) - f(u,y^i)g(u,y^i)\right]) < 0,\ i = 1,..,\alpha. \tag{71}$$

Again using hypothesis (i), we have that $b_i(x,u) > 0$, $i = 1,...,\alpha$. Thus, by Definition 1, inequalities (71) imply that the following inequalities

$$\Phi\left(x,u,(\xi_i,\rho_i)\right) < 0,\ i = 1,...,\alpha \tag{72}$$

hold for each $\xi_i \in \partial f(u,y^i)g(u,y^i) - f(u,y^i)\partial g(u,y^i)$, $i = 1,...,\alpha$. By the constraint (68), we have $\lambda_i \geq 0$, $\sum_{i=1}^{\alpha}\lambda_i = 1$. This implies, by inequalities (72), that the following inequalities

$$\sum_{i=1}^{\alpha}\lambda_i\Phi\left(x,u,(\xi_i,\rho_i)\right) < 0. \tag{73}$$

hold for each $\xi_i \in \partial f(u,y^i)g(u,y^i) - f(u,y^i)\partial g(u,y^i)$, $i = 1,...,\alpha$. By $x \in D$ and $(\alpha,\lambda,\overline{y},u,\mu,v) \in \Omega_{WD}$, we get that $h_j(x) \leq h_j(u)$, $j \in J(u)$. By hypothesis (ii), it follows that

$$\Psi_j^h\left(h_j(x) - h_j(u)\right) \leq 0,\ j \in J(u). \tag{74}$$

By Definition 1, we have that $b_{h_j}(x,u) > 0$, $j \in J(u)$. Therefore, inequalities (74) yield

$$b_{h_j}(x,u)\,\Psi_j^h\left(h_j(x) - h_j(u)\right) \leq 0,\ j \in J(u). \tag{75}$$

Using hypothesis (ii), by Definition 1, we conclude that the inequalities

$$\Phi\left(x,u,\left(\zeta_j,\rho_{h_j}\right)\right) \leq 0,\ j \in J(u)$$

hold for each $\zeta_j \in \partial h_j(u)$, $j \in J(u)$. Since $\mu_j \geq 0$, $j \in J$, the above inequalities yield

$$\sum_{j\in J(u)}\mu_j\Phi\left(x,u,\left(\zeta_j,\rho_{h_j}\right)\right) \leq 0. \tag{76}$$

By (73) and (76), it follows that

$$\sum_{i=1}^{\alpha}\lambda_i\Phi\left(x,u,(\xi_i,\rho_i)\right) + \sum_{j\in J(u)}\mu_j\Phi\left(x,u,\left(\zeta_j,\rho_{h_j}\right)\right) < 0. \tag{77}$$

The rest of this proof is similar to the proof of Theorem 4. ■

Theorem 9 *(Strong Duality). Let $\overline{x} \in D$ be an optimal point of the considered nonsmooth generalized fractional minimax problem (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then there exist $\left(\overline{\alpha},\overline{\lambda},\overline{y}\right) \in Q$ and $(\overline{x},\overline{\mu}) \in W_2\left(\overline{\alpha},\overline{\lambda},\overline{y}\right)$ such that $\left(\overline{\alpha},\overline{\lambda},\overline{y},\overline{x},\overline{\mu}\right)$ is optimal in (WD). If also the hypotheses of Theorem 8 are fulfilled, then the corresponding optimal values of (P) and (WD) are equal.*

Proof. By assumption, $\overline{x} \in D$ is an optimal point of (P) and the generalized Slater constraint qualification is satisfied at $\overline{x}$. Hence, by the nonparametric necessary optimality conditions (15)-(17), we conclude that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}\right)$ is feasible in the Weir dual problem (WD). Since

$$H\left(\overline{x}\right) = \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)},$$

then, using the weak duality theorem (Theorem 8), we conclude that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}\right)$ is optimal for (WD). Hence, the corresponding optimal values of (P) and (WD) are equal. ■

Theorem 10 *(Converse Duality). Let $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}\right)$ be an optimal point of (WD) such that $\overline{u} \in D$. Assume, furthermore, that one of the following two sets of hypotheses is satisfied:*

(a) (i) $f(\cdot, y^i)g(\overline{u}, y^i) - f(\overline{u}, y^i)g(\cdot, y^i)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;

(ii) $h_j(\cdot)$, $j \in J\left(\overline{u}\right)$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{u}$ on $D \cup S_{WD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$;

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j} \geq 0$.

(b) $f(\cdot, y^i)g(\overline{u}, y^i) - f(\overline{u}, y^i)g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;

Then $\overline{u}$ is optimal for the problem (P).

Proof. Proof follows directly from weak duality (Theorem 8). ■

Theorem 11 *(Strict Converse Duality). Let $\overline{x}$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}\right)$ be optimal solutions of the problems (P) and (WD), respectively, and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Assume, furthermore, that one of the two following sets of hypotheses is satisfied:*

(a) (i) $f(\cdot, y^i)g(\overline{u}, y^i) - f(\overline{u}, y^i)g(\cdot, y^i)$ is strictly $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$, Ψ_i is strictly increasing and $\Psi_i(0) = 0$;

(ii) $h_j(\cdot)$, $j \in J\left(\overline{u}\right)$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{u}$ on $D \cup S_{WD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$;

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j} \geq 0$.

(b) $f(\cdot, y^i)g(\overline{u}, y^i) - f(\overline{u}, y^i)g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is strictly $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$;

Then $\overline{x} = \overline{u}$ and the optimal values of (P) and (WD) are the same.

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{u}$. Since $\overline{x}$ is optimal for (P), there exist a positive integer $\overline{\alpha}^*$, scalars $\overline{\lambda}_i^* \geq 0$, $i = 1, ..., \overline{\alpha}^*$, scalars $\overline{\xi}_j^* \geq 0$, $j = 1, ..., k$, and vectors $\overline{y}^{*i}$, $i = 1, ..., \overline{\alpha}^*$, such that the nonparametric necessary optimality conditions are satisfied. This means that $\left(\overline{\alpha}^*, \overline{\lambda}^*, \overline{y}^*, \overline{x}, \overline{\xi}^*\right)$ is feasible in Weir dual problem (WD) and, moreover, by strong duality (Theorem 13), it is optimal for (WD). Now, proceeding as in the proof of weak duality theorem (Theorem 8) (replacing x by $\overline{x}$ and $(\alpha, \lambda, \overline{y}, u, \mu)$ by $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}\right)$, using strictly (b, Ψ, Φ, ρ)-univexity in place of (b, Ψ, Φ, ρ)-univexity), we obtain that the inequality

$$\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \Phi\left(x, u, (\xi_i, \rho_i)\right) + \sum_{j \in J(\overline{u})} \overline{\mu}_j \Phi\left(x, \overline{u}, \left(\zeta_j, \rho_{h_j}\right)\right) < 0.$$

holds for each $\xi_i \in \partial f(\overline{u}, y^i) g(\overline{u}, y^i) - f(\overline{u}, y^i) \partial g(\overline{u}, y^i)$, $i = 1, ..., \overline{\alpha}$, and for each $\zeta_j \in \partial h_j\left(\overline{u}\right)$, $j \in J\left(\overline{u}\right)$. The rest of this proof is similar to the proof of Theorem 8. ■

4.3. Bector duality

Now, following the lines of Bector (1989), we define for the considered nonsmooth generalized fractional minimax problem (P) its dual problem in the sense of Bector as follows:

$$\max_{(\alpha,\lambda,\overline{y}) \in Q} \quad \sup_{(u,\mu) \in W_3(\alpha,\lambda,\overline{y})} \frac{\sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right)}{\sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right)},$$

where $W_3\left(\alpha, \lambda, \overline{y}\right)$ is the set of all $(u, \mu) \in X \times R_+^m$, satisfying the following conditions:

$$0 \in G(u) \sum_{i=1}^{\alpha} \lambda_i \partial f\left(u, y^i\right) - F(u) \sum_{i=1}^{\alpha} \lambda_i \partial g\left(u, y^i\right) + \sum_{j=1}^{m} \mu_j \partial h_j(u), \text{ (BD)} \quad (78)$$

$$G(u) f(u, y^i) - F(u) g(u, y^i) \geq 0,\ i = 1, ..., \alpha, \quad (79)$$

$$\mu_j h_j(u) \geq 0,\ \ j = 1, ..., m, \quad (80)$$

$$\lambda_i \geq 0,\ \sum_{i=1}^{\alpha} \lambda_i = 1,\ \overline{y} = \left(y^1, ..., y^\alpha\right),\ y^i \in Y\left(u\right),\ i = 1, ..., \alpha,\ \mu_j \geq 0,\ j = 1, ..., m, \quad (81)$$

where, for the sake of convenience, we use the following denotations:

$$F(u) := \sum_{i=1}^{\alpha} \lambda_i f\left(u, y^i\right), \quad (82)$$

$$G(u) := \sum_{i=1}^{\alpha} \lambda_i g\left(u, y^i\right). \quad (83)$$

If, for a triplet $(\alpha, \lambda, \overline{y}) \in Q$, the set $W_3(\alpha, \lambda, \overline{y})$ is empty, then we define the supremum over it to be $-\infty$.

Let Ω_{BD} denote a set of all feasible solutions for the problem (BD). Further, $S_{BD} = \{u \in X : (\alpha, \lambda, \overline{y}, u, \mu) \in \Omega_{BD}\}$, and, for $u \in S_{BD}$, $J(u) = \{j \in J : h_j(u) \geq 0\}$.

THEOREM 12 *(Weak Duality). Let x and $(\alpha, \lambda, y, u, \mu)$ be feasible solutions for problems (P) and (BD), respectively. Assume, furthermore, that one of the following two sets of hypotheses is satisfied:*

a) (i) $G(u)f(\cdot, y^i) - F(u)g(\cdot, y^i)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at u on $D \cup S_{BD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$,

(ii) $h_j(\cdot)$, $j \in J(u)$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at u on $D \cup S_{BD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$,

(iii) $\sum_{i=1}^{\alpha} \lambda_i \rho_i + \sum_{j=1}^{m} \mu_j \rho_{h_j} \geq 0$,

b) $f(\cdot, y^i)G(u) - F(u)g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at u on $D \cup S_{WD}$, $a < 0 \Longrightarrow \Psi_i(a) < 0$ and $\sum_{i=1}^{\alpha} \lambda_i \rho_i \geq 0$.

Then

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{\sum_{i=1}^{\alpha} \lambda_i f(u, y^i)}{\sum_{i=1}^{\alpha} \lambda_i g(u, y^i)}.$$

PROOF. **Proof of theorem under hypothesis a).**
We proceed by contradiction. Suppose, contrary to the result, that

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} < \frac{\sum_{i=1}^{\alpha} \lambda_i f(u, y^i)}{\sum_{i=1}^{\alpha} \lambda_i g(u, y^i)}.$$

By (82) and (83), we have

$$\sup_{y \in Y} \frac{f(x, y)}{g(x, y)} < \frac{F(u)}{G(u)}.$$

Thus, the above inequality gives

$$\frac{f(x, \overline{y})}{g(x, \overline{y})} < \frac{F(u)}{G(u)}. \tag{84}$$

Hence, (84) can be rewritten as follows:

$$G(u)f(x, y^i) - F(u)g(x, y^i) < 0, \; i = 1, .., \alpha.$$

Then, the constraint (79) yields

$$G(u)f(x, y^i) - F(u)g(x, y^i) - \left[G(u)f(u, y^i) - F(u)g(u, y^i)\right] < 0, \; i = 1, .., \alpha. \tag{85}$$

By hypothesis (i), we have that $a < 0 \Longrightarrow \Psi_i(a) < 0$. Therefore, (85) gives

$$\Psi_i\left(G(u)f(x, y^i) - F(u)g(x, y^i) - \left[G(u)f(u, y^i) - F(u)g(u, y^i)\right]\right) < 0, \quad i = 1, .., \alpha. \tag{86}$$

As it follows from hypothesis (ii), $b_i(x,u) > 0$, $i = 1, ..., \alpha$. Thus, by Definition 1, inequalities (86) imply that the following inequalities

$$\Phi\left(x, u, (\xi_i, \rho_i)\right) < 0, \; i = 1, ..., \alpha \tag{87}$$

hold for each $\xi_i \in G(u)\partial f(u, y^i) - F(u)\partial g(u, y^i)$, $i = 1, ..., \alpha$. By the constraint (81), we have $\lambda_i \geq 0$, $\sum_{i=1}^{\alpha} \lambda_i = 1$. This implies, by inequalities (87), that the following inequalities

$$\sum_{i=1}^{\alpha} \lambda_i \Phi\left(x, u, (\xi_i, \rho_i)\right) < 0 \tag{88}$$

hold for each $\xi_i \in G(u)\partial f(u, y^i) - F(u)\partial g(u, y^i)$, $i = 1, ..., \alpha$. By $x \in D$ and $(\alpha, \lambda, \overline{y}, u, \mu, v) \in \Omega_{BD}$, it follows that $h_j(x) \leq h_j(u)$, $j \in J(u)$. By hypothesis (ii), we have that Ψ_j^h is increasing, and $\Psi_j^h(0) = 0$. Thus,

$$\Psi_j^h\left(h_j(x) - h_j(u)\right) \leq 0, \; j \in J(u). \tag{89}$$

By Definition 1, $b_{h_j}(x,u) > 0$, $j \in J(u)$. Hence, inequalities (74) yield

$$b_{h_j}(x,u)\, \Psi_j^h\left(h_j(x) - h_j(u)\right) \leq 0, \; j \in J(u). \tag{90}$$

Using hypothesis (ii), by Definition 1, we conclude that the inequalities

$$\Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0, \; j \in J(u)$$

hold for each $\zeta_j \in \partial h_j(u)$, $j \in J(u)$. Since $\mu_j \geq 0$, $j \in J$, the above inequalities imply

$$\sum_{j \in J(u)} \mu_j \Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) \leq 0. \tag{91}$$

By combining (88) and (91), we get

$$\sum_{i=1}^{\alpha} \lambda_i \Phi\left(x, u, (\xi_i, \rho_i)\right) + \sum_{j \in J(u)} \mu_j \Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) < 0. \tag{92}$$

Let us denote

$$\vartheta_i = \frac{\lambda_i}{\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j}, \; i = 1, ..., \alpha, \tag{93}$$

$$\delta_j = \frac{\mu_j}{\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j}, \; j \in J(u). \tag{94}$$

By (93) and (94), we have that $0 \leq \vartheta_i \leq 1$, $i = 1, ..., \alpha$, but for at least one $i \in \{1, ..., \alpha\}$, $\vartheta_i > 0$, $0 \leq \delta_j \leq 1$, $j \in J(u)$, and, moreover,

$$\sum_{i=1}^{\alpha} \vartheta_i + \sum_{j \in J(u)} \delta_j = 1. \tag{95}$$

Taking into account (93) and (94) in (92), we obtain

$$\sum_{i=1}^{\alpha} \vartheta_i \Phi\left(x, u, (\xi_i, \rho_i)\right) + \sum_{j \in J(u)} \delta_j \Phi\left(x, u, \left(\zeta_j, \rho_{h_j}\right)\right) < 0. \tag{96}$$

By Definition 1, $\Phi(x, u; (\cdot, \cdot))$ is convex. Since (95) holds, therefore, by the definition of convexity, (96) gives

$$\Phi\left(x, u, \sum_{i=1}^{\alpha} \vartheta_i \left(\xi_i, \rho_i\right) + \sum_{j \in J(u)} \delta_j \left(\zeta_j, \rho_{h_j}\right)\right) < 0.$$

Taking into account the Lagrange multipliers as being equal to 0 and again using (93) and (94), we get that the inequality

$$\Phi\left(x, u, \frac{1}{\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j} \left(\sum_{i=1}^{\alpha} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j, \right.\right.$$
$$\left.\left. \sum_{i=1}^{\alpha} \lambda_i \rho_i + \sum_{j \in J(u)} \mu_j \rho_{h_j}\right)\right) < 0 \tag{97}$$

holds for each $\xi_i \in G(u)\partial f(u, y^i) - F(u)\partial g(u, y^i)$, $i = 1, ..., \alpha$, and for each $\zeta_j \in \partial h_j\left(u\right)$, $j \in J$. Then, by the constraint (78), inequality (97) implies

$$\Phi\left(x, u, \frac{1}{\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j} \left(0\ ,\ \sum_{i=1}^{\alpha} \lambda_i \rho_i + \sum_{j \in J(u)} \mu_j \rho_{h_j}\right)\right) < 0. \tag{98}$$

By Definition 1, $\Phi(x, u, (0, a)) \geq 0$ for all $a \in R_+$. Then, by hypothesis (iii), the following inequality

$$\Phi\left(x, u, \frac{1}{\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j} \left(0\ ,\ \sum_{i=1}^{\alpha} \lambda_i \rho_i + \sum_{j \in J(u)} \mu_j \rho_{h_j}\right)\right) \geq 0$$

holds, contradicting (98). This completes the proof of this theorem under hypotheses (a).

Proof of this theorem under hypothesis b) is similar to the one under hypothesis a) and, therefore, it has been omitted in the paper. ■

THEOREM 13 *(Strong Duality). Let $\overline{x} \in D$ be an optimal solution of the considered nonsmooth generalized fractional minimax problem (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then, there exist $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right) \in Q$ and $\left(\overline{x}, \overline{\mu}\right) \in W_3\left(\overline{\alpha}, \overline{\lambda}, \overline{y}\right)$ such that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}\right)$ is an optimal solution of (BD). If also the hypotheses of Theorem 12 hold, then the corresponding optimal values of (P) and (BD) are equal.*

Proof. By assumption, $\overline{x} \in D$ is an optimal solution of (P) and the generalized Slater constraint qualification is satisfied at $\overline{x}$. Then, by the nonparametric necessary optimality conditions (15)-(17), we conclude that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}\right)$ is feasible for (BD). Since

$$\frac{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i f\left(\overline{x}, \overline{y}^i\right)}{\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i g\left(\overline{x}, \overline{y}^i\right)} = \sup_{y \in Y} \frac{f(\overline{x}, y)}{g(\overline{x}, y)},$$

therefore, using weak duality (Theorem 12), we get that $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{x}, \overline{\mu}\right)$ is optimal for (BD). Hence, the corresponding optimal values of (P) and (BD) are the same. ∎

Theorem 14 *(Converse Duality). Let $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}\right)$ be an optimal solution of (BD) such that $\overline{u} \in D$. Further, assume that one of the following two sets of hypotheses is satisfied:*

(a) (i) $G(\overline{u})f(\cdot, y^i) - F(\overline{u})g(\cdot, y^i)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{BD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$,

(ii) $h_j(\cdot)$, $j \in J(\overline{u})$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{u}$ on $D \cup S_{BD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$,

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j} \geq 0$,

(b) $f(\cdot, y^i)G(\overline{u}) - F(\overline{u})g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$, $a < 0 \Longrightarrow \Psi_i(a) < 0$ and $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i \geq 0$.

Then $\overline{u}$ is optimal for the problem (P).

Proof. Proof of this theorem follows directly from weak duality (see Theorem 12). ∎

Theorem 15 *(Strict Converse Duality). Let $\overline{x}$ and $\left(\overline{\alpha}, \overline{\lambda}, \overline{y}, \overline{u}, \overline{\mu}\right)$ be optimal solutions of (P) and (BD), respectively, and the generalized Slater constraint qualification be satisfied. Assume, furthermore, that one of the following two sets of hypotheses is satisfied:*

(a) (i) $G(\overline{u})f(\cdot, y^i) - F(\overline{u})g(\cdot, y^i)$ is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{BD}$ and $a < 0 \Longrightarrow \Psi_i(a) < 0$,

(ii) $h_j(\cdot)$, $j \in J(\overline{u})$, is $(b_{h_j}, \Psi_j^h, \Phi, \rho_{h_j})$-univex at $\overline{u}$ on $D \cup S_{BD}$, Ψ_j^h is increasing and $\Psi_j^h(0) = 0$,

(iii) $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i + \sum_{j=1}^{m} \overline{\mu}_j \rho_{h_j} \geq 0$,

(b) $f(\cdot, y^i)G(\overline{u}) - F(\overline{u})g(\cdot, y^i) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\cdot)$, $i = 1, ..., \overline{\alpha}$, is $(b_i, \Psi_i, \Phi, \rho_i)$-univex at $\overline{u}$ on $D \cup S_{WD}$, $a < 0 \Longrightarrow \Psi_i(a) < 0$ and $\sum_{i=1}^{\overline{\alpha}} \overline{\lambda}_i \rho_i \geq 0$.

Then $\overline{x} = \overline{u}$, that is, $\overline{u}$ is optimal for the problem (P).

Proof. Proofs of this theorem under hypotheses a) and b) are similar to the proofs of Theorem 12 and, therefore, they have been omitted in the paper. ∎

5. Conclusion

In this paper, the optimality conditions and several parametric and non-parametric duality results have been established for a new class of nonconvex nondifferentiable generalized fractional minimax programming problems. This paper extends highly significantly the earlier works, in which optimality conditions and duality results have been obtained for a generalized fractional minimax problem by applying a convexity assumption or under several generalized convexity notions, previously defined in the literature (see Remark 2). The results established here can be used also in the case of such a nonconvex nondifferentiable generalized fractional minimax programming problem, in which not all functions, constituting it, possess a fundamental property of convexity and most classes of generalized convex functions - namely, such that a stationary point of such a function is also its global minimizer. Evidently, all the optimality conditions and duality results, established in this paper for the considered generalized fractional minimax programming problem, extend earlier results for this type of optimization problems, for example, the results of Chandra and Kumar (1995), Zalmai (1995) and Antczak (2008). Furthermore, in the case when Y is a singleton, the considered generalized fractional minimax programming problems (P) becomes the standard fractional problem and duals reduce to the well known duals of Schaible (1976), Chandra and Bector (1989), and Weir (1986), respectively.

References

Ahmad, I. (2003) Optimality conditions and duality in fractional minimax programming involving generalized ρ-invexity. *International Journal of Management and System* **19**, 165-180.

Ahmad, I. and Husain, Z. (2006) Optimality conditions and duality in nondifferentiable minimax fractional programming with generalized convexity. *Journal of Optimization Theory and Applications* **129**, 255-275.

Ahmed, A. (2004) Sufficiency in nondifferentiable fractional minimax programming. *The Aligarh Bulletin of Mathematics* **23**, 117-122.

Antczak, T. (2008) Generalized fractional minimax programming with B-(p,r)-invexity. *Computers and Mathematics with Applications* **56**, 1505-1525.

Antczak, T. (2014) On nonsmooth (Φ,ρ)-invex multiobjective programming in finite-dimensional Euclidean spaces. *Journal of Advances Mathematical Studies* **7**, 127-145.

Antczak, T. and Stasiak, A. (2011) (Φ,ρ)-invexity in nonsmooth optimization. *Numerical Functional Analysis and Optimization* **32**, 1-25.

Bector, C.R., Chandra, S. and Bector, M.K. (1989) Generalized fractional programming duality: a parametric approach. *Journal of Optimization Theory and Applications* **60**, 243-260.

Bector, C.R., Chandra, S., Gupta, S. and Suneja, S.K. (1994) Univex sets, functions and univex nonlinear programming. In: Komolosi, S., Rapcsak, T. and Schaible, S. *Generalized Convexity. Lecture Notes in Economics and Mathematical Systems*, **405**, Springer Verlag, Berlin.

Bhatia, D. and Jain, P. (1994), Generalized (F, ρ)-convexity and duality for non smooth multi-objective programs. *Optimization* **31**, 153-164.

Chandra, S., Craven, B.D. and Mond, B. (1986) Generalized fractional programming duality: a ratio game approach. *Journal of Australian Mathematical Society* Ser. B **28**, 170-180.

Chandra, S. and Kumar, V. (1995) Duality in fractional minimax programming. *Journal of the Australian Mathematical Society* 58, 376–386.

Clarke, F.H. (1983) *Optimization and Nonsmooth Analysis.* A Wiley-Interscience Publication, John Wiley&Sons, Inc.

Craven, B.D. (2010) Kinds of vector invex. *Taiwanese Journal of Mathematics* **14**, 1925-1933.

Hanson, M.A. and Mond, B. (1982) Further generalization of convexity in mathematical programming. *Journal of Information and Optimization Science* **3**, 25-32.

Ho, S.C. and Lai, H.C. (2012) Optimality and duality for nonsmooth minimax fractional programming problem with exponential (p, r)-invexity. *Journal of Nonlinear and Convex Analysis* **13**, 433-447.

Ho, S.C. and Lai, H.C. (2014) Mixed type duality for nonsmooth minimax fractional programming involving exponential $(p; r)$-invexity. *Numerical Functional Analysis and Optimization* **35**, 1560-1578.

Jayswal, A., Kumar, R. and Kumar, D. (2013) Minimax fractional programming problem involving nonsmooth generalized univex functions. *International Journal of Control Theory and Applications* **3**, 7-22.

Jeyakumar, V. (1988) Equivalence of saddle-points and optima, and duality for a class of nonsmooth non-convex problems. *Journal of Mathematical Analysis and Applications* **130**, 334-343.

Jeyakumar, V. and Mond, B. (1992) On generalized convex mathematical programming. *Journal of Australian Mathematical Society* Ser.B **34**, 43-53.

Kim, D.S. and Schaible, S. (2004) Optimality and duality for invex nonsmooth multiobjective programming problems. *Optimization* **53**, 165-176.

Lee, G.M. (1994) Nonsmooth invexity in multiobjective programming. *Journal of Information and Optimization Sciences* **15**, 127-136.

Li, X.F., Dong, J.L. and Liu, Q.H. (1997) Lipschitz B-vex functions and nonsmooth programming. *Journal of Optimization Theory and Applications* **93**, 557-574.

Liang, Z.A. and Shi, Z.W. (2003) Optimality conditions and duality for a minimax fractional programming with generalized convexity. *Journal of Mathematical Analysis and Applications* **277**, 474-488.

Liu, J.C. (1996) Optimality and duality for generalized fractional programming involving nonsmooth (F, ρ)-convex functions. *Computers and Mathematics with Applications* **32**, 91-102.

LIU, J.C. (1996) Optimality and duality for generalized fractional programming involving nonsmooth pseudoinvex functions. *Journal of Mathematical Analysis and Applications* **202**, 667-685.

LIU, J.C. and WU, C.S. (1998a) On minimax fractional optimality conditions with (F, ρ)-convex. *Journal of Mathematical Analysis and Applications* **219**, 36-51.

LIU, J.C. and WU, C.S. (1998b) On minimax fractional optimality conditions with invexity. *Journal of Mathematical Analysis and Applications* **219**, 21-35.

LIU, J.C. WU, C.S. and R.L. SHEU (1997) Duality for fractional mimimax programming. *Optimization* **41**, 117-133.

MISHRA, S.K. (1997) Generalized fractional programming problems containing locally subdifferentiable ρ-univex functions. *Optimization* **41**, 135-158.

MISHRA, S.K., WANG, S.Y., LAI, K.K. and SHI, J.M. (2003) Nondifferentiable minimax fractional programming under generalized univexity. *Journal of Computational and Applied Mathematics* **158**, 379-395.

MISHRA, S.K., WANG, S.Y. AND LAI, K.K. (2008) On nonsmooth α-invex functions and vector variational-like inequality. *Optimization Letters* **2**, 91–98.

MISHRA, S.K. and UPADHYAY, B.B. (2014) Nonsmooth minimax fractional involving η-pseudolinear functions. *Optimization* **63**, 775–788.

MOND, B. and WEIR, T. (1981) Generalized concavity and duality. in: Schaible, S. and Ziemba, W.T. (eds.) *Generalized Concavity in Optimization and Economics.* Academic Press, New York.

MUKHERJEE, R.N. and PURNACHANDRA RAO, CH. (1996) Generalized F-convexity and its classification. *Indian Journal of Pure and Applications Mathematics* **27**, 1175-1183.

REILAND, T.W. (1990) Nonsmooth invexity. *Bulletin of the Australian Mathematical Society* **42**, 437-446.

SCHAIBLE, S. (1976) Fractional programming I, Duality. *Management Science* **22**, 858-867.

STANCU-MINASIAN, I.M. (1999) A fifth bibliography of fractional programming. *Optimization* **45**, 343-367.

SUNEJA, S.K. and LALITHA, C.S. (1993) Multiobjective fractional programming involving ρ-invex and related functions. *Opsearch* **30**, 1-14.

UPADHYAY, B.B. AND MISHRA, S.K. (2015) Nonsmooth semi-infinite minimax programming involving generalized (ϕ, ρ)-invexity. *Journal of Systems Science and Complexity* **28**, 857–875.

WEIR, T. (1986) A dual for multiobjective fractional programming. *Journal of Information and Optimization Science* **7**, 261-269.

WEIR, T. (1986) A duality theorem for a multiobjective fractional optimization problem. *Bulletin of Australian Mathematical Society* **34**, 376-386.

VIAL, J.P. (1983) Strong and weak convexity of sets and functions. *Mathematics of Operation Research* 8, 231-259.

YANG, X.M. and HOU, S.H (2005) On minimax fractional optimality and duality with generalized convexity. *Journal of Global Optimization* **31**, 235-252.

Zalmai, G.J. (1995) Optimality conditions and duality models for generalized fractional programming problems containing locally subdifferentiable and ρ-convex functions. *Optimization* **32**, 95-124.

Zalmai, G.J. (2007) Parametric duality models for discrete minmax fractional programming problems containing generalized (θ, η, ρ)-V- invex functions and arbitrary norms. *Journal of Applied Mathematics & Computing* **24**, 105-126.