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On optimal boundary and distributed control of partial integro–differential equations

ASATUR ZH. KHURSHUDYAN

To memoriam of my father Zhora M. Khurshudyan is dedicated.

A method of optimal control problems investigation for linear partial integro–differential equations of convolution type is proposed, when control process is carried out by boundary functions and right hand side of equation. Using Fourier real generalized integral transform control problem solution is reduced to minimization procedure of chosen optimality criterion under constraints of equality type on desired control function. Optimality of control impacts is obtained for two criteria, evaluating their linear momentum and total energy. Necessary and sufficient conditions of control problem solvability are obtained for both criteria. Numerical calculations are done and control functions are plotted for both cases of control process realization.

Key words: optimal control, integro–differential equation, equation of convolution type, Fourier generalized transform, impulsive control

1. Introduction

It is well known fact that the most part of rigorous techniques of optimal control systems investigation are non–applicable for solution of control problems for some special types of system uncertainties, and application of existing efficient numerical methods in those situations is quite onerous. For example, when the system of optimal control is nonlinear with respect to control vector (and even linear with respect to phase vector), the problem can be solved explicitly only in some exceptional cases, and problems of control by coefficients or by moving loads are rigorously irresolvable [1, 2]. There are many other examples of control problems, when system‘s investigation is associated with additional difficulties and costs. Quite a common example of control system with uncertainty is a system containing convolution of unknown function, which occurs in various areas of mathematical physics and includes integral (Fredholm equations of the first and second kinds), as well as integro–differential equations (in ordinary and partial derivatives).

A.Z. Khurshudyan is with Faculty of Mathematics and Mechanics, Yerevan State University, Alex Manoogian Str. 1, 0025 Yerevan, Armenia, email: khurshudyan@ysu.am

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Many real processes are very often roughly modeled by ordinary or partial differential equations. However, local character of differential equations does not allow us to take into account such real–world phenomena, as, for instance, processes with memory (prehistory). Introduction of integral terms containing unknown function into differential equation, thereby transforming it into integro–differential, is of help in this case. As a common example of such equations in finite interval, the following one may serve (in one dimensional case):

$$\mathcal{H}[w]+\int_a^b \mathcal{K}(x,s,t)\mathcal{N}[w]ds=\mathcal{F}(x,t), \tag{1}$$

where $w=w(x,t)$ is the unknown function, $\mathcal{K}:[a,b]\times[a,b]\times\mathrm{T}\to\mathbb{R}$ is a given function, called kernel of equation, T is finite or infinite interval, $\mathcal{H}[\cdot]$ and $\mathcal{N}[\cdot]$ are differential operators, acting on unknown function, and $\mathcal{F}:[a,b]\times\mathrm{T}\to\mathbb{R}$ is the given right hand side, satisfying certain conditions. If operators $\mathcal{H}[\cdot]$ and $\mathcal{N}[\cdot]$ are linear, contains ordinary or partial derivatives, then equation (1) is called linear, ordinary or partial, respectively. Furthermore, equation (1) is called symmetric, if its kernel is symmetric: $\mathcal{K}(x,s,\cdot)=\mathcal{K}(s,x,\cdot)$, and difference or convolution type, if $\mathcal{K}(x,s,\cdot)=\mathcal{K}(x-s,\cdot)$.

The monograph [3] is devoted to thorough investigation and classification of integro–differential operators. Main areas of integro–differential equations application are described, and numerous problems mathematically formulated by those equations, particularly in theories of elasticity and viscoelasticity, continuum mechanics, contact interactions mechanics, growing body mechanics and fracture mechanics are considered ibid. In [4] the description of some financial processes by partial integro–differential equations are investigated. In [5] the explicit solution of linear difference partial integro–differential equations in quite general form is obtained via Laplace transform. Some examples of integro–differential equations occurring in applications are solved and investigated. Numerical method of nonlinear singular partial integro–differential equations solution is constructed in [6].

Accounting of such irreversible processes of wave energy transfer to medium particle as dispersion and dissipation, is accompanied with introduction in ordinary wave equation of some additional linear terms containing the unknown function, the form of which depends only on physical mechanism of interaction between wave and medium. In general, the description of wave propagation in a homogeneous isotropic medium with dissipation or dispersion properties occupied domain $\Omega\subset\mathbb{R}^3$ in 3–dimensional formulation is mathematically equivalent to solution of the following wave equation:

$$\Delta w=\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}+\mathcal{L}[w],\ \ (x,y,z)\in\Omega,\ t>0, \tag{2}$$

where $w=w(x,y,z,t)$ is the unknown function, Δ is 3–dimensional Laplace operator, and $\mathcal{L}[w]$ is some linear operator, acting on function $w(x,y,z,t)$. Particularly, in sound wave propagation problem in a dissipative medium [7]

$$\mathcal{L}[w] \equiv -\frac{b}{c_0^2\rho_0}\frac{\partial^2 \Delta w}{\partial t^2},$$

where b is the medium dissipation factor. In propagation problem of electromagnetic wave in a dispersive medium [7]

$$\mathcal{L}[w] \equiv \frac{4\pi}{c_1^2}\frac{\partial^2}{\partial t^2}\int_0^\infty \kappa(\tau)w(x,y,z,t-\tau)d\tau, \tag{3}$$

where $\kappa(t),\ t>0$, is a dimensionless bounded function, characterizing dielectric susceptibility of medium. Typical examples of electromagnetic waves include radio waves, TV signals, radar beams, and light rays.

It turns out, that taking into account such simple (in the sense of linearity), but natural phenomena considerably complicates the investigation, and numerical calculations makes more time consuming. Investigation of control problems for such systems is also extremely complicated.

The main purpose of this investigation is development of optimal control problem solution technique namely for such systems, when control process is carried out by a boundary function and right hand side of equation. Simplicity of numerical application of technique is also one of our aims. Taking into consideration the uncertainty type of those equations, one may easily see, that control algorithm to be developed should be based on some integral transform. Optimal control problems for integro–differential equations are considered also in [8, 9, 10, 11].

2. Mathematical statement of the problem

So, we begin our investigation with non–homogeneous, one–dimensional analogue of equation (2) in finite interval (for simplicity, symmetric: it can always be done by linear transformation of variable t) when (3) is taken into account:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} + \alpha\frac{\partial^2}{\partial t^2}\int_{-\theta}^{\theta} \kappa(\tau)w(x,t-\tau)d\tau + U(x,t), \tag{4}$$

$$(x,t) \in (0,1)\times(-\theta,\theta),$$

where $\alpha \in \mathbb{R}^+$ is a dimensionless constant, subjected to boundary conditions

$$w(0,t) = \varphi(t),\ w(l,t) = \psi(t),\ t \in (-\theta,\theta), \tag{5}$$

(all quantities and variables are dimensionless).

From mathematics point of view, control process of system (4)–(5) may be carried out either by one of boundary functions $\varphi(t)$ and $\psi(t)$, or by function $U(x,t)$ from right hand side of equation (4). The case when control process is realized by function $\kappa(t)$ is considerably difficult. Let us suppose for definiteness, that [12, 13] $U(x,t) = p(x)u(t)$, where $p(x)$ is a given nonzero function, and control process may be realized by function $u(t)$.

The following initial data at $t = -\theta$ and terminal data at $t = \theta$ are considered:

$$w(x,-\theta) = w_{-\theta}(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=-\theta} = \dot{w}_{-\theta}(x),\ x \in (0,1), \tag{6}$$

$$w(x,\theta) = w_{\theta}(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=\theta} = \dot{w}_{\theta}(x),\ x \in (0,1). \tag{7}$$

In the framework of accepted physical interpretation system (4)–(5), in particular, describes the transfer of one–dimensional electromagnetic signal (impulse) on a finite distance in dispersive medium, at that $\varphi(t)$ and $\psi(t)$ are input and output signals, respectively. Then, unlike traditional statement of optimal control problems for distributed parameter systems [1], demanding ensuring of given terminal data for given initial ones, we may demand to ensure one boundary condition by appropriate choice of the other one when functions (6), (7) are given.

The main purpose of this investigation is to construct an efficient for numerical reasons algorithm of the following two problems solution.

The first problem or *the problem of boundary control* requires determination of a function $\varphi^o(t)$ optimal in the sense of given optimality criterion among all admissible controls $\varphi \in \mathcal{D}$ ensuring given output signal $\psi(t)$ when data (6) and (7) are given, and find necessary and sufficient conditions of its existence.

The second problem or *the problem of distributed control* requires determination of a function $u^o(t)$ optimal in the sense of given optimality criterion among all admissible controls $u \in \mathcal{D}$ ensuring given output signal $\psi(t)$ when input signal $\varphi(t)$ and data (6) and (7) as well, are given, as well as find necessary and sufficient conditions of its existence.

Hereafter, we shall call a real–valued function *admissible control*, if it satisfies the system (4)–(6) solution existence and uniqueness conditions in considered function space. Further, we shall call a controlled system *fully controllable* in a certain space, if there exists an admissible control function, resolving *the first* or *second* problems for that system in that space [12]– [14].

Further, in order to write equation (4) for all real t, let us introduce operator $\mathcal{A}_\theta[\cdot]$ defined on whole real axis and acting as

$$\mathcal{A}_\theta[f(\cdot,t)] = \begin{cases} f(\cdot,t), & t \in [-\theta,\theta]; \\ 0, & t \notin [-\theta,\theta]. \end{cases}$$

The explicit form of operator $\mathcal{A}_\theta[\cdot]$ can be constructed in several manners. For example, it can be done using characteristic function (indicator) of segment $[-\theta,\theta]$:

$$\chi_{[-\theta,\theta]}(t) = \begin{cases} 1, \ t \in [-\theta,\theta]; \\ 0, \ t \notin [-\theta,\theta]. \end{cases}$$

Indeed, it is obvious, that $\mathcal{A}_\theta[f] = \chi_{[-\theta,\theta]}(t) f(\cdot,t)$. Expressing characteristic function $\chi_{[-\theta,\theta]}(t)$ in terms of Heaviside unit–step function, defined as

$$H(t) = \begin{cases} 1, \ t > 0; \\ 0, \ t < 0, \end{cases}$$

we can write operator $\mathcal{A}_\theta[\cdot]$ as follows:

$$\mathcal{A}_\theta[f(\cdot,t)] = [H(t+\theta) - H(t-\theta)] f(\cdot,t) \equiv f_1(\cdot,t), \ t \in \mathbb{R}.$$

Then using this expression, one can write equation (2.1) in space of generalized functions for all real t, and include initial and terminal data (2.3), (2.4) in it:

$$\frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w_1}{\partial t^2} + \alpha \frac{\partial^2}{\partial t^2}[\kappa_1 * w_1] + U_1(x,t) + W(x,t), \ \ (x,t) \in [0,1] \times \mathbb{R}, \tag{8}$$

$$W(x,t) = W_-(x,t) - W_+(x,t),$$

$$W_\pm(x,t) = [\delta(t \pm \theta) + \alpha\kappa_1(t \pm \theta)] \dot{w}_{\pm\theta}(x) + \frac{d}{dt}[\delta(t \pm \theta) - \alpha\kappa_1(t \pm \theta)] w_{\pm\theta}(x),$$

where $\delta(t)$ is the Dirac delta function, defined as

$$\delta(t) = H'(t) = \begin{cases} 0, \quad t \neq 0; \\ \infty, \ t = 0, \end{cases}$$

at that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \ \ \delta(-t) = \delta(t),$$

with derivative

$$\frac{d\delta(t)}{dt} = \delta'(t),$$

which is taken in generalized sense [15]. It should be noted, that function $W(x,t)$ containing delay and advance of argument t is identically zero when $t \pm \theta \notin [-\theta,\theta]$, at that

$$W(x,t) = \begin{cases} -W_+(x,t), \ \ t \in [-\theta,0); \\ W_0(x), \ \ t = 0; \\ W_-(x,t), \ \ t \in (0,\theta], \end{cases} \qquad x \in [0,1],$$

where

$$W_0(x) = \alpha\left[\kappa_1(-\theta)\dot{w}_{-\theta}(x) - \kappa_1'(-\theta)w_{-\theta}(x) - \kappa_1(\theta)\dot{w}_\theta(x) + \kappa_1'(\theta)w_\theta(x)\right].$$

When obtaining right hand side of equation (8) the following relation was used [15]:

$$f(t)\delta'(t-\theta) = f(\theta)\delta'(t-\theta) - f'(\theta)\delta(t-\theta).$$

Symbol $*$ in right hand side of equation (8) denotes operator of convolution acting as follows:

$$\kappa_1 * w_1 = \int_{-\infty}^{\infty} \kappa_1(\tau)w_1(x,t-\tau)d\tau = \int_{-\infty}^{\infty} \kappa_1(t-\tau)w_1(x,\tau)d\tau.$$

In transformed form equation (8) often occurs also in optics, mechanics and theory of probability [17, 16, 7].

In the same way from boundary conditions (5) we will obtain

$$w_1(0,t) = \varphi_1(t),\ w_1(1,t) = \psi_1(t),\ t \in \mathbb{R}. \tag{9}$$

3. Boundary control

It is obvious that functions $w_1(x,t)$, $\varphi_1(t)$, $\psi_1(t)$ and $U_1(x,t)$ are defined for all $t \in \mathbb{R}$ and identically zero outside interval $[-\theta,\theta]$, i.e. are compactly supported in that interval, where they coincide with functions $w(x,t)$, $\varphi(t)$, $\psi(t)$ and $U(x,t)$.

Applying now Fourier real generalized integral transform with respect to variable t in the sense of [15] to equation (8) and conditions (9), using the formulae of convolution transform, after some algebraic transformations we will obtain:

$$\frac{d^2\overline{w}_1(x,\sigma)}{dx^2} + \sigma^2[1+\alpha\overline{\kappa}_1(\sigma)]\overline{w}_1(x,\sigma) = \overline{U}_1(x,\sigma) + \overline{W}(x,\sigma), \tag{10}$$

$$(x,\sigma) \in (0,1)\times\mathbb{R},$$

$$\overline{w}_1(0,\sigma) = \overline{\varphi}_1(\sigma),\ \overline{w}_1(1,\sigma) = \overline{\psi}_1(\sigma),$$

where

$$\mathcal{F}_t[f(\cdot,t)] \equiv \overline{f}(\cdot,\sigma) = \int_{-\infty}^{\infty} f(\cdot,t)e^{i\sigma t}dt,\ \ \sigma \in \mathbb{R},$$

is the Fourier real generalized transform, at that

$$\mathcal{F}_t[g_1(\cdot,t)] = \mathcal{F}_t\{\mathcal{A}_\theta[g(\cdot,t)]\} = \int_{-\theta}^{\theta} g(\cdot,t)e^{i\sigma t}dt,\ \ \sigma \in \mathbb{R},$$

$\mathcal{F}_t[\cdot]$ is the Fourier operator, and σ is the spectral parameter of Fourier transform.

Solution of *the first problem* gives the following

Theorem 1 *Function* $\varphi_1(t) \in \mathcal{D}$ *is the Fourier inverse transform of function* $\overline{\varphi}_1(z)$, $z = \sigma + i\varsigma$, *determining from countable system*

$$\overline{\varphi}_1(z_k) = [\overline{\psi}_1(z_k) - \Phi(z_k)]\, e^{\chi(z_k)}, \quad k = 1,2,3,..., \tag{11}$$

where complex numbers z_k *are determined from characteristic equation*

$$e^{2\chi(z)} = 1. \tag{12}$$

Here

$$\chi(z) = i\sqrt{z^2 + \alpha z^2 \overline{\kappa}_1(z)},$$

$$\Phi(z) = \frac{1}{\chi(z)} \int_0^1 \left[\overline{U}_1(\xi, z) + \overline{W}(\xi, z)\right] \sinh[(1-\xi)\chi(z)]d\xi.$$

From Theorem 1 immediately follows

Corollary 1 *If kernel* $\kappa_1(t)$ *of equation* (8) *satisfies conditions*

a) $\mathcal{F}_t[\kappa_1(t)]$ *is a real–valued function,*

b) $1 + \alpha\overline{\kappa}_1(\sigma) > 0$, $\sigma \in \mathbb{R}$,

then function $\varphi_1(t) \in \mathcal{D}$ *is the Fourier inverse transform of function* $\overline{\varphi}_1(z)$, $z = \sigma + i\varsigma$, *determining from countable system*

$$\overline{\varphi}_1(z_k) = (-1)^k\, [\overline{\psi}_1(z_k) - \Phi(z_k)], \quad k = 1,2,3,..., \tag{13}$$

$$\sin(\chi(z)) = 0. \tag{14}$$

Here

$$\Phi(z) = \frac{1}{\chi(z)} \int_0^1 \left[\overline{U}_1(\xi, z) + \overline{W}(\xi, z)\right] \sin[(1-\xi)\chi(z)]d\xi,$$

$$\chi(z) = \sqrt{z^2 + \alpha z^2 \overline{\kappa}_1(z)}.$$

Remark 1 Separating real and imaginary parts of expression $\chi(z) = \chi_1(\sigma,\varsigma) + i\chi_2(\sigma,\varsigma)$, for desired roots σ_k, ς_k determination from (14) we will obtain system $\chi_1(\sigma_k,\varsigma_k) = \pi k$, $\chi_2(\sigma_k,\varsigma_k) = 0$, $k = 1,2,3,...$, where $\sigma_k + i\varsigma_k = z_k$.

It is well known [15], that condition a) of Corollary 1 holds, particularly, when $\kappa_1(t)$, $t \in \mathbb{R}$, is an even function: $\kappa_1(-t) = \kappa_1(t)$. Let us add that a condition similar to condition b) of Corollary 1 is obtained also in [17] when introducing the solvability of Riemann problem in theory of analytical functions. Well posedness of system (8), (9), when $\kappa_1 \in \mathrm{L}^1[-\theta,\theta]$ function is even and U_1, $W \in \mathrm{L}^1[-\theta,\theta]$ is proved in [18].

Note, that equalities (11) can be considered as interpolation conditions in nodes $z_k,\ k=1,2,3,...$, for $\overline{\varphi}_1(z),\ z\in\mathbb{C}$, function determination. Solution of that interpolation problem can be attacked by different efficient methods of interpolation, application of Fourier generalized inverse transform when minimum of chosen optimality criterion will be achieved, will derive us to solution of optimal control problem under study. However, here we will proceed in a different way [1, 12, 13]. Taking into account that function $\varphi_1(t)$ is compactly supported, one may separate real and imaginary parts of equalities (11) and as a result get the following countable system of equalities:

$$\int_{-\theta}^{\theta}\varphi(t)e^{-\varsigma_k t}\cos(\sigma_k t)dt=M_{1k},\ \int_{-\theta}^{\theta}\varphi(t)e^{-\varsigma_k t}\sin(\sigma_k t)dt=M_{2k}, \tag{15}$$

$$k=1,2,3,...,$$

where

$$M_{1k}+iM_{2k}\equiv M_k=\left[\overline{\psi}_1(\sigma_k+i\varsigma_k)-\Phi(\sigma_k+i\varsigma_k)\right]e^{\chi(\sigma_k+i\varsigma_k)}.$$

Remark 2 As the characteristic equation (12) is symmetric with respect to roots $z_k,\ k=1,2,3,...$: together with z_k for all k, $-z_k$ also satisfies that equation, then taking into account properties of Fourier integrals [15] one can prove, that $\overline{\varphi}_1(-z_k)=\overline{\overline{\varphi}_1(z_k)}$ and $M(-z_k)=\overline{M(z_k)}$, where the line over expression means its complex adjoint, therefore consideration of system (15) may be limited only for roots $z_k=\sigma_k+i\varsigma_k,\ k=1,2,3,...$

Thus, solution of *the first problem* can be reduced to minimization procedure of chosen optimality criterion under integral constraints (15) on unknown function $\varphi(t)$.

4. Distributed control

Let us proceed now to solution of *the second problem*. Instead of resolving system (11) in this case we will obtain

$$\overline{u}_1(z_k)=\frac{\left[\overline{\psi}_1(z_k)-\overline{\varphi}_1(z_k)e^{\chi(z_k)}\right]\chi(z_k)-\Gamma(z_k)}{\Pi(z_k)},\ \ k=1,2,3,... \tag{16}$$

for the same roots $z_k,\ k=1,2,3,...$, where

$$\Gamma(z_k)=\int_0^1\overline{W}(\xi,z_k)\sinh\left[(1-\xi)\chi(z_k)\right]d\xi,$$

$$\Pi(z_k)=\int_0^1 p(\xi)\sinh\left[(1-\xi)\chi(z_k)\right]d\xi.$$

A system of equalities of (15) type with respect to unknown control function can also be obtained in this case with same kernels, but with other right hand sides.

From applications point of view, electromagnetic signals controlled by a source concentrated at isolated point of medium are especially important. This case corresponds to substitution $p(x) = \delta(x - x_0)$, where $x_0 \in (0,1)$, in resolving conditions (16). Then relation

$$\Pi(z_k) = \sinh\left[(1 - x_0)\chi(z_k)\right],\ k = 1,2,3,...,$$

should be taken into account. Note also, that when $x_0 \to 0$ or $x_0 \to 1$ i.e. when the source "reaches" anyone of the medium boundaries, $\Pi(z_k) \to 0$, while numerator of fraction (16) remains bounded, therefore $\overline{u}_1(z_k) \to \infty$ which should be expected [13].

Let us add that under conditions of Corollary 1 in this case we will obtain:

$$\overline{u}_1(z_k) = \frac{\left[\overline{\psi}_1(z_k) + (-1)^{k+1}\overline{\varphi}_1(z_k)\right]\pi k - \Gamma(z_k)}{\Pi(z_k)},\ \ k = 1,2,3,...,$$

for the same roots $z_k,\ k = 1,2,3,...,$ where

$$\Gamma(z_k) = \int_0^1 \overline{W}(\xi, z_k)\sin\left[(1-\xi)\pi k\right]d\xi,\ \Pi(z_k) = \int_0^1 p(\xi)\sin\left[(1-\xi)\pi k\right]d\xi.$$

5. L^1 and L^2 optimal controls

In fact, solutions of both problems under investigation are reduced to minimization procedure of chosen optimality criterion under constraints of equality type on desired function. This problem can be attacked by different as rigorous techniques, as well as efficient numerical methods of nonlinear programming [19]. However, taking into account the important point that kernels of system (15) are bounded, reduced problem is convenient to solve via moments problem [14], treating those constraints as moments equalities with respect to unknown function.

First, let us determine solution of obtained moments problem for two given criteria. For that purpose, we will deal with truncated part of countable system (15) for some finite n:

$$\int_{-\theta}^{\theta} u(t)e^{-\varsigma_k t}\cos(\sigma_k t)dt = M_{1k},\ \int_{-\theta}^{\theta} u(t)e^{-\varsigma_k t}\sin(\sigma_k t)dt = M_{2k}, \tag{17}$$

$$k = \overline{1;n}.$$

Convergence of solution of finite system of moments problem to solution of infinite one is investigated as it is done in [1].

First, let us consider the case of minimization of a functional, evaluating summary "linear momentum" of control function [12]– [14]:

$$\Upsilon[u]=\int_{-\theta}^{\theta}|u(t)|dt,\ \ u\in\mathcal{D}.$$

As the space of measurable functions $L^1[-\theta,\theta]$ is a Banach space with respect to norm $||u(t)||_{L^1[-\theta,\theta]}=\Upsilon[u]$, then solution of finite system (17) when chosen criterion should be minimized is advisable to determine in that space. Our aim is to determine a function $u^o(t)$ optimal in the sense of chosen optimality criterion $\Upsilon[u]$ in set of measurable, compactly supported in $[-\theta,\theta]$ functions

$$\mathcal{D}=\{u\in L^1[-\theta,\theta]:\mathrm{u}\equiv 0,\mathrm{t}\notin[-\theta,\theta]\}.$$

Such controls we shall call L^1–optimal. Note, that the set $\mathcal{D}$ is everywhere dense in $L^1[-\theta,\theta]$ [15].

According to moments problem solution technique under integral criterion, outlined in [14], in this case we will obtain [12, 13]:

$$u_n^o(t)=\sum_{j=1}^{m}u_{nj}^o\delta(t-t_j^o),\ \ t\in[-\theta,\theta], \tag{18}$$

where intensities of control impacts $u_{nj}^o,\ j=\overline{1;m}$, are constrained by conditions

$$\mathrm{sgn}u_{nj}^o=\mathrm{sgn}h_n^o(t_j^o),\ \ j=\overline{1;m},$$

sgn$x$ is the well known sign function, and are determined from system of equations

$$\sum_{j=1}^{m}u_{nj}^o e^{-\varsigma_k t_j^o}\cos(\sigma_k t_j^o)=M_{1k},\ \sum_{j=1}^{m}u_{nj}^o e^{-\varsigma_k t_j^o}\sin(\sigma_k t_j^o)=M_{2k},\ \ k=\overline{1;n}, \tag{19}$$

at that

$$h_n^o(t)=\sum_{k=1}^{n}e^{-\varsigma_k t}\left[l_{1k}^o\cos(\sigma_k t)+l_{2k}^o\sin(\sigma_k t)\right],\ t\in[-\theta,\theta],$$

and moments $t_j^o,\ j=\overline{1;m}$, of those impacts application are determined from the following maximum condition:

$$\sup_{t\in[-\theta,\theta]}\left|\sum_{k=1}^{n}e^{-\varsigma_k t}\left[l_{1k}^o\cos(\sigma_k t)+l_{2k}^o\sin(\sigma_k t)\right]\right|=\left[\sum_{j=1}^{m}|u_{nj}^o|\right]^{-1}. \tag{20}$$

Optimal coefficients l_{1k}^o and $l_{2k}^o,\ k=\overline{1;n}$, are determined from the following problem of conditional minimum:

$$\sum_{k=1}^{n}e^{-\varsigma_k t_j^o}\left[l_{1k}^o\cos(\sigma_k t_j^o)+l_{2k}^o\sin(\sigma_k t_j^o)\right]\xrightarrow[\{\Lambda_{1k},\Lambda_{2k}\}]{}\min,$$

when

$$\sum_{k=1}^{n} [l_{1k}M_{1k} + l_{2k}M_{2k}] = 1.$$

Number m of control impacts is determined from inclusion conditions $\{t_j^o\}_{j=1}^m \subset [-\theta, \theta]$ uniquely.

Necessary and sufficient conditions of moments problem (17) solvability when chosen optimality criterion should be minimized gives the following

Theorem 2 *Finite system* (17) *is resolvable if and only if the condition*

$$\rho_n^o = \sum_{j=1}^{m} |u_{nj}^o| \tag{21}$$

holds.

Using results of monograph [1] we can conclude.

Theorem 3 *System* (8)–(9) *is* fully controllable *in* $L^1[-\theta, \theta]$ *if and only if the quantity (20) differs from zero for all* $n \in \mathbb{N}$.

Remark 3 Condition of Theorem 2 is equivalent to requirement that at least one of controls intensities u_{nj}^o, $j = \overline{1;m}$, differs from zero. It gives us corresponding constraints on the main determinant of system (19).

Let us proceed now to solution of problem under study when the quadratic criterion

$$\Upsilon[u] = \int_{-\theta}^{\theta} u^2(t)dt, \quad u \in \mathcal{D},$$

evaluating "full energy" expending on control process [13, 14], should be minimized.

It is well known, that square root of that functional defines norm in Hilbert space $L^2[-\theta, \theta]:\ ||u(t)||_{L^2[-\theta,\theta]} = \Upsilon^{1/2}[u]$, therefore solution of moments problem (17) when chosen criterion should be minimized is advisable to determine in that space. Our aim is to obtain a function $u^o(t)$ optimal in the sense of chosen criterion $\Upsilon[u]$ in set of square measurable, compactly supported in $[-\theta, \theta]$ functions

$$\mathcal{D} = \{u \in L^2[-\theta, \theta] : u \equiv 0, t \notin [-\theta, \theta]\}.$$

Such controls we shall call L^2–optimal.

According to moments problem solution technique under quadratic criterion, outlined in [14], in this case we will obtain [13]:

$$u_n^o(t) = \sum_{k=1}^{n} e^{-\varsigma_k t} \left[\Lambda_{1k}^o \cos(\sigma_k t) + \Lambda_{2k}^o \sin(\sigma_k t)\right],\ t \in [-\theta, \theta], \tag{22}$$

where coefficients $\Lambda^o_{pk},\ p=1;2$, are determined from system of linear algebraic equations

$$\mathbf{J}\boldsymbol{\Lambda}^o=\mathbf{M}, \tag{23}$$

where $\boldsymbol{\Lambda}^o=(\Lambda^o_{11}\dots\Lambda^o_{1n}\ \Lambda^o_{21}\dots\Lambda^o_{2n})^T$, $\mathbf{M}=(M_{11}\dots M_{1n}\ M_{21}\dots M_{2n})^T$, upper index T denotes transposition,

$$\mathbf{J}=\begin{pmatrix} J^+_{11} & J^+_{12} & \dots & J_{11} & J_{12} & \dots \\ J^+_{21} & J^+_{22} & \dots & J_{21} & J_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{11} & J_{21} & \dots & J^-_{11} & J^-_{12} & \dots \\ J_{12} & J_{22} & \dots & J^-_{21} & J^-_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$J^{\pm}_{jk}=\int_{-\theta}^{\theta}e^{-(\varsigma_j+\varsigma_k)t}\begin{pmatrix}\cos(\sigma_j t)\cos(\sigma_k t)\\ \sin(\sigma_j t)\sin(\sigma_k t)\end{pmatrix}dt,$$

$$J_{jk}=\int_{-\theta}^{\theta}e^{-(\varsigma_j+\varsigma_k)t}\cos(\sigma_j t)\sin(\sigma_k t)dt.$$

As analogue of Theorem 3 in this case will serve the following one.

Theorem 4 *System* (8)–(9) *is* fully controllable *in* $\mathrm{L}^2[-\theta,\theta]$ *if and only if the quantity*

$$\begin{aligned}\rho^o_n\equiv{}&\sum_{k=1}^{n}(\Lambda^o_{1k})^2J^+_{kk}+2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\Lambda^o_{1j}(\Lambda^o_{1k}J^+_{jk}+\Lambda^o_{2k}J_{jk})+\\ &+\sum_{k=1}^{n}(\Lambda^o_{2k})^2J^-_{kk}+2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\Lambda^o_{2j}(\Lambda^o_{2k}J_{jk}+\Lambda^o_{2k}J^-_{jk})\end{aligned} \tag{24}$$

is positive for all $n\in\mathbb{N}$.

From positivity condition of quantity ρ^o_n (24) one will be able to obtain corresponding restrictions on parameters of system (8)–(9) for its *fully controllability*.

Remark 4 If all roots $z_k,\ k=1,2,3,\dots$, of characteristic equation (12) are real, i.e. $\varsigma_k=0,\ k=1,2,3,\dots$, then instead of formulas (22)–(24) the following should be used: optimal controls (22) reads as

$$u^o_n(t)=\sum_{k=1}^{n}\left[\Lambda^o_{1k}\cos(\sigma_k t)+\Lambda^o_{2k}\sin(\sigma_k t)\right],\ t\in[-\theta,\theta], \tag{25}$$

system (23) is separated into two independent systems of linear algebraic equations with respect to coefficients $\Lambda^o_{pk},\ p = 1;2$, correspondingly

$$\mathbf{J}^{\pm}\boldsymbol{\Lambda}^o_p = \mathbf{M}_p,\ p = 1;2,$$

$$\boldsymbol{\Lambda}^o_p = (\Lambda^o_{p1}\dots\Lambda^o_{pn})^T,\ \mathbf{M}_p = (M_{p1}\dots M_{pn})^T,\ \mathbf{J}^{\pm} = \{J^{\pm}_{jk}\}^n_{j,k=1},$$

$$J^{\pm}_{jk} = \int_{-\theta}^{\theta}\begin{pmatrix}\cos(\sigma_j t)\cos(\sigma_k t)\\ \sin(\sigma_j t)\sin(\sigma_k t)\end{pmatrix}dt,$$

and

$$\begin{aligned}\rho^o_n \equiv \sum_{k=1}^{n}(\Lambda^o_{1k})^2 J^{+}_{kk} + 2\sum_{j=1}^{n-1}\Lambda^o_{1j}\sum_{k=j+1}^{n}\Lambda^o_{1k}J^{+}_{jk} + \\ + \sum_{k=1}^{n}(\Lambda^o_{2k})^2 J^{-}_{kk} + 2\sum_{j=1}^{n-1}\Lambda^o_{2j}\sum_{k=j+1}^{n}\Lambda^o_{2k}J^{-}_{jk}.\end{aligned} \tag{26}$$

6. Numerics

To illustrate constructed algorithm, let us consider two examples of optimal controls determination. First, determine L^2–optimal solution of system (17) for *the first problem*, when the control process is considered in time–interval $t \in [-\pi,\pi]$, $U(x,t) \equiv 0$, the second boundary condition, initial and terminal data read as follows

$$\psi(t) = \cos(2t),\ w_{-\pi}(x) = -\cos(\pi x),\ \dot{w}_{-\pi} = \sin(\pi x),$$

$$w_{\pi}(x) = \cos(2\pi x),\ \dot{w}_{\pi} = \sin(2\pi x),\ \ (x,t) \in [0,1]\times[-\pi,\pi],$$

respectively, and the kernel of equation (8)– $\kappa(t) = e^{-a|t|},\ t \in [-\pi,\pi],\ a = const \in \mathbb{R}^+$ (Debye dispersion model). Note, that transmission conditions, concerning chosen data are satisfied.

It is interesting to add, that such kernels occur in waves diffraction problems investigating by Fourier transform [20]. It, obviously, satisfies condition a) of Corollary 1. Furthermore, as $\overline{\kappa}(\sigma) = a(a^2+\sigma^2)^{-1}$, therefore $1+\alpha\overline{\kappa}_1(\sigma) > 0$ for all $\sigma \in \mathbb{R}$ and $\alpha,\ a \in \mathbb{R}^+$. Then, from (14) we will obtain

$$\sigma_k = \left[\sqrt{\alpha_k^2 + a^2(\pi k)^2} + \alpha_k\right]^{\frac{1}{2}},\ \varsigma_k = \left[\sqrt{\alpha_k^2 + a^2(\pi k)^2} - \alpha_k\right]^{\frac{1}{2}},$$

$$2\alpha_k = (\pi k)^2 - a(a+2\alpha),\ \ k = 1,2,3,\dots$$

It is easy to see, that for large k roots σ_k do not depend on parameter a, whereas $\lim_{k\to\infty}\varsigma_k = a$. But as ς_k are controls damping factor (see (22)), then one may conclude, that as faster the kernel $\kappa(t)$ decreases, as faster controls damp.

Let us add, that $\sigma_k = O(k)$ when $k \to \infty$, i.e. for k large enough they became equidistant. Moreover, as a result of calculations was observed, that $M_{pk} = O(k^{-3}),\ p = 1;2,$ when $k \to \infty$, which justifies truncation of infinite system (15), at that they do not depend on factor α in range $[0.01, 10]$ for large k. Boundary optimal control function (22) is plotted in Fig. 1–2, when $n = 80$ and $a = 0.25;\ 0.5;\ 1;\ 2$. From this graphs it is obvious, that when parameter a increases, absolute value of controls also increases.

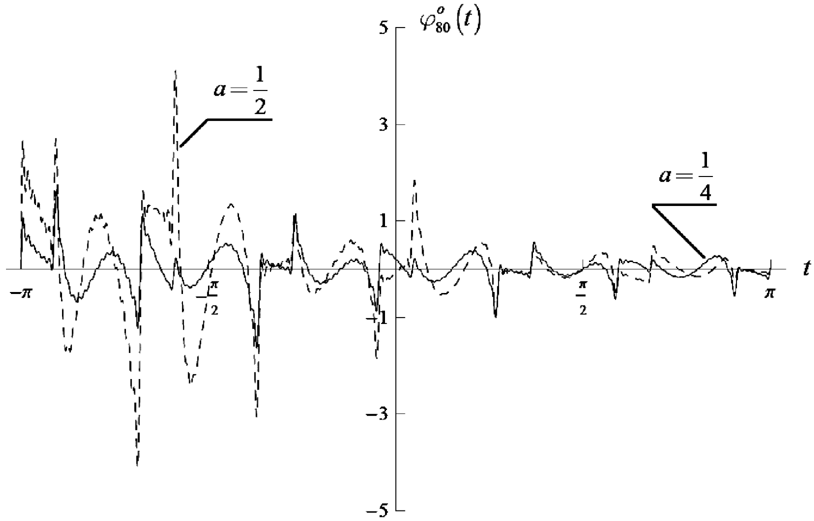

Figure 1. Optimal control function for $n = 80$: $u^o_{80}(t)$.

As second illustrative example, let us consider L^2–optimal solution of system (17) for *the second problem*, when

$$\varphi(t) = \cos(t),\ \psi(t) = \sin(2t),\ w_{-\pi}(x) = (x-1)\cos(\pi x),\ \dot{w}_{-\pi} = -2x\cos(\pi x),$$

$$w_{\pi}(x) = (x-1)\cos(2\pi x),\ \dot{w}_{\pi} = 2x\cos(2\pi x),\ \ (x,t) \in [0,1]\times[-\pi,\pi],$$

and $p(x) = \delta(x-x_0),\ \kappa(t) = -b|t|^{-1},\ t \in [-\pi,\pi],\ b = const \in \mathbb{R}^+$. Note, that transmission conditions, concerning chosen data are satisfied. Kernel $\kappa(t)$ in this case also satisfies condition a) of Corollary 1. Moreover, $\overline{\kappa}_1(\sigma) = 2b(\gamma + \ln|\sigma|)$, where γ is Euler‘s constant, and, therefore choosing parameter b in a certain manner one may satisfy condition b) of Corollary 1 as well. Substituting results in (14), we will obtain

$$z_k = \frac{\beta_k}{\sqrt{\Lambda\left(\beta_k^2 \cdot e^{\frac{1}{\alpha b}+2\gamma}\right)}},\ \ \beta_k^2 = \frac{(\pi k)^2}{\alpha b},\ \ k = 1,2,3,...,$$

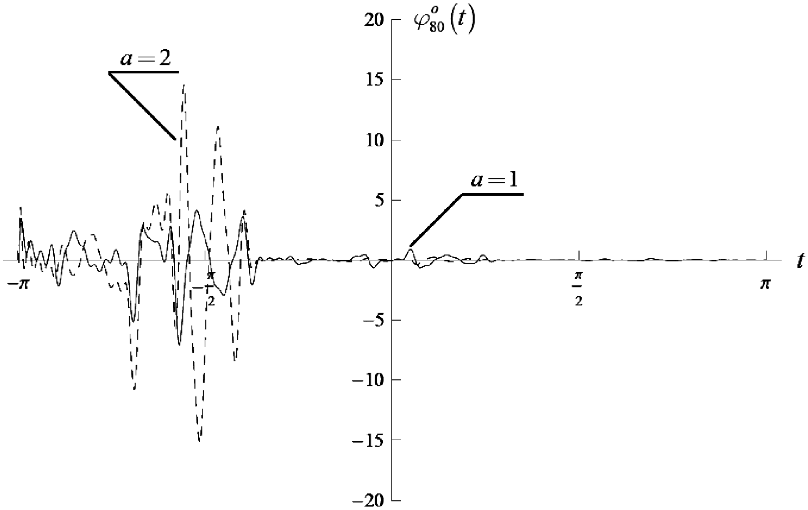

Figure 2. Optimal control function for $n = 80$: $u_{80}^{o}(t)$.

where $\Lambda(x)$ is Lambert‘s function [21]. Taking into account properties of that function one may prove, that all roots $z_k,\ k = 1,2,3,...$, in this case are real, i.e. $\varsigma_k = 0$, and $z_k = O(k)$ and $M_{pk} = O(k^{-3}),\ p = 1;2$, when $k \to \infty$ as well, at that they do not depend on factor α in range $[0.01,10]$ for large k. Optimal control function (25) is plotted in Fig. 3–6 when $n = 80$ and $x_0 = 0.685$.

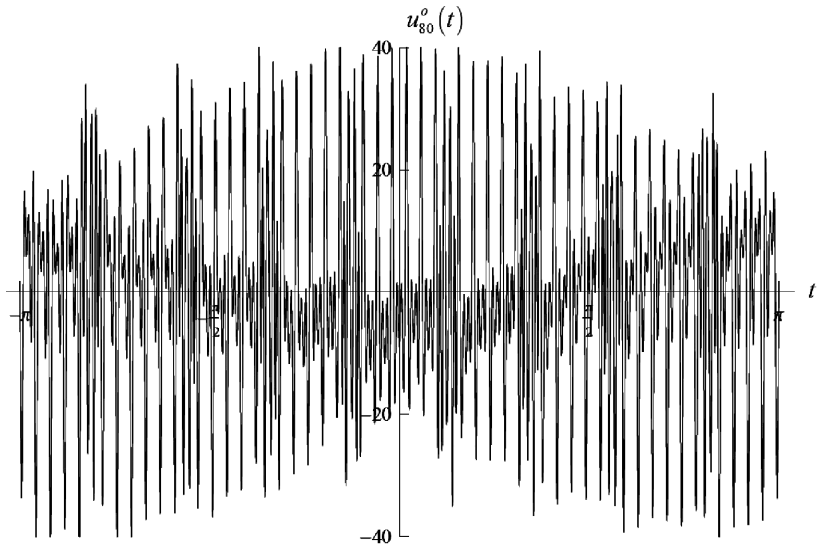

Figure 3. Optimal control function for $b = 0.001$.

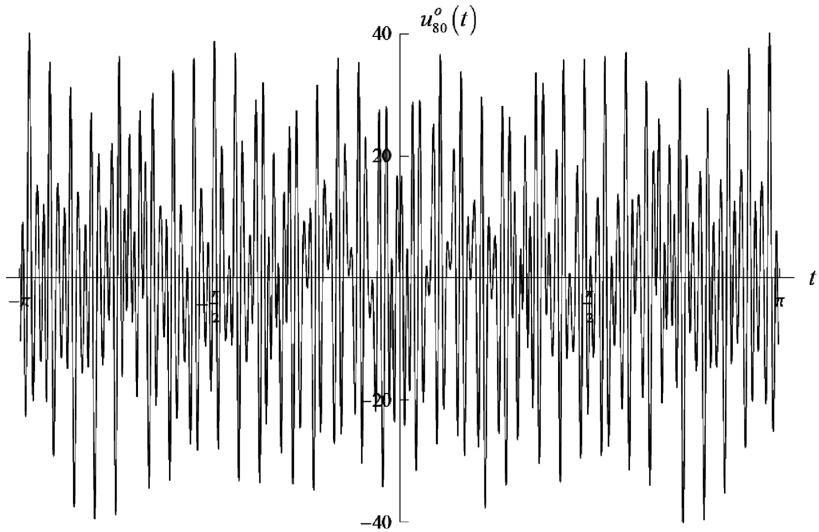

Figure 4. Optimal control function for $b = 0.01$.

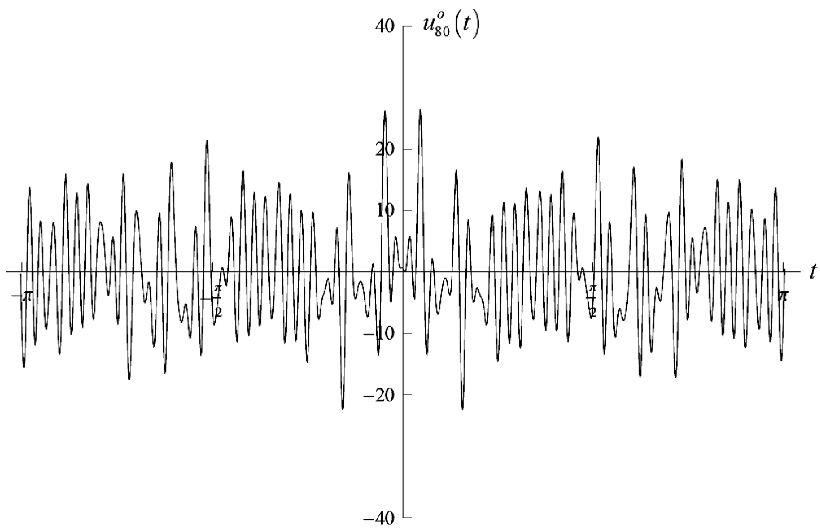

Figure 5. Optimal control function for $b = 0.1$.

It was discovered that when parameter b increases, absolute value and frequency of control impacts decrease, and for large values of that parameter coefficients Λ_{1k}^{o} of sines‘ in expression (25) dominate coefficients Λ_{2k}^{o} of cosines‘, which is connected with choice of boundary functions. It was also observed, that absolute value of control function does not signally depend on point x_0 of control concentration except cases $x_0 \to 0$ and $x_0 \to 1$ mentioned above.

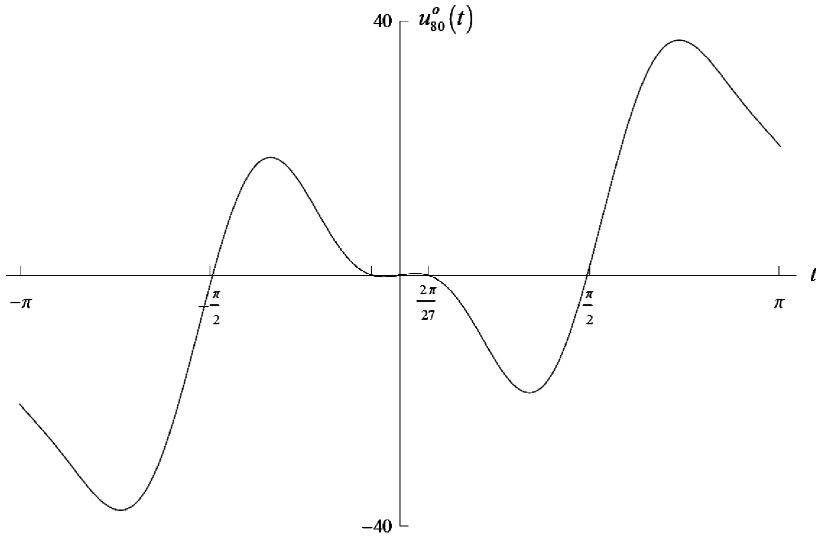

Figure 6. Optimal control function for $b = 20$.

Note that for chosen parameters of system (8)–(9) the quantities (24) and (26) are always positive.

7. Conclusions

In the paper solution of control problem of partial integro-differential equation of convolution type when control process is realized by boundary functions (*the first problem*) or right hand side (*the second problem*) is reduced to minimization procedure of chosen optimality criterion under constraints of equality type on unknown function and is obtained in explicit form (see (18) and (22) or (25)). According to the main result (Theorem 1) for determination of resolving optimal control is required:

Step 1. To find roots of a characteristic transcendent equation, containing only Fourier transform of equation kernel (see (12)),

Step 2. Using those roots to separate real and imaginary parts of a system of equalities with respect to unknown function Fourier transform (see (11)) and to obtain the corresponding system of moments problem (see (15)),

Step 3. Applying control formulae for corresponding optimality criterion (see (18) and (22) or (25)), to determine desired function of optimal control,

Step 4. To check if the necessary and sufficient condition of control problem solvability is satisfied (see (20) and (24) or (26)).

One of the main aims of this paper concerning efficiency of suggested control algorithms from numerical calculations viewpoint, we think, is achieved. As a matter of fact, L^1–optimal solution determination was reduced to conditional minimum problem, which can be attacked by well–known techniques of variational calculus [19], and L^2–optimal solution determination was reduced to solution of a system of linear algebraic equations.

What concerns to criteria considered above, let us note that in fact, the first criterion is convenient to set, for instance, when one needs to have an electromagnetic output pulse, and the second criterion -on the contrary, when one needs to have a continuous output signal from discrete input signal. We hope that this method will be applied with the same efficiency to control problems investigation for other equations, containing convolution of unknown function, as ordinary as well as partial integro-differential equations with variable limit of integration.

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Appendix

Proof [Proof of Theorem 1]

According to method of Cauchy, the fundamental solution of system (10) can be written as follows:

$$\overline{w}_1(x,\sigma) = a_+ e^{\chi(\sigma)x} + a_- e^{-\chi(\sigma)x} + $$
$$+\frac{1}{\chi(\sigma)}\int_0^x \left[\overline{U}_1(\xi,\sigma)+\overline{W}(\xi,\sigma)\right]\sinh[(x-\xi)\chi(\sigma)]d\xi, \tag{A1}$$
$$(x,\sigma)\in(0,1)\times\mathbb{R},$$

where

$$a_\pm(\sigma) = \frac{[\overline{\psi}_1(\sigma)-\Phi(\sigma)]\,e^{\pm\chi(\sigma)}-\overline{\varphi}_1(\sigma)}{e^{\pm2\chi(\sigma)}-1},\ \chi(\sigma)=i|\sigma|\sqrt{1+\alpha\overline{\kappa}_1(\sigma)}, \tag{A2}$$

$$\Phi(\sigma) = \frac{1}{\chi(\sigma)}\int_0^1 \left[\overline{U}_1(\xi,\sigma)+\overline{W}(\xi,\sigma)\right]\sinh[(1-\xi)\chi(\sigma)]d\xi,$$

As $w_1(x,t)$ is compactly supported function in $[-\theta,\theta]$ then it is well known [15], that its Fourier generalized transform $\overline{w}_1(x,z)$ is an analytical entire function in whole complex plane $z=\sigma+i\varsigma$, satisfying inequality

$$|z^\rho\cdot\overline{w}_1(x,z)|\leqslant A_\rho e^{\theta|\varsigma|}$$

for all $x\in[0,1]$; $\rho=0,1,2,\ldots$, and some corresponding constant $A_\rho\geqslant0$.

In view of $W(\cdot,t)$ is compactly supported in $[-\theta,\theta]$, one may show that if $U_1(\cdot,t)$ is compactly supported in $[-\theta,\theta]$ as well, integral term on the right hand side of (A1) extended for all $z=\sigma+i\varsigma$ is analytic entire function. Therefore, using that extension one may prove that the function $\overline{w}_1(x,z)$ satisfies recalled theorem conditions if and only if $a_\pm=a_\pm(z)$ are analytic entire functions. From expressions (A2) extended for all $z\in\mathbb{C}$ it is easy to see, that they are entire or not simultaneously. Thus, for example, $a_+=a_+(z)$ is an entire analytical function if and only if conditions (11) and (12) are satisfied.

The same reasoning was made in the case of distributed control.

When proving Corollary 1, instead of formula (A1) and (A2) the following must be used:

$$\overline{w}_1(x,\sigma) = \overline{\varphi}_1(\sigma)\cos(\chi(\sigma)x) +$$
$$+\frac{[\overline{\psi}_1(\sigma)-\Phi(\sigma)]-\overline{\varphi}_1(\sigma)\cos(\chi(\sigma))}{\sin(\chi(\sigma))}\sin(\chi(\sigma)x)+$$
$$+\frac{1}{\chi(\sigma)}\int_0^x \left[\overline{U}_1(\xi,\sigma)+\overline{W}(\xi,\sigma)\right]\sin[(x-\xi)\chi(\sigma)]d\xi,\ \ (x,\sigma)\in(0,1)\times\mathbb{R},$$

where

$$\Phi(\sigma) = \frac{1}{\chi(\sigma)} \int_0^1 \left[\overline{U}_1(\xi,\sigma) + \overline{W}(\xi,\sigma)\right] \sin[(1-\xi)\chi(\sigma)] d\xi,$$

$$\chi(\sigma) = |\sigma| \sqrt{1 + \alpha\overline{\kappa}_1(\sigma)} > 0.$$

The rest part of the proof is similar to proof of Theorem 1.

It should be noted, that conditions a) and b) of Corollary 1 have simple physical treatment. It turns out, that the quantity $\varepsilon(\sigma) = 1 + \alpha\overline{\kappa}_1(\sigma)$ is complex dielectric permittivity of isotropic medium with dispersion property [7, 16], and those conditions are equivalent to assumption, that quantity $\varepsilon(\sigma),\ \sigma \in \mathbb{R}$, is real–valued, because if so it is positive for all known materials. On the other hand, with increase of propagating signal frequency to values similar to eigenfrequency of medium, the difference between dielectric and conducting abilities of medium decreases, and it turns out, that existence of imaginary part in dielectric permittivity expression from macroscopic point of view is indistinguishable from conducting ability; they both lead to heat evaluation. Thus, both conditions of Corollary 1 are practically realizable.

If one needs to obtain controlled electromagnetic wave, he has to apply Fourier inverse generalized transform to (A1).

Proof [Proof of Theorem 3]

Necessary and sufficient condition (20) is obtained from general existence theorem for moments problem solution [14]. According to it, for solvability of system (17) it is necessary and sufficient, that the norm of functions $h_n^o(t)$ in space, adjoint to the space of control function to be finite and non–zero. Since the space $\mathrm{L}^1[-\theta,\theta]$ is taken as control function space, and the space adjoint to it is the space of infinite measurable functions $\mathrm{L}^\infty[-\theta,\theta]$ with norm $||h_n^o||_{\mathrm{L}^\infty[-\theta,\theta]} = \sup_{t\in[-\theta,\theta]} |h_n^o(t)| \equiv [\rho_n^o]^{-1}$, therefore in view of (18) and (19) we will get (20). Note, that topology in L^∞ instead of sup–norm might be introduced through ess sup–norm denoting essential sup of a function [14].

Further, the countable system (15) is resolvable if and only if finite system (17) is resolvable for all $n \in \mathbb{N}$ [1]. Thus, theorem is proved.

Proof [Proof of Theorem 4]

is similar to proof of Theorem 3, but in this case in view of selfadjointness of space $\mathrm{L}^2[-\theta,\theta]$ we have $\rho_n^o = \left[||u_n^o(t)||_{\mathrm{L}^2[-\theta,\theta]}\right]^{-2}$.