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On the Existence of Continuous Positive Monotonic Solutions of a Self-Reference Quadratic Integral Equation

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ABSTRACT: In this work we study the existence of positive monotonic solutions of a self-reference quadratic integral equation in the class of continuous real valued functions. The continuous dependence of the unique solution will be proved. Some examples will be given.

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1. Introduction

Most papers of differential and integral equations with deviating arguments introduce the deviation of the arguments only on the time itself, however, the case of the deviating arguments depend on both the state variable x and the time t is important in theory and practice. These kinds of equations play an important role in nonlinear analysis and have many applications (see [1], [7]-[11] and [13]- [16]).
Buică [8] studied the existence, uniqueness and continuous dependence of the solution of the integral equation

$$x(t) = x_0 + \int_a^t f(s, x(x(s)))ds$$

corresponding to the initial value problem

$$\frac{d}{dt}x(t) = f\big(t, x\big(x(t)\big)\big), \quad t \in (a, b], \quad x(a) = x_0$$

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where $f \in C([a,b] \times [a,b])$ and Lipschitz continuous in the second argument.
Here we relax the assumptions and generalize the results of [8] for the self-reference quadratic integral equation

$$x(t) = a(t) + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big)ds, \quad t \in [0,T]. \qquad (1)$$

Quadratic integral equations have been studied by some authors, see for examples [2]-[6] and [9] and references therein.
Let $C[0,T]$ be the Banach space consisting of all functions which are defined and continuous on the interval $[0,T]$. Our aim in this paper is to study the existence of continuous positive monotonic solutions $x \in C[0,T]$ of the self-reference quadratic integral equation (1). The uniqueness of the solution will be studied also. Moreover we prove that the unique solution of (1) depends continuously on the the functions a, f_1 and f_2.

2. Existence of solution

Consider the quadratic integral equation (1) under the following assumptions:

(*i*) a:$[0,T] \to R^+$ and there exists a positive constant a such that

$$|a(t_2) - a(t_1)| \le a|t_2 - t_1|, \quad t_1, t_2 \in [0,T].$$

(*ii*) $f_i : [0,T] \times [0,T] \to R^+$ satisfies Carathéodory condition, i.e. f_i are measurable in t for all $x \in C[0,T]$ and continuous in x for almost all $t \in [0,T]$, $i = 1,2$.

(*iii*) There exist two constants $b_1,\ b_2 \ge 0$ and two bounded measurable functions $m_i : [0,T] \to R,\ |m_i(t)| \le c_i$ such that

$$|f_i(t,x)| \le |m_i(t)| + b_i|x|, \ i = 1,2.$$

(*iv*) $\phi_i : [0,T] \to [0,T]$ such that $\phi_i(0) = 0$ and

$$|\phi_i(t) - \phi_i(s)| \le |t - s|, \quad i = 1,\ 2.$$

This assumption implies that $\phi_i(t) \le t,\ i = 1,\ 2$ and $x(0) = a(0)$.

(*v*) $LT + |a(0)| \le T$ and $L = a + 2M_1M_2T < 1$ where

$$M_1 = c_1 + b_1T,\ M_2 = c_2 + b_2T.$$

Define the set S_L by

$$S_L = \big\{x \in C[0,T] : \ |x(t) - x(s)| \le L|t - s|\big\} \subset C[0,T].$$

It clear that S_L is nonempty, closed, bounded and convex subset of $C[0,T]$.

Now we can prove the following existence theorem

Theorem 1. *Let the assumptions (i) − (v) be satisfied, then the self-reference quadratic integral equation (1) has at least one positive solution* $x \in S_L \subset C[0,T]$.

Proof. Define the operator F associated with equation (1) by

$$Fx(t) = a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t)} f_2(s, x(x(s)))ds, \quad t \in [0,T].$$

Let $x \in S_L \subset C[0,T],\ t \in [0,T]$. Then, from our assumptions we have

$$\begin{aligned}
|Fx(t)| &= |a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t)} f_2(s, x(x(s)))ds| \\
&\le |a(t)| + \int_0^{\phi_1(t)} |f_1(s, x(x(s)))|ds \int_0^{\phi_2(t)} |f_2(s, x(x(s)))|ds \\
&\le |a(t)| + \int_0^{\phi_1(t)} \{|m_1(s)| + b_1|x(x(s))|\}ds \int_0^{\phi_2(t)} \{|m_2(s)| + b_2|x(x(s))|\}ds \\
&\le |a(t)| + \big[c_1\phi_1(t) + b_1 \int_0^{\phi_1(t)} \{L|x(s)| + |x(0)|\}ds\big] \\
&\quad \big[c_2\phi_2(t) + b_2 \int_0^{\phi_2(t)} \{L|x(s)| + |x(0)|\}ds\big] \\
&\le |a(t)| + \big[c_1 T + b_1(LT + |a(0)|)\phi_1(t)\big]\big[c_2 T + b_2(LT + |a(0)|)\phi_2(t)\big] \\
&\le |a(t)| + \big[c_1 + b_1 T\big]\big[c_2 + b_2 T\big]T^2 \\
&\le |a(t)| + M_1 M_2 T^2 \le a\ T + |a(0)| + M_1 M_2 T^2 \\
&< L\ T + |a(0)| \le T.
\end{aligned}$$

This proves that the class $\{Fx\}$ is uniformly bounded.

Now let $x \in S_L$ and $t_1,\ t_2 \in [0,T]$ such that $t_1 < t_2$ and $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= \big|a(t_2) + \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_2)} f_2(s, x(x(s)))ds \\
&- a(t_1) - \int_0^{\phi_1(t_1)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds\big| \\
&= \big|a(t_2) - a(t_1) \\
&+ \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_2)} f_2(s, x(x(s)))ds \\
&- \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \\
&+ \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \\
&- \int_0^{\phi_1(t_1)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds\big|
\end{aligned}$$

$$
\begin{aligned}
&\le |a(t_2)-a(t_1)| \\
&+ \Big|\int_0^{\phi_1(t_2)} f_1(s,x(x(s)))ds\Big[\int_0^{\phi_2(t_2)} f_2(s,x(x(s)))ds - \int_0^{\phi_2(t_1)} f_2(s,x(x(s)))ds\Big]\Big| \\
&+ \Big|\int_0^{\phi_2(t_1)} f_2(s,x(x(s)))ds\Big[\int_0^{\phi_1(t_2)} f_1(s,x(x(s)))ds - \int_0^{\phi_1(t_1)} f_1(s,x(x(s)))ds\Big]\Big| \\
&\le |a(t_2)-a(t_1)| \\
&+ \int_0^{\phi_1(t_2)} |f_1(s,x(x(s)))|ds \ \Big|\int_{\phi_2(t_1)}^{\phi_2(t_2)} f_2(s,x(x(s)))ds\Big| \\
&+ \int_0^{\phi_2(t_1)} |f_2(s,x(x(s)))|ds \ \Big|\int_{\phi_1(t_1)}^{\phi_1(t_2)} f_1(s,x(x(s)))ds\Big| \\
&\le a\ |t_2-t_1| \\
&+ \int_0^{\phi_1(t_2)} \{|m_1(s)|+b_1|x(x(s))|\}ds\Big(\Big|\int_{\phi_2(t_1)}^{\phi_2(t_2)} \{|m_2(s)|+b_2|x(x(s))|\}ds\Big|\Big) \\
&+ \Big(\int_0^{\phi_2(t_1)} \{|m_2(s)|+b_2|x(x(s))|\}ds\Big)\Big(\Big|\int_{\phi_1(t_1)}^{\phi_1(t_2)} \{|m_1(s)|+b_1|x(x(s))|\}ds\Big|\Big) \\
&\le a\ |t_2-t_1| \\
&+ \Big[c_1\phi_1(t_2)+b_1\int_0^{\phi_1(t_2)} \{L|x(s)|+|x(0)|\}ds\Big] \\
&\quad \Big[c_2\ \big|\phi_2(t_2)-\phi_2(t_1)\big| + b_2\ \Big|\int_{\phi_2(t_1)}^{\phi_2(t_2)} \{L|x(s)|+|x(0)|\}ds\Big|\Big] \\
&+ \Big[c_2\phi_2(t_1)+b_2\int_0^{\phi_2(t_1)} \{L|x(s)|+|x(0)|\}ds\Big|\Big] \\
&\quad \Big[c_1\ \big|\phi_1(t_2)-\phi_1(t_1)\big| + b_1\Big|\int_{\phi_1(t_1)}^{\phi_1(t_2)} \{L|x(s)|+|x(0)|\}ds\Big|\Big] \\
&\le a\ |t_2-t_1| \\
&+ \big[c_1+b_1\{L\ T+|a(0)|\}\big]\ \big[c_2+b_2\{L\ T+|a(0)|\}\big]\ \phi_1(t_2)\ \big|\phi_2(t_2)-\phi_2(t_1)\big| \\
&+ \big[c_2+b_2\{L\ T+|a(0)|\}\big]\ \big[c_1+b_1\{L\ T+|a(0)|\}\big]\ \phi_2(t_1)\ \big|\phi_1(t_2)-\phi_1(t_1)\big| \\
&\le a\ |t_2-t_1| \\
&+ \big[c_1+b_1\{L\ T+|a(0)|\}\big]\ \big[c_2+b_2\{L\ T+|a(0)|\}\big]\ T|t_2-t_1| \\
&+ \big[c_2+b_2\{L\ T+|a(0)|\}\big]\big[c_1+b_1\{L\ T+|a(0)|\}\big]\ T\ |t_2-t_1| \\
&\le a\ |t_2-t_1| + 2T\big(c_1+b_1\ T\big)\big(c_2+b_2\ T\big)|t_2-t_1| \\
&= a\ |t_2-t_1| + 2TM_1M_2|t_2-t_1| = L|t_2-t_1|.
\end{aligned}
$$

This proves that $F: S_L \to S_L$ and the class $\{Fx\}$ is equicontinuous.
Now the class of continuous functions $\{Fx\} \subset S_L \subset C[0,T]$ is uniformly bounded and equicontinuous on S_L. Hence, applying Arzela-Ascoli Theorem [12] we deduce that the operator F is compact.

Finally we show that F is continuous. Let $\{x_n\} \subset S_L$ such that $x_n \to x_0$ on $[0, T]$, then

$$\begin{aligned} |f_i(t, x_n(x_n(t))))| &\leq |m_i(t)| + b_i|x_n(x_n(t))| \\ &\leq |m_i(t)| + b_i T, \quad i = 1,\ 2 \end{aligned}$$

and

$$\begin{aligned} |x_n(x_n(t)) - x_0(x_0(t))| &= |x_n(x_n(t)) - x_n(x_0(t)) + x_n(x_0(t)) - x_0(x_0(t))| \\ &\leq |x_n(x_n(t)) - x_n(x_0(t))| + |x_n(x_0(t)) - x_0(x_0(t))| \\ &\leq L|x_n(t) - x_0(t)| + |x_n(x_0(t)) - x_0(x_0(t))|. \end{aligned}$$

This implies that

$$x_n(x_n(t))) \to (x_0(x_0(t)).$$

From the continuity of $f_i,\ i = 1,\ 2$ in the second argument we have

$$f\big(t, x_n(x_n(t))\big) \to f\big(t, x_0(x_0(t))\big).$$

Now by Lebesgue's dominated convergence Theorem [12] we obtain

$$\begin{aligned} \lim_{n\to\infty} \big(Fx_n\big)(t) &= \lim_{n\to\infty} a(t) + \lim_{n\to\infty} \int_0^{\phi_1(t)} f_1\big(s, x_n(x_n(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x_n(x_n(s))\big)ds \\ &= a(t) + \int_0^{\phi_1(t)} f_1\big(s, x_0(x_0(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x_0(x_0(s))\big)ds \\ &= \big(Fx_0\big)(t). \end{aligned}$$

Then F is continuous. Using Schauder fixed point Theorem ([12]), then the operator F has at least one fixed point $x \in S_L$. Consequently there exist at leat one solution $x \in C[0, T]$ of equation (1).
Finally, from our assumptions we have

$$x(t) = a(t) + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big)ds > 0, \quad t \in [0, T].$$

and the solution of the quadratic integral equation (1) is positive. □

Now the following two corollaries can be easily proved.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied. If the functions a, ϕ_1 and ϕ_2 are nondecreasing, then the solution of the quadratic integral equation (1) is positive and nondecreasing.*

Corollary 2. *Let the assumptions of Corollary 1 be satisfied. If, in addition* $\phi_i(t) = t,\ i = 1,\ 2$, *then the quadratic integral equation*

$$x(t) = a(t) \ + \int_0^t f_1\big(s, x(x(s))\big)ds \int_0^t f_2\big(s, x(x(s))\big)ds, \quad t \in [0, T] \tag{2}$$

has at least one positive and nondecreasing solution $x \in C[0, T]$.

Example 1. Consider the following quadratic integral equation

$$\begin{aligned} x(t) \quad &= \quad \left(\frac{1}{4} + \frac{1}{8}t\right) + \int_0^{\beta_1 t} \left(\frac{1}{3}s^3 e^{-s^2} + \frac{\ln(1 + |x(x(s)))|}{4 + s^2}\right) ds \\ & \int_0^{\beta_2 t^\zeta} \left(\frac{1}{12}|\cos(3(s+1))| + \frac{3}{24}|x(x(s))|\right) ds, \end{aligned} \tag{3}$$

where $t \in [0, 1],\ \beta_1 \in (0, 1],\ \zeta > 1$ and $\beta_2\zeta < 1$.
Here we have

$$f_1\big(t, x(x(t))\big) = \frac{1}{3}t^3 e^{-t^2} + \frac{\ln(1 + |x(x(t))|)}{4 + t^2},$$

$$|f_1\big(t, x(x(t))\big)| \leq \frac{1}{3}t^3 e^{-t^2} + \frac{1}{4}|x(x(t))| \quad \text{and} \quad m_1(t) = \frac{1}{3}t^3 e^{-t^2},$$

$$f_2\big(t, x(x(t))\big) = \frac{1}{12}\cos(3(t+1)) + \frac{3}{24}|x(x(t))|,$$

$$|f_2\big(t, x(x(t))\big)| = \frac{1}{12}|\cos(3(t+1))| + \frac{3}{24}|x(x(t))| \quad \text{and} \quad m_2(t) = \frac{1}{12}|\cos(3(t+1))|.$$

Also we have $\phi_1(t) = \beta_1 t,\ \phi_2(t) = \beta_2 t^\zeta,\ a(t) = \frac{1}{4} + \frac{1}{8}t,\ a = \frac{1}{8},\ b_1 = \frac{1}{4},\ b_2 = \frac{3}{24}$, $c_1 = \frac{1}{3},\ c_2 = \frac{1}{12}$, and $M_1 = \frac{7}{12},\ M_2 = \frac{5}{24}$.
Hence $L \simeq 0.368 < 1$ and $L\ T + |a(0)| = 0.618 \leq T = 1$.
Now it is clear that all assumptions of Theorem 1 are satisfied, then equation (3) has at least one solution.

3. Uniqueness of the solution

In this section we study the uniqueness of the solution $x \in C[0, T]$ of the quadratic integral equation (1).
Consider the following assumption

(ii^*) $f_i : [0, T] \times [0, T] \to R^+$ are measurable in t for all $x \in C[0, T]$, satisfy the Lipschitz condition

$$|f_i(t, x) - f_i(t, y)| \leq b_i\ |x - y| \quad i = 1,\ 2$$

$|f_i(t, 0)| \leqslant c_i,\ \forall_{t \in [0.T]}$.

Theorem 2. *Let the assumptions (i), (iv), (v) and (ii*) be satisfied, if*

$$(\gamma_1\ b_2 + \gamma_2\ b_1)\ T\ (L+1) < 1,$$

where $\gamma_i = (c_i + b_i T)T,\ i = 1,\ 2,$ *then equation (1) has a unique solution* $x \in C[0,T]$.

Proof. From assumption (ii^*) we can deduced that

$$|f_i(t,x)| \le b_i\ |x| + |f_i(t,0)|\ \le b_i\ |x| + c_i, \quad i = 1,2,$$

then all assumptions of Theorem 1 are satisfied and the integral equation (1) has at least one solution. Let $x,\ y$ be two solutions of (1), then obtain

$$\begin{aligned}
|x(t)-y(t)| &= \Big|a(t)\ + \int_0^{\phi_1(t)} f_1\big(s,x(x(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s,x(x(s))\big)ds \\
&- \ a(t) - \int_0^{\phi_1(t)} f_1\big(s,y(y(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s,y(y(s))\big)ds\Big| \\
&= \Big| \int_0^{\phi_1(t)} f_1\big(s,x(x(s))\big)\ ds\ \Big[\int_0^{\phi_2(t)} \big\{f_2\big(s,x(x(s))\big) - f_2\big(s,y(y(s))\big)\big\}ds\Big] \\
&+ \int_0^{\phi_2(t)} f_2\big(s,y(y(s))\big)\ ds\ \Big[\int_0^{\phi_1(t)} \big\{f_1\big(s,x(x(s))\big) - f_1\big(s,y(y(s))\big)\big\}ds\Big]\Big| \\
&\le \int_0^{\phi_1(t)} |f_1\big(s,x(x(s))\big)|\ ds\ \int_0^{\phi_2(t)} |f_2\big(s,x(x(s))\big) - f_2\big(s,y(y(s))\big)|ds \\
&+ \int_0^{\phi_2(t)} |f_2\big(s,y(y(s))\big)|\ ds\ \int_0^{\phi_1(t)} |f_1\big(s,x(x(s))\big) - f_1\big(s,y(y(s))\big)|ds \\
&\le \int_0^{\phi_1(t)} |f_1\big(s,x(x(s))\big)|\ ds\ b_2 \int_0^{\phi_2(t)} |x(x(s)) - y(y(s))|ds \\
&+ \int_0^{\phi_2(t)} |f_2\big(s,y(y(s))\big)|\ ds\ b_1 \int_0^{\phi_1(t)} |x(x(s)) - y(y(s))|ds, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\int_0^{\phi_i(t)} |f_i\big(s,x(x(s))\big)|ds &\le b_i \int_0^{\phi_i(t)} |x(x(s))|ds + \int_0^{\phi_i(t)} |f_i(t,0)|ds \\
&\le b_i \int_0^{\phi_i(t)} \big\{L\ T + |x(0)|\big\}ds + c_i\phi_i(t) \\
&\le b_i\phi_i(t)\ T\ +\ c_i\phi_i(t) \\
&\le (b_iT + c_i)T = \gamma_i,\ i = 1,2 \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
|x(x(s)) - y(y(s))| &= |x(x(s)) - y(y(s)) + x(y(s)) - x(y(s))| \\
&\le |x(x(s)) - x(y(s))| + |x(y(s)) - y(y(s))| \\
&\le L|x(s)) - y(s)| + |x(y(s)) - y(y(s))|. \qquad (6)
\end{aligned}$$

Substituting (5) and (6) in (4) we can get

$$\begin{aligned} |x(t)-y(t)| &\leq \gamma_1\ b_2 \int_0^{\phi_2(t)} \big\{L|x(s)-y(s)|+|x(y(s))-y(y(s))|\big\}ds \\ &+ \gamma_2\ b_1 \int_0^{\phi_1(t)} \big\{L|x(s)-y(s)|+|x(y(s))-y(y(s))|\big\}ds \\ &\leq \gamma_1\ b_2\ \|x-y\|\ (L+1)\ \phi_2(t)\ +\gamma_2\ b_1\ \|x-y\|\ (L+1)\ \phi_1(t) \\ &\leq (\gamma_1\ b_2+\gamma_2\ b_1)\ T\ (L+1)\ \|x-y\| \end{aligned}$$

and

$$\big[1-(\gamma_1\ b_2+\gamma_2\ b_1)\ T\ (L+1)\big]\ \|x-y\| \leq 0,$$

then $x(t)=y(t),\ t\in[0,T]$ and equation (1) has a unique solution $x\in C[0,T]$. □

Example 2. Let $T=1,\ t\in[0,1]$ and $\alpha,\ \beta,\ \mu,\ \rho\in(0,1]$ are parameters. Consider the following quadratic integral equation

$$x(t)=\left(\frac{2}{7}+\frac{1}{7}t\right)+\int_0^{\alpha t}\left(\frac{\mu}{8-s}+\frac{1}{14}|x(x(s))|\right)ds\int_0^{\beta t}\left(\frac{\rho}{6}\ln(1+|s|)+\frac{1}{2}|x(x(s))|\right)ds. \tag{7}$$

Here we have

$$f_1\big(t,x(x(t))\big)=\frac{\mu}{8-t}+\frac{1}{14}|x(x(t))|,$$

$$|f_1\big(t,x)-f_1\big(t,y)| \leq \frac{1}{14}|x-y|,$$

$$f_2\big(t,x(x(t))\big)=\frac{\rho}{6}\ln(1+|t|)+\frac{1}{2}|x(x(t))|,$$

and

$$|f_2\big(t,x)-f_2\big(t,y)| \leq \frac{1}{2}|x-y|.$$

Also, $m_1(t)=\frac{\mu}{8-t},\ c_1=\frac{1}{7},\ m_2(t)=\frac{\rho}{6}\ln(1+|t|),\ c_2=\frac{1}{6},\ \phi_1(t)=\alpha t,\ \phi_2(t)=\beta t$ and $a(t)=\frac{2}{7}+\frac{1}{7}t$, then we obtain $a=\frac{1}{7}$, $b_1=\frac{1}{14}$, $b_2=\frac{1}{2}$, $M_1=\frac{3}{14}$ and $M_2=\frac{2}{3}$.
Hence $L=\frac{3}{7}<1$ and $L\ T+|a(0)|=\frac{5}{7}\leq T=1$.
Moreover we have $\gamma_1=\frac{3}{14},\ \gamma_2=\frac{2}{3}$ and

$$(\gamma_1\ b_2+\gamma_2\ b_1)\ T\ (L+1)\simeq 0.2210<1.$$

Now all assumptions of Theorem 2 are satisfied, then equation (7) has a unique solution.

4. Continuous dependence

In this section we prove that the solution of equation (1) depends continuously on the functions $a,\ f_1, f_2$.

4.1. Continuous dependence on the function a

Definition 1. The solution of the integral equation (1) depends continuously on the function a if $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ such that

$$|a(t) - a^*(t)| \leq \delta \qquad \Rightarrow \|x - x^*\| \leq \epsilon \tag{8}$$

where x^* is the unique solution of equation

$$x^*(t) = a^*(t) + \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big)ds, \quad t \in [0, T]. \tag{9}$$

Theorem 3. *Let the assumptions of Theorem 2 be satisfied, assume that* $|a(t) - a^*(t)| \leq \delta$, *then the solution of (1) depends continuously on the function* a.

Proof. Let $|a(t) - a^*(t)| \leq \delta$, then we can get

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big|a(t) + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big)ds \\
&- a^*(t) - \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big)ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big)ds\Big| \\
&= \Big|a(t) - a^*(t) + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big)ds \\
&\times \Big[\int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big)ds - \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big)ds\Big] \\
&+ \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big)ds \\
&\times \Big[\int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big)ds - \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big)ds\Big]\Big| \\
&\leq |a(t) - a^*(t)| \\
&+ \int_0^{\phi_1(t)} |f_1\big(s, x(x(s))\big)|ds \int_0^{\phi_2(t)} |f_2\big(s, x(x(s))\big) - f_2\big(s, x^*(x^*(s))\big)|ds \\
&+ \int_0^{\phi_2(t)} |f_2\big(s, x^*(x^*(s))\big)|ds \int_0^{\phi_1(t)} |f_1\big(s, x(x(s))\big) - f_1\big(s, x^*(x^*(s))\big)|ds \\
&\leq \delta + \int_0^{\phi_1(t)} (c_1 + b_1|x(x(s))|)ds\ b_2 \int_0^{\phi_2(t)} |x(x(s)) - x^*(x^*(s))|ds \\
&+ \int_0^{\phi_2(t)} (c_2 + b_2|x^*(x^*(s))|)ds\ b_1 \int_0^{\phi_1(t)} |x(x(s)) - x^*(x^*(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta + M_1\ \phi_1(t)\ b_2 \int_0^{\phi_2(t)} |x(x(s)) - x^*(x^*(s))| ds \\
&+ M_2\ \phi_2(t)\ b_1 \int_0^{\phi_1(t)} |x(x(s)) - x^*(x^*(s))| ds \\
&\leq \delta + M_1\ T\ b_2\ (L+1)\|x - x^*\|\ \phi_2(t) \\
&+ M_2\ T\ b_1\ (L+1)\|x - x^*\|\ \phi_1(t) \\
&\leq \delta + (\gamma_1 b_2 + \gamma_2 b_1)(L+1)\ T\ \|x - x^*\|,
\end{aligned}
$$

$$\|x - x^*\|\big(1 - (\gamma_1 b_2 + \gamma_2 b_1)(L+1)\ T\big) \quad \leq \quad \delta$$

and

$$\|x - x^*\| \quad \leq \quad \frac{\delta}{1 - (\gamma_1 b_2 + \gamma_2 b_1)(L+1)T} = \epsilon.$$

4.2. Continuous dependence on the functions f_1

Here we prove that the solution of the equation (1) depends continuously on the function f_1.

Definition 2. The solution of the integral equation (1) depends continuously on the function f_1 if $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ such that

$$|f_1\big(t, x(x(t))\big) - f_1^*\big(t, x(x(t))\big)| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon \tag{10}$$

where x^* is the unique solution of equation

$$x^*(t) \ = \ a(t) \ + \int_0^{\phi_1(t)} f_1^*\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds, \qquad t \in [0, T].$$

Theorem 4. *Let the assumptions of Theorem 2 be satisfied, assume that*

$$|f_1\big(t, x(x(t))\big) - f_1^*\big(t, x(x(t))\big)| \leq \delta,$$

then the solution of (1) depends continuously on the functions f_1.

Proof. Let $|f_1\big(t, x(x(t))\big) - f_1^*\big(t, x(x(t))\big)| \leq \delta,$ then we obtain

$$
\begin{aligned}
|x(t) - x^*(t)| \ &= \ \Big|a(t) \ + \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
&- \ a(t) \ - \int_0^{\phi_1(t)} f_1^*\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
= \; & \Big| \int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
- \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
+ \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
- \; & \int_0^{\phi_1(t)} f_1^*\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \Big| \\
= \; & \Big| \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
\times \; & \Big[\int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big) ds - \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \Big] \\
+ \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \\
- \; & \int_0^{\phi_1(t)} f_1^*\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \\
+ \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \\
- \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \Big| \\
= \; & \Big| \int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds \Big[\int_0^{\phi_1(t)} f_1\big(s, x(x(s))\big) ds - \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \Big] \\
+ \; & \int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds \Big[\int_0^{\phi_2(t)} f_2\big(s, x(x(s))\big) ds - \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \Big] \\
+ \; & \int_0^{\phi_2(t)} f_2\big(s, x^*(x^*(s))\big) ds \; \Big[\int_0^{\phi_1(t)} f_1\big(s, x^*(x^*(s))\big) ds - \int_0^{\phi_1(t)} f_1^*\big(s, x^*(x^*(s))\big) ds \Big] \Big| \\
\leq \; & \int_0^{\phi_2(t)} |f_2\big(s, x(x(s))\big)| ds \int_0^{\phi_1(t)} |f_1\big(s, x(x(s))\big) - f_1\big(s, x^*(x^*(s))\big)| ds \\
+ \; & \int_0^{\phi_1(t)} |f_1\big(s, x^*(x^*(s))\big)| ds \int_0^{\phi_2(t)} |f_2\big(s, x(x(s))\big) - f_2\big(s, x^*(x^*(s))\big)| ds \\
+ \; & \int_0^{\phi_2(t)} |f_2\big(s, x^*(x^*(s))\big)| ds \int_0^{\phi_1(t)} |f_1\big(s, x^*(x^*(s))\big) - f_1^*\big(s, x^*(x^*(s))\big)| ds \\
\leq \; & \int_0^{\phi_2(t)} |f_2\big(s, x(x(s))\big)| ds \int_0^{\phi_1(t)} b_1 |x\big(x(s)\big) - x^*\big(x^*(s)\big)| ds \\
+ \; & \int_0^{\phi_1(t)} |f_1\big(s, x^*(x^*(s))\big)| ds \int_0^{\phi_2(t)} b_2 |x\big(x(s)\big) - x^*\big(x^*(s)\big)| ds \\
+ \; & \int_0^{\phi_2(t)} |f_2\big(s, x^*(x^*(s))\big)| ds \int_0^{\phi_1(t)} |f_1\big(s, x^*(x^*(s))\big) - f_1^*\big(s, x^*(x^*(s))\big)| ds.
\end{aligned}
$$

Using (5) and (6) we obtain

$$|x(t) - x^*(t)| \leq \gamma_2 b_1 (L+1)T\|x - x^*\| + \gamma_1 b_2 (L+1)T\|x - x^*\| + \gamma_2 T\delta,$$

$$\|x - x^*\|\big[1 - (\gamma_2 b_1 + \gamma_1 b_2)(L+1)T\big] \leq \gamma_2 T\delta$$

and

$$\|x - x^*\| \leq \frac{\gamma_2 T\delta}{1 - (\gamma_2 b_1 + \gamma_1 b_2)(L+1)T} = \epsilon.$$

□

Corollary 3. *Let the assumptions of Theorem 4 be satisfied. In Example 2 if* μ *changed to* μ^**, then the solution of equation (7) depends continuously on* μ *(the function* f_1*).*

4.3. Continuous dependence on the functions f_2

By the same way, as in Theorem 4 we can prove that the solution of equation (1) dependence continuously on the function f_2.

Theorem 5. *Let the assumptions of Theorem 2 be satisfied, assume that*

$$|f_2\big(t, x(x(t))\big) - f_2^*\big(t, x(x(t))\big)| \leq \delta,$$

then the solution of (1) depends continuously on the functions f_2.

Corollary 4. *Let the assumptions of Theorem 5 be satisfied. In Example 2 if* ρ *changed to* ρ^**, then the solution of equation (7) depends continuously on* ρ *(the function* f_2*).*

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