

ON THE PATH PARTITION OF GRAPHS

Mekkia Kouider and Mohamed Zamime

Communicated by Gyula O.H. Katona

Abstract. Let G be a graph of order n . The maximum and minimum degree of G are denoted by Δ and δ , respectively. The path partition number $\mu(G)$ of a graph G is the minimum number of paths needed to partition the vertices of G . Magnant, Wang and Yuan conjectured that

$$\mu(G) \leq \max \left\{ \frac{n}{\delta + 1}, \frac{(\Delta - \delta)n}{(\Delta + \delta)} \right\}.$$

In this work, we give a positive answer to this conjecture, for $\Delta \geq 2\delta$.

Keywords: path, partition.

Mathematics Subject Classification: 05C20.

1. INTRODUCTION

Throughout the paper, all graphs are finite, simple and undirected. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We denote by n the order of G . The *neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$. The *degree* of v , denoted by $d(v)$, is the size of its neighborhood. The *minimum degree* of the graph G is denoted by $\delta(G)$, and the *maximum degree* by $\Delta(G)$.

Let A and B be two subsets of $V(G)$. Let $\varepsilon(A, B)$ be the number of edges with one end vertex in the set A the other one in the set B .

In this work, we deal with the partition problem. The cover problem and the partition problem constitute a large and important class of well studied problems in the fields of graph theory. A *cycle cover* of a graph (resp. a *path cover*) is a set \mathcal{C} of cycles (resp. paths) of the graph such that each vertex belongs to at least one cycle (resp. one path) of \mathcal{C} . Many results on these concepts, have been given in the literature. For example, Kouider [5, 6], and Kouider and Lonc [7] studied the problem of covering a graph by a minimum number of cycles. More details and references can be found in the survey of Manuel [12].

Among the many variations of the partition problem, we mention the *path partition* that has been studied intensively for about sixty years. A family \mathcal{P} of paths is called

a *path partition* of a graph G if its members cover the vertices of the graph and are vertex disjoint. Its cardinality $|\mathcal{P}|$ is the number of paths of \mathcal{P} . The *path partition number* of G is

$$\mu(G) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a path partition of } G\}.$$

The concept of path partition number was introduced by Ore [13] in 1961. Several works have been done in this topic. See for example [1, 2, 4, 8, 9].

In 1996, Reed proved in [14] the following result.

Theorem 1.1 ([14]). *Let G be a connected cubic graph on n vertices. Then*

$$\mu(G) \leq \left\lceil \frac{n}{9} \right\rceil.$$

Furthermore, for 2-connected graphs, a better bound is established by Yu [15].

Theorem 1.2. *Let G be a 2-connected cubic graph on n vertices. Then*

$$\mu(G) \leq \left\lceil \frac{n}{10} \right\rceil.$$

For regular graphs, in 2009, Magnant and Martin [10] conjectured the following.

Conjecture 1.3 ([10]). *Let G be a d -regular graph on n vertices. Then*

$$\mu(G) \leq \frac{n}{d+1}.$$

They verified this last conjecture for the case $d \leq 5$ (see [10]). In 2018, Han obtained an asymptotic answer.

Theorem 1.4 ([4]). *For every c , $0 < c < 1$ and $\alpha > 0$, there exists n_0 such that if $n \geq n_0$, $d \geq cn$ and G is a d -regular graph on n vertices, then $n/(d+1)$ vertex-disjoint paths cover all vertices of G except αn .*

Gruskys and Letzter [3] improved this result by allowing to take $\alpha = 0$.

In 2016, Magnant, Wang and Yuan [11] extended Conjecture 1.3 to general graphs as follows.

Conjecture 1.5 ([11]). *Let G be a graph on n vertices. Then*

$$\mu(G) \leq \max \left\{ \frac{n}{\delta+1}, \frac{(\Delta-\delta)n}{(\Delta+\delta)} \right\}.$$

If true, the last conjecture would be sharp. For $\delta+2 \leq \Delta$, the bound is achieved by the collection of disjoint copies of $K_{\delta,\Delta}$. For $\delta = \Delta$, it is achieved by the collection of disjoint copies of complete graphs $K_{\delta+1}$. This conjecture is proved in [11] for the case $\delta = 1$ and $\delta = 2$.

In this work, we prove Conjecture 1.5 for all graphs with maximum degree Δ at least 2δ .

Theorem 1.6. *Let G be a graph of order n of minimum degree δ , ($\delta \geq 2$), and maximum degree Δ with $\Delta \geq 2\delta$. Then*

$$\mu(G) \leq \frac{(\Delta - \delta)n}{(\Delta + \delta)}.$$

We remark that $\frac{n}{\delta+1} \leq \frac{(\Delta-\delta)n}{(\Delta+\delta)}$ if and only if $\delta + 2 \leq \Delta$. So for $\delta \geq 2$ and $\Delta \geq 2\delta$, the inequality of the theorem is equivalent to

$$\mu(G) \leq \max \left\{ \frac{n}{\delta + 1}, \frac{(\Delta - \delta)n}{(\Delta + \delta)} \right\}$$

which is the inequality of Conjecture 1.5.

2. PRELIMINARIES

Let us introduce the following notations and definitions. Let \mathcal{P} be a minimum path partition of $V(G)$. So, $|\mathcal{P}| = \mu(G)$. Let p_i be the number of paths of order $i \in \{1, 2\}$ in \mathcal{P} .

We may suppose that $p_1 + p_2 \neq 0$, otherwise we have $\mu(G) \leq \frac{n}{3}$. As $\Delta \geq 2\delta$, we get $\mu(G) \leq \frac{(\Delta-\delta)n}{(\Delta+\delta)}$ and the problem is resolved.

Let V_1 be the set of isolated vertices of \mathcal{P} and V_2 be the set of end vertices of the isolated edges of \mathcal{P} . We denote by R any path in \mathcal{P} , and we write $R = R[a, b] = [a, \dots, b]$ if a and b are the end vertices of R . We set $End(R) = \{a, b\}$. Let $Int(R)$ be the set of internal vertices of R . Let $\mathcal{A} \subseteq \mathcal{P}$. We denote by $Int(\mathcal{A})$ (resp. $End(\mathcal{A})$) the set of internal (resp. end) vertices of the paths of \mathcal{A} . For i fixed, we denote by R_i any path of order i . We set \mathcal{R}_i the set of paths of order i . By $abcd$ or $[a, b, c, d]$ we denote a path with 4 vertices. For i odd, $i \geq 3$, let us set $C_i = \bigcup_{R \in \mathcal{R}_i} c(R)$, where $c(R)$ denotes the central vertex of the path R .

Example 2.1. Let us illustrate the above notations relative to a partition in Figure 1. We consider

$$\begin{aligned} \mathcal{R}_1 &= \{x_1\}, & \mathcal{R}_2 &= \{[x_2, x_3], [x_4, x_5], [x_6, x_7]\}, \\ \mathcal{R}_3 &= \{[x_8, \dots, x_9]\}, \\ \mathcal{R}_4 &= \{[x_{10}, \dots, x_{11}], [x_{12}, \dots, x_{13}]\}, \\ \mathcal{R}_5 &= \{[x_{14}, \dots, x_{15}], [x_{16}, \dots, x_{17}], [x_{18}, \dots, x_{19}], [x_{20}, \dots, x_{21}]\}, \\ End(\mathcal{R}_3) &= \{x_8, x_9\}, & End(\mathcal{R}_4) &= \{x_{10}, x_{11}, x_{12}, x_{13}\}, \\ End(\mathcal{R}_5) &= \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}\}, \\ C_3 &= \{w_1\}, & C_5 &= \{w_5, w_6, w_7, w_8\}. \end{aligned}$$

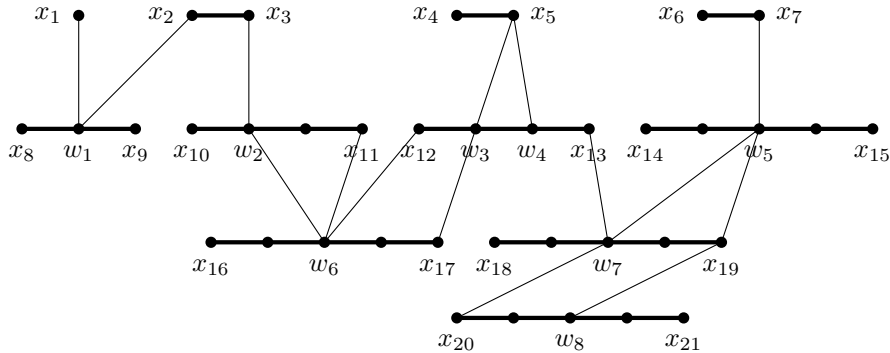


Fig. 1. Illustration of the definitions

For $x \in \text{End}(R)$, $N_{\text{ext}}(x)$ is the set of non path neighbors (neighbors of x outside its own path R) and $N_{\text{ext}}(X') = \bigcup_{x \in X'} N_{\text{ext}}(x)$ with $X' \subset \text{End}(\mathcal{P})$. Now using $N_{\text{ext}}(V_1 \cup V_2)$, we define a subset X of $\text{End}(\mathcal{P})$, and we denote $N_{\text{ext}}(X)$ by W . Let $X_1 = V_1 \cup V_2$, $W_1 = N_{\text{ext}}(x_1)$ and for $t \geq 1$, X_t being defined, let

$$X_{t+1} = X_t \cup \left(\bigcup_{N_{\text{ext}}(X_t) \cap \text{Int}(R) \neq \emptyset, R \in \mathcal{R}} \text{End}(R) \right).$$

Let $s \geq 1$ the first integer such that $X_s = X_{s+1}$. Let us set $X = X_s$, $W = N_{\text{ext}}(x)$ and for $t \in \{1, \dots, s\}$, let $W_{t+1} = N_{\text{ext}}(X_{t+1}) \setminus N_{\text{ext}}(X_t)$. Then $W = \bigcup_{i=1}^s W_i$. Here is an example of that construction.

Example 2.2. For the partition in Figure 1, we have

$$\begin{aligned} X_1 &= \{x_1, x_2, \dots, x_7\}, \\ X_2 &= X_1 \cup \{x_8, x_9, \dots, x_{15}\}, \\ X_3 &= X_2 \cup \{x_{16}, x_{17}, x_{18}, x_{19}\}, \\ X_4 &= X_3 \cup \{x_{20}, x_{21}\} = X, \\ W_1 &= \{w_1, w_2, \dots, w_5\}, \quad W_2 = \{w_6, w_7\}, \quad W_3 = \{w_8\}, \\ W &= \{w_1, w_2, \dots, w_8\}. \end{aligned}$$

Let $X_0 = \emptyset$. Pick $w_r \in W_r$ for some r . By definition of w_r , there exists a sequence

$$\alpha(w_r) = x_1 w_1, x_2 w_2, \dots, x_r w_r,$$

where for each $t \in \{1, \dots, r\}$, $x_t \in X_t - X_{t-1}$, $w_t \in W_t$ and $x_t w_t$ is an edge joining two paths of the partition. In addition, for each $t \in \{1, \dots, r-1\}$, w_t and x_{t+1} are in the same path of the partition.

The sequence $\alpha(w_r)$ has a good order if the vertex w_r belongs to a path R with end vertices, say x_{r+1} and x'_{r+1} , in $X_{r+1} - X_r$. The vertex w_r is then said to be of good order. Using a sequence $\alpha(w_r)$ with good order, we can define two new partitions as follows.

For each $i \in \{1, \dots, r + 1\}$, we orient the paths of \mathcal{P} such that each x_i is the terminal extremity. We denote by w_t^+ and w_t^- the successor and the predecessor of w_t , respectively.

(1) $\mathcal{P}_1(w_r)$ is obtained from \mathcal{P} by deleting the edges $w_t w_t^+$, $1 \leq t \leq r$ and adding the edges $x_t w_t$ for $1 \leq t \leq r$;

(2) $\mathcal{P}_2(w_r)$ is obtained from \mathcal{P} by deleting the edges $w_t w_t^+$, $1 \leq t \leq r - 1$ and the edge $w_r w_r^-$ and adding the edges $x_t w_t$ for $1 \leq t \leq r$.

If we consider the sets of edges of these partitions we note that

$$E(\mathcal{P}_2) = (E(\mathcal{P}_1) - w_r w_r^-) \cup w_r w_r^+.$$

Furthermore, $|\mathcal{P}_2| = |\mathcal{P}_1| = \mu(G)$.

For example, the sequence $\alpha(w_2)$ in the graph of Figure 2, defines two partitions. We have

$$\mathcal{P}_1(w_2) = \{x'_1 x_1 w_1 w_1^-, w_1^+ x_2 w_2 w_2^- x'_3, x_3 w_2^+, x'_4 w_3^- w_3 w_3^+ x_4\}$$

and

$$\mathcal{P}_2(w_2) = \{x'_1 x_1 w_1 w_1^-, w_1^+ x_2 w_2 w_2^+ x_3, w_2^- x'_3, x'_4 w_3^- w_3 w_3^+ x_4\}.$$

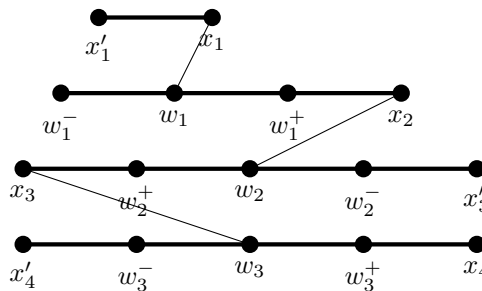


Fig. 2. Graph with $\mu(G) = 4$

We denote by $R_i[x', x]$ any path of order i oriented from x' to x . So x' is the initial end of R_i and x is its terminal end.

Observation 2.3. If $w \in W_i$, then for some i , w belongs to some path $R_i[x', x]$. The path $R_{i_1}[w^+, x]$ is in $\mathcal{P}_1(w)$ and the path $R_{i_2}[x', w^-]$ is in $\mathcal{P}_2(w)$. Note that the subpath $R_{i_1}[w^+, x]$ (resp. $R_{i_2}[x', w^-]$) is of order i_1 (resp. i_2) such that $i_1 + i_2 + 1 = i$.

3. PROOF OF THEOREM 1.6

We choose a minimum path-partition \mathcal{P}_0 such that:

- (1) p_1 is minimum,
- (2) if (1) is satisfied, then p_2 is minimum.

Let \mathcal{P}' be the set of paths with end vertices in X . Let $p = |\mathcal{P}'|$. Let p_i be the number of paths of order i in \mathcal{P}' .

Let us outline the sketch of the proof.

In view to bound $\mu(G)$ we want to bound p_1 and p_2 . We consider the set X generated by $V_1 \cup V_2$, and therefore the two sets $W = N_{ext}(X)$ and $\varepsilon(X, W)$. Note that the cardinality of X is $2p - p_1$. The proof of our theorem is done through the following steps. We want to bound in two manners the number of edges $\varepsilon(X, W)$. The upper bound will use W and Δ , the lower bound will use X and δ .

In the first part of the proof, we show some claims relative to the set W and one relative to the lower bound of $\varepsilon(x, W)$ for $x \in X$.

In the second part of the proof, we calculate the bounds of $\varepsilon(X, W)$. We get finally an upper bound for $p_1 + 2p_2$ in function of p, δ and Δ , and, then an upper bound for $\mu(G)$.

3.1. CLAIMS

Claim 3.1.

- (1) For each $v \in V_1$, $N(v) \subset C_3$.
- (2) For each $a \in V_2$, $N(a) \subset C_3 \cup \text{Int}(\mathcal{R}_4) \cup C_5$.

$$So, \begin{cases} |N(a) \cap R| \leq 1, & \text{for every path } R \text{ of order 3 or 5,} \\ |N(a) \cap R| \leq 2, & \text{for every path } R \text{ of order 4.} \end{cases}$$

Proof. (1) Let v be a vertex of V_1 . By the minimality of \mathcal{P}_0 , v is not adjacent to an end vertex of another path in \mathcal{P}_0 . Let w be a neighbor of v in a path oriented from x' to x . We have two partitions. In \mathcal{P}_0 , we replace the path v and the path $R_i[x', x]$ either by the pair of paths $vR_{i_1}[w, x]$, $R_{i_2}[x', w^-]$ or by the pair of paths $R_{i'_1}[x', w]v$, $R_{i'_2}[w^+, x]$. By the minimality of p_1 , w is both predecessor of x and successor of x' . So the order of $R_i[x', x]$ is 3, and w is the center of $R_i[x', x]$. Thus, $N(v) \subset C_3$.

(2) Let w be a neighbor of a in $R_i[x', x]$. As precedently, we get two partitions and by definition of \mathcal{P}_0 , each of $R_i[x', w^-]$ and $R_i[w^+, x]$ should be of order at most two. So $N(a) \subset C_3 \cup C_5 \cup \text{Int}(\mathcal{R}_4)$, completing the proof of Claim 3.1. \square

Claim 3.2. Let W_a be the set of vertices of good order in W . Then:

- (1) $W_a \subset C_3 \cup \text{Int}(\mathcal{R}_4) \cup C_5$,
- (2) $W = W_a$.

Proof. (1) Suppose that there exists $w \in W_a$ such that w is in the path $R_i[x', x]$. By Observation 2.3, we have $i - 1 = i_1 + i_2$. If $i_1 \geq 3$ (resp. $i_2 \geq 3$), then $\mathcal{P}_1(w)$ (resp. $\mathcal{P}_2(w)$) contains $p_1 - 1$ paths of order 1 or p_1 paths of order 1 and $p_2 - 1$ paths of order 2. A contradiction with the definition of \mathcal{P}_0 . Thus, $W_a \subset C_3 \cup \text{Int}(\mathcal{R}_4) \cup C_5$.

(2) Suppose that $W \neq W_a$. In $W_b - W_a$ there exists necessarily a vertex $w = w_r$ with sequence $\alpha(w) = x_1w_1, x_2w_2, \dots, x_rw_r$ with $x_t \in X_t - X_{t-1}$ and x_r is an end vertex of some path $R = [x'_r, \dots, x_r]$. By the definition of w , the vertex w belongs to a path $R' = [x'_j, \dots, x_j]$ with $x_j \in X_j, j \leq r$. By the definition of X_j , the path R' contains one element of W_a , say w_a . By the definition of W_a , w_a is adjacent to a vertex x'' of X_{j-1} , end vertex of a path R'' . Since $W_a \subset C_3 \cup \text{Int}(\mathcal{R}_4) \cup C_5$, then $|R'| = 4$ or 5 and $w = w_a^-$ or $w = w_a^+$. The end vertex $x_r \in X_r$ and the end vertex $x'' \in X_{j-1}$ are adjacent respectively to w and w_a , successive vertices of the same path R' .

We get a partition with $p - 1$ paths. We replace the three paths R, R' and R'' by the two paths composed by $R \cup R' \cup R'' \cup \{x_rw, x''w_a\} - \{ww_a\}$ (see Figure 3), a contradiction with the minimality of \mathcal{P}_0 . Thus, $W_b = \emptyset$ and so, $W = W_a$. This completes the proof of Claim 3.2.

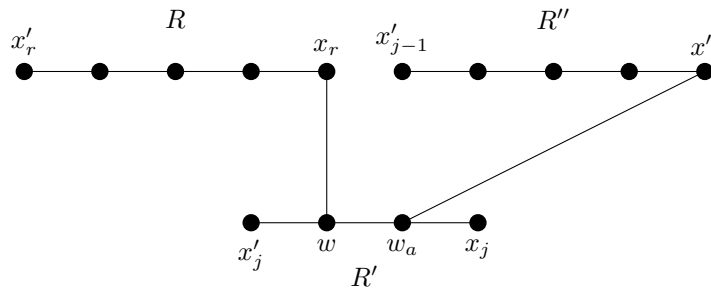


Fig. 3. Paths R, R', R''

□

Claim 3.3. For each path R of order 4 in \mathcal{P}_0 , we have $|W \cap V(R)| \leq 2$. Furthermore, if $|W \cap V(R)| = 2$, then there is a unique $x \in X$ such that $W \cap V(R) \neq \emptyset$ and thus $\varepsilon(X, R) = 4$.

Proof. By the minimality of \mathcal{P}_0 for each path $R \in \mathcal{R}_4$, we have $|W \cap V(R)| \leq 2$. Now, assume that there exists a path $R \in \mathcal{R}_4$ such that $W \cap V(R) = \{w_1, w_2\}$ where $w_i \in N_{ext}(x_i), i = 1, 2$. By taking off the edge w_1w_2 and adding the edges x_1w_1 and x_2w_2 we obtain a partition with $p - 1$ paths, a contradiction. So $x_1 = x_2$. Let $R'(x_1)$ be the path of extremity x_1 in the partition. We may suppose $R = [x'_0, w_1, w_2, x_0]$. If x_0 is neighbor of w_1 , then we replace the two paths R and $R'(x_1)$ by the path $R'(x_1) \cup [w_2, x_0, w_1, x'_0]$. We get a partition with $p - 1$ paths, a contradiction. So, there is no edge x_0w_1 . Similarly, there is no edge x'_0w_2 . It follows that $\varepsilon(X, R) = 4$, completing the proof of Claim 3.3. □

For $i \in \{1, 2\}$, let \mathcal{R}_4^i be the set of paths of order 4, which contain exactly i elements of W .

For the lower bound we shall need the following claim.

Claim 3.4.

- (1) If $x \in X \cap V(\mathcal{R}_1 \cup \mathcal{R}_3)$, then x has $d(x)$ neighbors in W .
- (2) If $x_1, x_2 \in X \cap V(\mathcal{R}_4)$, then the set $\{x_1, x_2\}$ has $d(x_1) + d(x_2) - 1$ neighbors in W if x_1, x_2 belong to $V(\mathcal{R}_4^1)$, and it has $d(x_1) + d(x_2)$ neighbors if x_1, x_2 belong to $V(\mathcal{R}_4^2)$.
- (3) If $x \in X \cap V(\mathcal{R}_2 \cup \mathcal{R}_5)$, then x has $d(x) - 1$ neighbors in W .

Proof. (1) If $x \in V_1$, then by Claim 3.1, $N(x) \subset W$. If $x \in X \cap V(\mathcal{R}_3)$, then let $R = xyx'$. By the minimality of μ , x and x' are not adjacent. It follows that $N(x) \subset W$.

(2) Let $R' = x_1yx_2$. By the minimality of p , there is no edge x_1x_2 .

First assume that $W \cap R' = \{y\}$. By definition of W , y is neighbor of an end vertex, say x of path $R = [x', \dots, x]$. If $x_1z \in E(G)$, then we replace the two paths R and R' by exactly one path $x_2zx_1yx \dots x'$, a contradiction with the minimality of the path partition. It follows that $N(x_1) \subset W$. Note that $(N(x_2) - \{z\}) \subset W$.

Now assume that $W \cap V(R') = \{y, z\}$, then $N(x_1) \subset W$ and $N(x_2) \subset W$.

(3) Clearly if $x \in V_2$, then x has $d(x) - 1$ neighbors in W . Let $R'' = x_1yzt_2x_2$. By Claim 3.2, we have $W \cap V(R'') = \{z\}$. It follows that $N(x_1) - \{y\} \subset W$ and $N(x_2) - \{t\} \subset W$, completing the proof of Claim 3.4. □

Now we bound first $|\mathcal{P}'|$ then $\mu(G)$.

3.2. CALCULATIONS OF THE BOUNDS

- (1) The bound of $p_1 + 2p_2$.

Let $k = \frac{\Delta}{\delta}$. We shall prove the following inequality.

Claim 3.5.

$$p_1 + 2p_2 \leq (p_3 + p_4 + p_5)(k - 2) + \frac{2}{\delta}p_2,$$

where p_i is the number of paths of order i in \mathcal{P}' .

Proof. Put $p'_4 = |\mathcal{R}_4^1|$ and $p''_4 = |\mathcal{R}_4^2|$. Let $\varepsilon(X, W)$ be the number of edges between X and W . Let w be any vertex of W . Observe that for $w \in \text{Int}(\mathcal{R}_5)$, w has at most $\Delta - 2$ neighbors in X . For $w \in \text{Int}(\mathcal{R}_4^1)$, then w has at least one neighbor which does not belong to X and so, w has at most $\Delta - 1$ neighbors in X . By Claim 3.3, if $w \in \text{Int}(\mathcal{R}_4^2)$, then w has exactly two neighbors in X . If $w \in W \cap \text{Int}(\mathcal{R}_3)$ then w has at most Δ neighbors in X . By Claims 3.1 and 3.2, for each $i \geq 6$, $W \cap \text{Int}(\mathcal{R}_i) = \emptyset$ and $X \cap \text{End}(\mathcal{R}_i) = \emptyset$. It follows that

$$\varepsilon(X, W) \leq p_3\Delta + p'_4(\Delta - 1) + 4p''_4 + p_5(\Delta - 2).$$

On the other hand, by Claim 3.4,

$$\begin{aligned} \varepsilon(X, W) &\geq \sum_{x \in \text{End}(\mathcal{R}_1)} d(x) + \sum_{x \in \text{End}(\mathcal{R}_2)} (d(x) - 1) \\ &\quad + \sum_{x \in \text{End}(\mathcal{R}_3) \cap X} d(x) \\ &\quad + \sum_{x_1, x_2 \in \text{End}(\mathcal{R}_4^1) \cap X} (d(x_1) + d(x_2) - 1) \\ &\quad + \sum_{x_1, x_2 \in \text{End}(\mathcal{R}_4^2) \cap X} (d(x_1) + d(x_2)) \\ &\quad + \sum_{x \in \text{End}(\mathcal{R}_5) \cap X} (d(x) - 1). \end{aligned}$$

As for each $x \in X$ we have $\delta \leq d(x)$, it follows that

$$\begin{aligned} p_1 \delta + 2p_2(\delta - 1) + 2p_3 \delta + p'_4(2\delta - 1) + p''_4 2\delta + 2p_5(\delta - 1) \\ \leq p_3 \Delta + p'_4(\Delta - 1) + 4p''_4 + p_5(\Delta - 2). \end{aligned}$$

As $k \geq 2$ and $\delta \geq 2$, then $\Delta \geq 4$ and we replace $4p''_4$ by $\Delta p''_4$ in the last inequality. Then

$$\begin{aligned} p_1 \delta + 2p_2(\delta - 1) + 2p_3 \delta + p'_4(2\delta - 1) + p''_4 2\delta + 2p_5(\delta - 1) \\ \leq p_3 \Delta + p'_4(\Delta - 1) + p''_4 \Delta + p_5(\Delta - 2). \end{aligned}$$

As $p_4 = p'_4 + p''_4$, we get

$$p_1 \delta + 2p_2(\delta - 1) \leq p_3(\Delta - 2\delta) + p_4(\Delta - 2\delta) + p_5(\Delta - 2\delta).$$

Since $\Delta = k\delta$, then

$$p_1 + 2p_2 \leq (p_3 + p_4 + p_5)(k - 2) + \frac{2}{\delta} p_2.$$

This completes the proof of Claim 3.5. □

(2) *Calculation of the bound of $|\mathcal{P}'| = p$.*

By Claim 3.5, there exists $r \leq k - 2$, such that

$$p_1 + 2p_2 = r(p_3 + p_4 + p_5) + \frac{2}{\delta} p_2. \tag{3.1}$$

Let us call n_1 the order of $V(\mathcal{P}')$. Recall that \mathcal{P}' is the set of paths with end vertices in X , $p = |\mathcal{P}'|$ and p_i is the number of paths of order i in \mathcal{P}' . By Claims 3.1 and 3.2,

\mathcal{P}' contains no path of order at least 6. It follows that

$$p = p_1 + p_2 + p_3 + p_4 + p_5$$

and

$$n_1 = p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5.$$

Using equality (3.1), we obtain

$$p = (r + 1)(p_3 + p_4 + p_5) + \left(\frac{2}{\delta} - 1\right)p_2.$$

As $\delta \geq 2$, we get $p \leq (r + 1)(p_3 + p_4 + p_5)$. Again by equality (3.1), we have

$$n_1 = (r + 3)p_3 + (r + 4)p_4 + (r + 5)p_5 + \frac{2}{\delta}p_2.$$

This yields

$$n_1 \geq (r + 3)(p_3 + p_4 + p_5).$$

Since $r \leq (k - 2)$, we get the inequality $p \leq \frac{k-1}{k+1}n_1$.

(3) *The bound of $\mu(G)$.*

Let $G_2 = G - V(\mathcal{P}')$. Let $n_2 = n - n_1$. Clearly $\mu(G) \leq p + \mu(G_2)$. We know that $p \leq \frac{k-1}{k+1}n_1$. Recall that each path of \mathcal{P}_0 contained in G_2 has order at least 3. It follows that $\mu(G_2) \leq \frac{n_2}{3}$. Since $k \geq 2$, we have $\frac{1}{3} \leq \frac{k-1}{k+1}$ and so $\mu(G_2) \leq \frac{k-1}{k+1}n_2$. Thus, $\mu(G) \leq \frac{k-1}{k+1}n$. This finishes the proof of the theorem.

Acknowledgements

The authors are grateful to the anonymous referee whose careful reading gave them the opportunity to improve the quality of the paper.

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Mekhia Kouider (corresponding author)
km@lri.fr

University Paris Sud
France

Mohamed Zamime
zamimemohamed@yahoo.com

University Yahia Fares of Medea
Department of Technology
Mathematics Laboratory and its Applications
Medea, Algeria

Received: January 17, 2023.

Revised: June 24, 2023.

Accepted: July 7, 2023.