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Static Large Games with Finitely Many Types of Players and Strategies

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Abstract. The paper briefly presents a theory of games with finitely many infinite populations (types) each of whom has finitely many available strategies; the payoff of an individual player depends on the distribution of choices of strategies in all populations and his own particular choice. We give specific examples of applications of the theory in several areas: spatial allocation (of species), economic models – household economy and transportation networks. We also briefly discuss questions of computation of equilibria and relations of large games, as understood in the present paper, to ordinary matrix games, games with continuum of players and evolutionary game theory.

Keywords and phrases: large game, equilibrium, strategy, type of player, spatial allocation, household economy, road traffic model, directed graph, Fibonacci numbers

Statyczne duże gry ze skończoną liczbą typów graczy i strategii

Streszczenie. Praca zawiera skrótowy opis gier ze skończoną liczbą nieskończonych populacji (typów), z których każda dysponuje skończoną liczbą strategii; wypłata pojedynczego gracza zależy od rozkładu wyborów strategii wszystkich populacji i jego własnego wyboru strategii. Podajemy konkretne przykłady zastosowań w kilku dziedzinach: alokacja przestrzenna (gatunków), modele ekonomiczne – model gospodarki drobnotowarowej i model ruchu drogowego. Dyskutujemy też krótko zagadnienia obliczeniowe i związki dużych gier, jak przedstawiono je w tym artykule, ze zwykłymi grami macierzowymi, grami z continuum graczy i z ewolucyjną teorią gier.

Słowa i frazy kluczowe: duża gra, równowaga, strategia, typ gracza, alokacja przestrzenna, gospodarka drobnotowarowa, model ruchu drogowego, graf skierowany, liczby Fibonacciego

1. Introduction

„Large games” is a concept intended to formalize and create precise mathematical background to study situations involving a large number of participants undertaking decisions of importance for them. Obviously, we are often faced with such situations in scientific research and usually in cases where some data is presented in an aggregate way; for instance, if we describe an economy involving millions of households then we do not give a complete description of those households but we use a sort of aggregation. We rather classify the considered households into types, for instance according to their size, location etc. We then assume that the so classified objects (households) have some common features and they usually may undertake similar actions resulting in similar outcomes. This is to be understood as a large game: a finite number of infinite uniform populations; the payoff of an individual player depends on his own decision and the distributions of the decisions chosen by all populations. Allowing the populations to be infinite also gives a chance to use a broader arsenal of mathematical tools.

More formally:

in an ordinary noncooperative game, the players′ payoffs are defined for sequences of pure strategies and have the form: $F^i: S^1 \times \cdots \times S^n \to \mathbb{R}$;

in the mixed extension, the players′ payoffs are defined for sequences of mixed strategies and have the form: $F^i: \Delta_{k_1} \times \cdots \times \Delta_{k_n} \to \mathbb{R}$; (the Δ_s to be defined soon);

in a large game, the players′ payoffs are defined for individual decisions and sequences of the types′ strategy distributions (which are identical with mixed strategies, but interpreted differently) and have the form: $F^i: S^i \times \Delta_{k_1} \times \cdots \times \Delta_{k_n} \to \mathbb{R}$.

In this paper we briefly present a theory of large games with finitely many (usually pairwise differing) infinite populations and applications of this theory in the areas of biology (spatial allocation of species), economics (household economy) and engineering and computer science (flows in networks). Partly it is a survey of previously published results which are however rearranged and presented in a unified manner. The paper also presents new results, especially in parts concerning transportation networks (Section 6).

We also briefly discuss questions of computation of equilibria and relations of large games, as understood in the present paper, to ordinary matrix games, games with continuum of players and evolutionary game theory.

Actually, the paper describes a procedure which can be generally applied to prove the existence of equilibria in models arising in various disciplines of science. This procedure formally describes what is actually often done in practice: aggregation and processing of the data. The procedure is mainly applicable in situations where the number of decision subjects is very large and: (1) they can only influence their own payoff but not the payoffs of the others; (2) a precise description of characteristics of all individuals is impossible or simply senseless.

2. Notation and basic definitions

$|V|$ – the number of elements of a finite set V;
χ – the characteristic function, i. e. $\chi(A) = 1$ if A is true and otherwise $\chi(A) = 0$;
for any ordered pair $e = (v,w)$ we denote $\tilde{e} = (w,v)$;
$(A_t|t\in T))$ – a family indexed by elements of the set T;
Δ_n – the $(n–1)$-dimensional *standard simplex*, i. e. $\Delta_n = \{x\in\mathbb{R}^n{}_+ \mid \sum_i x_i = 1\}$;
$<x; y>$ – the *inner product* of x and y;
supp p – the support of p, i. e. the set of indices i for which p_i is different from 0;
$x \leq y$, applied to vectors x, y, means $x_i . y_i$ for all i;
x^T denotes the transposition of the vector x.
Generally, upper indices are used to numerate players, lower indices – to numerate strategies.

A *directed graph* is a pair $G = (V, E)$ such that V is a finite set (of objects called *vertices*) while E is a set of ordered pairs of distinct elements of V (the elements of E are called *edges*).
A *undirected graph* is a directed graph in which for all $(v,w)\in V^2$, $(v,w)\in V$ implies $(w,v)\in V$.
We assume in this paper that graphs have no *loops* (edges of the form (v, v)). A *path* in a graph G is a sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that $(v_{i–1}, v_i)\in E$ for $i = 1, \ldots, n$; v_0 is a *beginning* while v_n is an *end* of the path. We say that a graph is *connected* if for every distinct vertices v and w there exists a path beginning at v and ending at w. A path is *straight* if all v_i, $i = 0, \ldots, n$, are different; its *length* is n. A path is a *cycle* if all v_i, $i = 1, \ldots, n$, are different and $v_0 = v_n$. We denote by $\mathcal{P}(v,w)$ the set of all straight paths beginning at v and ending at w; this set is obviously finite.
The graph which can be arranged into a path in which there are no other edges is a *chain*.
If (v,w) belongs to E then w is a *neighbor* of v; the set of all neighbors of v is denoted by Nb(v).

3. Large games (with finitely many types of players and strategies)

A *large game* is a system $\Gamma = (G, (S^g|g\in G), (F^g|g\in G))$, where G is a finite set of the *types* of players, for $g\in G$, S^g is an at least two-element finite set of *strategies* available for players of type g, F^g: $S^g \times \boldsymbol{S} \to \mathbb{R}$ are *payoff* functions of players of type g, for $g\in G$; in this case $\boldsymbol{S}$ denotes the product $\Pi_{g\in G}\, \Delta_{|S^g|}$.

A *profile* in a large game Γ is an element of $\boldsymbol{S} = \Pi_{g\in G}\, \Delta_{|S^g|}$. An *equilibrium* in a large game Γ is a profile $\mathbf{p} = (p^g|g\in G)$ such that for every $g\in G$, for all $s\in$supp p^g and all $s' \in S^g$, there is $F^g(s'; \mathbf{p}) . F^g(s; \mathbf{p})$.

Remark. *A profile* $\mathbf{p} = (p^g|g\in G)$ *is an equilibrium if and only if there exist numbers* $(C^g|g\in G)$ *such that for all g and i belonging to the support of* p^g, $F^g(i;\mathbf{p}) = C^g$ *and for the remaining i*, $F^g(i;\mathbf{p}) . C^g$. □

THEOREM 1. *Let a game* $\Gamma = (G, (S^g|g\in G), (F^g|g\in G))$ *be given. If for all types g and all strategies* $s\in S^g$, *the functions* $F^g(s; \cdot)$ *are continuous then the game has an equilibrium*.

The **proof** of this theorem relies on a direct application of the Kakutani Theorem. The details can be found in Wieczorek [2004]. □

4. Application: spatial allocation

Here we are interested in problems in which members of several populations (species) have to select a habitat from a set of possible locations. So, jointly each population generates a frequency distribution on the set of all available locations formally, it is a vector p having $|V|$ nonnegative coordinates which sum up to 1. The payoff of each individual depends on his own decision and distributions of the choice of all populations.

Neighborhood games

In the neighborhood games the payoffs only depend on the location of the given individual and the volumes of settlements of all populations in this and in the neighboring (second-order neighboring etc.) locations.

Formally, in this paper we understand a *single-population neighborhood game* as follows:

There is given an undirected connected graph $G = (V, E)$ and a real number α.

The payoff of a player who chose a vertex v while a generated distribution was p, is equal to

$$F(v;\mathbf{p}) = p_v + \alpha \cdot \Sigma_{w \in \mathrm{Nb}(v)}\, p_w.$$

Such a game will be denoted by $[G, \alpha]$.

Examples of equilibria in single-population neighborhood games

Example 1. (Equilibria in $[\mathrm{I}_n, \frac{1}{3}]$.) Game of single-population with underlying graph being a chain I_n (with n vertices $v_1, v_2, \ldots, v_n$), $\alpha = \frac{1}{3}$, so we have the game $[\mathrm{I}_n, \frac{1}{3}]$.

So the payoff of individuals who chose v_1 is equal to $p_1 + \frac{1}{3} \cdot p_2$; the payoff of those who chose v_n is equal to $p_n + \frac{1}{3} \cdot p_{n-1}$; those who chose one of the remaining v_i will receive $p_i + \frac{1}{3} \cdot (p_{i-1} + p_{i+1})$.

Finding a full support equilibrium in $[\mathrm{I}_n, \frac{1}{3}]$

We present a method to construct an equilibrium whose support are all vertices. (There is exactly one such equilibrium.) The illustration is given for the case of $n = 7$; for the other n's the construction is analogous.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
Step 1.	1	1	2	3	5	8	13
Step 2.	13	8	5	3	2	1	1
Step 3.	13	8	10	9	10	8	13
Step 4.	$\frac{13}{71}$	$\frac{8}{71}$	$\frac{10}{71}$	$\frac{9}{71}$	$\frac{10}{71}$	$\frac{8}{71}$	$\frac{13}{71}$

In step 1 we assign consecutive Fibonacci numbers to consecutive vertices.
In step 2 the same numbers are written in the reversed order below.
In step 3 the two numbers above are multiplied.
In step 4 we simply normalize the sequence in the previous row, multiplying the numbers by the inverse of their sum.

So we found a distribution $\left(\frac{13}{71}, \frac{8}{71}, \frac{10}{71}, \frac{9}{71}, \frac{10}{71}, \frac{8}{71}, \frac{13}{71}\right)$ which occurs to be an equilibrium in the game $[I_7, \frac{1}{3}]$. To prove it, we check that the payoff, at the choice of any v_i, $i = 1, \ldots, 7$, is the same and it is equal to $\frac{47}{213}$.

The last step 4 can be skipped if we intend to execute the procedure in the next paragraph.

Finding all equilibria in $[I_n, \frac{1}{3}]$

We first note that for every nonempty subset S of the set of all vertices there is exactly one equilibrium with support equal to S.

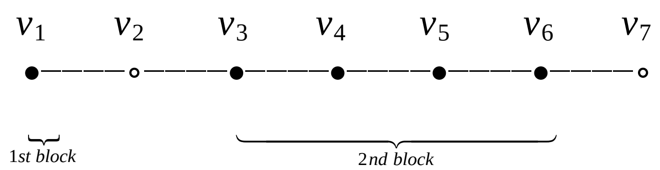
Step 1. We find equilibria of the games $[I_i, \frac{1}{3}]$, $i = 1, 2, \ldots, n$, with full supports, in the analogous manner as it was done in the previous section.
Step 2. We identify all maximal connected subsets of S, we call them blocks.
Step 3. We select a support and find the blocks which constitute it. We assign consecutive numbers of respective equilibria to consecutive elements of the blocks. (This step must be repeated for each potential support; their number is obviously 2^n-1.)
Step 4. We then normalize the numbers assigned to the blocks so as to make the payoffs corresponding to all blocks equal.
Step 5. Finally, we normalize again so as to get the sum of all numbers equal 1.

To illustrate this procedure, we get back to the example of $[I_7, \frac{1}{3}]$; we find

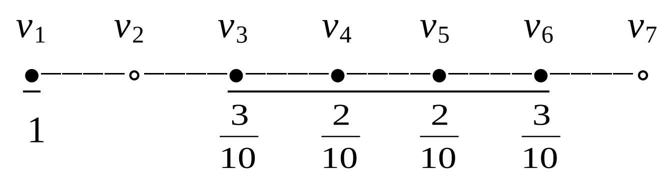
the equilibrium $\left(\frac{8}{38}, \frac{5}{38}, \frac{6}{38}, \frac{6}{38}, \frac{5}{38}, \frac{8}{38}\right)$ for $[I_6, \frac{1}{3}]$ with the payoff $\frac{29}{114}$;
the equilibrium $\left(\frac{5}{20}, \frac{3}{20}, \frac{4}{20}, \frac{3}{20}, \frac{5}{20}\right)$ for $[I_5, \frac{1}{3}]$ with the payoff $\frac{3}{10}$;
the equilibrium $\left(\frac{3}{10}, \frac{2}{10}, \frac{2}{10}, \frac{3}{10}\right)$ for $[I_4, \frac{1}{3}]$ with the payoff $\frac{11}{30}$;
the equilibrium $\left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$ for $[I_3, \frac{1}{3}]$ with the payoff $\frac{7}{15}$;
the equilibrium $\left(\frac{1}{2}, \frac{1}{2}\right)$ for $[I_2, \frac{1}{3}]$ with the payoff $\frac{2}{3}$;

the equilibrium (1) for $[I_1, \frac{1}{3}]$ with the payoff 1.

We select a subset of the set of all vertices and identify the blocks which constitute it. For example, we take into account the set of vertices $W = \{v_1, v_3, v_4, v_5, v_6\}$. So we have



and we identify the blocks: $\{v_1\}$ and $\{v_3,v_4,v_5,v_6\}$. We then assign the coefficients associated with respective equilibria to consecutive blocks:



We then normalize the terms assigned to respective blocks so as to get equal payoffs in consecutive blocks:

$$\frac{11}{30} \qquad \frac{3}{10} \quad \frac{2}{10} \quad \frac{2}{10} \quad \frac{3}{10}$$

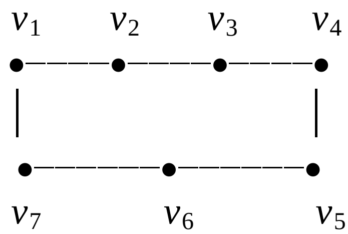
and finally we normalize once again to obtain a probability distribution:

$$\frac{11}{41} \qquad \frac{9}{41} \quad \frac{6}{41} \quad \frac{6}{41} \quad \frac{9}{41}.$$

So the constructed (unique) equilibrium with the support $\{v_1, v_3, v_4, v_5, v_6\}$ is $\left(\frac{11}{41}, 0, \frac{9}{41}, \frac{6}{41}, \frac{6}{41}, \frac{9}{41}, 0\right)$.

Example 2. (Equilibria in $[O_n, \frac{1}{3}]$.) The situation is similar in the case of neighborhood games on graphs being cycles; the difference is only in the case where the support of an equilibrium is to be the set of all vertices; in this case the equilibrium is the distribution whose all terms are equal to $|V|^{-1}$. The symbol O_n denotes the n-element cycle.

In the case of the game $[O_7, \frac{1}{3}]$ (the graph is drawn below)



this equilibrium is equal to $\left(\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}\right)$.

Finding all equilibria in games $[\mathrm{I}_n, \alpha]$ **and** $[\mathrm{O}_n, \alpha]$ **for** $\alpha \in [0, \frac{1}{2})$

Finding all full support equilibria in $[\mathrm{L}_n, \alpha]$ can be proceeded as follows:
For any real number A we first define an infinite sequence, to be called *generalized Fibonacci sequence with parameter A* = 2.
The sequence $(F^A{}_1, F^A{}_2, F^A{}_3, \ldots)$ is defined as follows: $F^A{}_1 = F^A{}_2 = 1$; then, having already defined $F^A{}_1, F^A{}_2, \ldots, F^A{}_{k-1,}$ we define $F^A{}_k = A{\cdot}F^A{}_{k-1} + F^A{}_{k-2}$ for odd k and $F^A{}_k = F^A{}_{k-1} + F^A{}_{k-2}$ for even k.
In the next step we define, for natural positive k sequences $(f^{Ak}{}_1, \ldots, f^{Ak}{}_k)$ as follows:
if k is even then, for $i = 1, \ldots, k$, $f^{Ak}{}_i = F^A{}_i \cdot F^A{}_{k+1-i}$;
if k is odd then, for odd $i = 1, 3, 5, \ldots, k$, $f^{Ak}{}_i = F^A{}_i \cdot F^A{}_{k+1-i}$;
if k is odd then, for even $i = 2, 4, 6, \ldots, k-1$, $f^{Ak}{}_i = A{\cdot}F^A{}_i \cdot F^A{}_{k+1-i}$.
The so defined sequence $(f^{Ak}{}_1, \ldots, f^{Ak}{}_k)$ is called a *standard A-sequence of length k.*
The *normalized standard A-sequence of length k* is the sequence

$$\mathrm{N}(f^{Ak}{}_1, f^{Ak}{}_2, \ldots, f^{Ak}{}_k) = (B{\cdot}f^{Ak}{}_1, B{\cdot}f^{Ak}{}_2, \ldots, B{\cdot}f^{Ak}{}_k), \text{ where } B = f^{Ak}{}_1 + f^{Ak}{}_2 + \cdots + f^{Ak}{}_k.$$

THEOREM 2. *We are given a game* $[\mathrm{I}_n, \alpha]$ with $\alpha \in (0, \frac{1}{2})$. *The unique equilibrium with full support is the normalized standard A-sequence of length n, where* $A = 1/\alpha - 2$.

Details of the **proof** can be found in Wieczorek [2009].□

To find all equilibria in $[\mathrm{I}_n,\alpha]$ and $[\mathrm{O}_n,\alpha]$ we must mimic the procedure previously performed to find all equilibria in $[\mathrm{I}_n, \frac{1}{3}]$ and $[\mathrm{O}_n, \frac{1}{3}]$: we choose we first identify all blocks, then we assign the already found equilibrium for the respective blocks, then we proportionally adopt the payoffs in all blocks and finally we normalize the obtained sequence to have the sum of its all elements equal to 1.

Example 3. The generalized Fibonacci sequence with parameter $A = 2$ is
(1, 1, 3, 4, 11, 15, 41, 56, 153, 209, …). The sum of the first 6 elements of this sequence is 76; the sum of the first 7 elements is 240.
For $A = 1$ we get the ordinary Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, …).

Example 4. Equilibria in $[\mathrm{I}_6, \frac{1}{4}]$. The standard A-sequence of length 6,

$$(f^{A,6}{}_1, f^{A,6}{}_2, f^{A,6}{}_3, f^{A,6}{}_4, f^{A,6}{}_5, f^{A,6}{}_6),$$

is equal to

$$(F^A{}_1 \cdot F^A{}_6,\ F^A{}_2 \cdot F^A{}_5,\ F^A{}_3 \cdot F^A{}_4,\ F^A{}_4 \cdot F^A{}_3,\ F^A{}_5 \cdot F^A{}_1,\ F^A{}_6 \cdot F^A{}_1);$$

for $A = 2$ this sequence becomes (15, 11, 12, 12, 11, 15) and after normalization $\left(\frac{15}{76},\frac{11}{76},\frac{12}{76},\frac{12}{76},\frac{11}{76},\frac{15}{76},\right)$. By Theorem 2 this is the unique equilibrium with full support in $[\mathrm{I}_6,\frac{1}{4}]$.

Example 5. Equlibria in $[\mathrm{I}_7,\frac{1}{4}]$. The standard A-sequence of length 7,

$$(f^{A,7}{}_1, f^{A,7}{}_2, f^{A,7}{}_3, f^{A,7}{}_4, f^{A,7}{}_5, f^{A,7}{}_6, f^{A,7}{}_7),$$

is equal to

$$(F^A{}_1 \cdot F^A{}_7,\ A\cdot F^A{}_2 \cdot F^A{}_6,\ F^A{}_3 \cdot F^A{}_5,\ A\cdot F^A{}_4 \cdot F^A{}_4,\ F^A{}_5 \cdot F^A{}_3,\ A\cdot F^A{}_6 \cdot F^A{}_2,\ F^A{}_7 \cdot F^A{}_1);$$

for $A = 2$ this sequence becomes (41, 30, 33, 32, 33, 30, 41) and after normalization $\left(\frac{41}{240},\frac{30}{240},\frac{33}{240},\frac{32}{240},\frac{33}{240},\frac{30}{240},\frac{41}{240}\right)$. By Theorem 2 this is the unique equilibrium with full support in the game $[\mathrm{I}_7,\frac{1}{4}]$.

Example 5. For any connected graph G and any nonempty set of its vertices W with k elements the unique equilibrium in the game $[G,0]$ whose support is W is a profile assigning the number k^{-1} to elements of W and 0 to the remaining vertices. This game has exactly 2^n-1 equilibria.

5. Application: economic models

Household economies

The model of a *household economy* consists of the following elements:

- a positive integer n which is the number of *types of households*;
- a positive integer k which is the number of different *activities* (equal to the number of goods at the market);
- a vector $\mathbf{q} = (q^1, q^2, \ldots, q^n) \in \mathbb{R}^n$ describing the *structure* of the population of respective types, for convenience it may be assumed that $\mathbf{q} \in \Delta_n$;
- a matrix $R = (r^i{}_j)$ $(i = 1, \ldots, n, j = 1, \ldots, k)$, with nonnegative entries (matrix of *coefficients of efficiency*); it is understood that the household of type i undertaking the j-th activity produces $r^i{}_j$ units of the good number j, for $i = 1, \ldots, n,\ j = 1, \ldots, k$;
- (*demand*) *functions* d^i: $\mathbb{R}_+ \times \Delta_k \to \mathbb{R}^k{}_+$ of respective types of households; the actual demand of a household depends on its income $I = r^i{}_j \cdot \pi_j$, where π_j is the unit price of the j-th good and we assume that at the prevailing price system $\pi = (\pi_1, \ldots, \pi_k)$ the equation $\langle d^i(I, \pi); \pi\rangle = I$ is satisfied for all arguments involved, i.e. the value of individual demand at given income at prevailing prices is equal to the actual income. (We assume that all prices are nonnegative, for convenience we also assume that prices are normalized, i. e. $\pi = (\pi_1, \ldots, \pi_k) \in \Delta_k$.)

The so described economy will be shortly denoted by $\mathfrak{E} = (n, k, \mathbf{q}, R, (d^i|i=1, \ldots, n))$.
A distribution of actions of the households of a fixed type, determined by a vector $p^i = (p^i{}_1, \ldots, p^i{}_k) \in \Delta_k$, describes a situation where the fraction $p^i{}_j$ (for instance understood as 25% if $p^i{}_j = 0.25$) of all households of type i decided to undertake the j-th activity, for $j = 1, \ldots, k$ and $i = 1, \ldots, n$. The sequence of vectors $(p^1, \ldots, p^n)$, describing activity of households of all types will be shortly denoted by $\mathbf{p}$.
Given a distribution of activities $\mathbf{p} = (p^1, \ldots, p^n)$ and a price system $\pi = (\pi_1, \ldots, \pi_k)$, the *aggregated demand* (for all goods) is the vector

$$\mathrm{D}(\mathbf{p},\pi) = \sum_{i=1}^{n} q^i \cdot \sum_{j=1}^{k} d_i\left(r_j^i \cdot \pi_j, \pi\right) \cdot p_j^i$$

while the *aggregated supply* (of all goods, hence it is a k-vector) is defined by

$$S(\mathbf{p}) = \left(\sum_{i=1}^{n} q^i \cdot r_1^i \cdot p_1^i, \ldots, \sum_{i=1}^{n} q^i \cdot r_k^i \cdot p_k^i \right)$$

(note that the latest term does not depend on π).
A *state of the economy* $\mathfrak{E}$ is formally defined as a pair $(\mathbf{p},\pi)$: a vector of distributions of activities of the households of all types $\mathbf{p} = (p^1, \ldots, p^n)$ and a price vector π. A *competitive equilibrium* is a state of the economy $(\mathbf{p},\pi)$ such that

$$D(\mathbf{p},\pi) \leq S(\mathbf{p})$$

and for all $i = 1,\ldots, n$ and all $j \in \operatorname{supp} p^i$ there is

$$r^i{}_j \cdot \pi_j = \max_{l=1,\ldots,k} r^i{}_l \cdot \pi_l$$

(we do not require that at equilibrium the demand must be equal to supply but this will be the case if the prices are positive, see Theorem 3 below). So a competitive equilibrium is a state of the economy, such that there is no good the demand for which is larger than its supply and such that (almost) all households are choosing activities giving them the maximal possible income, the same for all households of their type. We allow for an excess supply but this may only happen if the price of such a good is zero; this situation cannot happen at equilibrium if the demand functions (for each fixed good) are strictly decreasing in price of this good.

Auxiliary large game

One way to prove the existence of a competitive equilibrium for a household economy has been proposed by Wieczorek in [1996]; it is based on a construction of an auxiliary large game

$$\Gamma_{\mathfrak{E}} = (G, (S^g|g \in G), (F^g|g \in G))$$

with $n + 1$ types of small players, each of whom has k available actions with appropriately defined payoff functions; players of the auxiliary $(n + 1)$-st type are responsible for clearing the market (this is a novelty, usually in general equilibrium literature there is just one player responsible for clearing the market). Formally:
the set of types G is equal to $\{1, \ldots, n, n+1\}$;
all strategy sets S^g are equal to $\{1, \ldots, k\}$;
for $g = 1, \ldots, n$ and $j = 1, \ldots, k$, $F^g(j; \mathbf{p},\pi) = r^g{}_j \cdot \pi_j$;
for $j = 1, \ldots, k$, $F^{n+1}(j; \mathbf{p},\pi) = D_j(\mathbf{p},\pi) - S_j(\mathbf{p})$.

The following theorem was formulated by Wieczorek in [1996], the idea going back to the seminal paper by Arrow and Debreu [1950]:

THEOREM 3. *Let* $\mathfrak{E} = (n, k, \mathbf{q}, R, (d^i|i=1, \ldots, n))$ *be a household economy.*
(i) *Equilibria of the auxiliary game* $\Gamma_{\mathfrak{E}}$ *are the same as competitive equilibria of* $\mathfrak{E}$.
(ii) *If all demand functions* d^i *are continuous then the auxiliary game has an equilibrium; hence there exists a competitive equilibrium of* $\mathfrak{E}$.
(iii) (Walras Law) *If all demand functions* d^i *are continuous and satisfy the condition*

$$<d^i(I,\pi);\pi> = I \text{ for all } i \text{ and } \pi \text{ in the domain of } d^i \qquad (\#)$$

then, at any competitive equilibrium $(\mathbf{p}^*,\pi^*)$, *for all* $j = 1, \ldots, k$,

$$D_j(\mathbf{p}^*,\pi^*) < S^j(\mathbf{p}^*) \text{ implies } \pi^*_j = 0.$$

(iv) *If all demand functions* d^i *are continuous and satisfy the condition* (#) *while all coefficients of efficiency are positive then, at any competitive equilibrium* $(\mathbf{p}^*,\pi^*)$,
there is $D(\mathbf{p}^*,\pi^*) < S(\mathbf{p}^*)$. □

6. Road traffic model

Let $G = (V, E)$ be a directed graph without loops. Because of the intended interpretation, we shall rather call vertices *towns*, edges – *roads* and paths – *routes*. The graph itself is then called a *traffic net*. Another possible interpretation would be in terms of information theory: the routes are communication channels in which data is remitted in measurable units.
For a path $t = (v_0, v_1, \ldots, v_n)$, we denote by Parts($t$) the set of all roads of the form (v_{i-1}, v_i), for some $i = 1, \ldots, n$.
A *type* (of travelers) is any nonempty subset T of some $\mathcal{P}(v,w)$.
A *simple traffic problem* over G is a system

$$\Gamma = (\mathcal{T}, (r^T|T\in\mathcal{T}), (f_e|e\in E)),$$

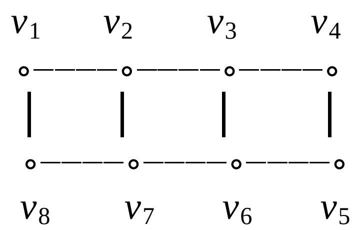
such that $\mathcal{T}$ is a nonempty collection of types, the numbers r^T are positive (we denote $R = \sum_{T\in\mathcal{T}} r^T$) and, for $e\in E$, f_e: $[0,R]\times[0,R] \to \mathbb{R}_+$. Those functions are called *partial cost functions* while r^T are the *types' volumes*.
A *strategy profile* of population of type T is a probability distribution on the set of elements of T, so it can be identified with an element of $\Delta_{|T|}$. A *global strategy profile* is an $\mathbf{s} = (s^T|T\in\mathcal{T})$, where s^T is a strategy profile of population T; it can be regarded as an element of $\Pi_{T\in\mathcal{T}}\Delta_{|T|}$.
Intensity of traffic along edge e, subject to a global strategy profile $\mathbf{s}$, is the number

$$O(e,\mathbf{s}) = \sum_{T\in\mathcal{T}} r^T \cdot\sum_{t\in T} s^T_t \cdot \chi(e\in\text{Parts}(t)).$$

If needed, we take $O(e,\mathbf{s}) = 0$ for e not belonging to E.
Finally, we define, for a *global strategy profile* $\mathbf{s}$ and a traveler of type T who chose a strategy t, his *total cost* to be equal to

$$\sum_{e\in\text{Parts}(t)} f_e(O(e,\mathbf{s}),O(\tilde{e},\mathbf{s})).$$

Example of a road traffic net:



Remark. The model is possibly simple, it can be arbitrarily extended, if necessary. A natural and rather simple generalization is by taking into account the case where the cost of the choice of a route also depends on the cost of passing through a town (vertex); also the entrance to the destination and exit from the departure place may cost; those situations can be modeled analogously, as in the previous construction.

THEOREM 4. *If all functions $f_v(\cdot, \cdot)$ are continuous then there is an equilibrium in the model.*

To prove this theorem just note that Γ is already a large game, one only has to verify the assumptions. □

7. Remarks on computation methods

In this section we only list the computation methods that may been used to compute equilibria in large games and related models. Their detailed description extends the scope of the present paper and it rather deserves a presentation in a separate paper.
Finding all equilibria in a game $\Gamma = (G, (S^g|g\in G), (F^g|g\in G))$ is equivalent to finding all solutions $\boldsymbol{s}$ of the system of equations

$$\Theta^g(\mathbf{s}) = \max_{j\in S^g} F^g(j,\mathbf{s}) - \Sigma_{j\in S^g} (s^g_j \cdot F^g(j,\mathbf{s})) = 0, \text{ for } g\in G.$$

Since all $\Theta^g(\boldsymbol{s})$ are nonnegative, the above problem is equivalent to the following:

$$\text{Find } \mathbf{s} \text{ such that } \Sigma_{g\in G}\Theta^g(\mathbf{s}) = 0.$$

Because of the max operator appearing above, the optimization problem is usually nonsmooth which complicates the application of standard search methods.
Nevertheless, the following methods of finding equilibria often prove successful:

- (quasi-)analytic;
- (modified) iterative;
- „intuitive" – guess and verify;
- finite element;
- artificial intelligence.

For references concerning computation of equilibria see Maćkiewicz and Wieczorek [2002], Doup and Talman [1985, 1987], Zaifu Yang [1999], Scarf and Hansen [1973] and Van der Laan and Talman [1982].

Appendix 1. Relations to ordinary games

To allow for immediate comparison, we recall some definitions concerning classic noncooperative games.
A *finite k-person game* (k>2) is a system $\Gamma = (G, (S^g|g\in G), (F^g|g\in G))$, where G is a k-element set of *players*, all *strategy* sets S^g of respective players have at least two but finitely many elements while F^g: $\Pi_{g\in G}S^g \to \mathbb{R}$ are called *payoff functions* of respective players.
In the case of k = 2 the payoffs are usually presented by a pair of matrices:

$$\begin{array}{c} A \\ 1 \\ 2 \\ \vdots \\ m \end{array}\begin{array}{c} \begin{array}{cccc} 1' & 2' & \cdots & n' \end{array} \\ \begin{bmatrix} a_{11'} & a_{12'} & \cdots & a_{1n'} \\ a_{21'} & a_{22'} & \cdots & a_{2n'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2'} & \cdots & a_{mn'} \end{bmatrix} \end{array}, \qquad \begin{array}{c} B \\ 1 \\ 2 \\ \vdots \\ m \end{array}\begin{array}{c} \begin{array}{cccc} 1' & 2' & \cdots & n' \end{array} \\ \begin{bmatrix} b_{11'} & b_{12'} & \cdots & b_{1n'} \\ b_{21'} & b_{22'} & \cdots & b_{2n'} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn'} \end{bmatrix} \end{array}$$

A *profile* in a k-person game Γ is a system of strategies $\mathbf{s} = (s^g|g\in G)$, $s^g\in S^g$ (one for each player).
Equilibrium in a k-person game Γ is a profile $\mathbf{s} = (s^g|g\in G)$, $s^g\in S^g$, such that for every player g and every profile $\mathbf{s}'$ differing from $\mathbf{s}$ only so that in $\boldsymbol{s}$ the strategy of the player g is replaced by another strategy $\mathbf{s}'$ from the set S^g, there is $F^g(\mathbf{s}') \,.\, F^g(\mathbf{s})$, i. e. no player can increase his payoff unilaterally changing his strategy.

A *mixed strategy* of player i is a probability distribution on the set of his strategies $S^i = \{1, 2, \ldots, k_i\}$, so it is an element of the $\left(\left|S^{k_i}\right|-1\right)$-dimensional standard simplex $\Delta_{\left|S^{k_i}\right|}$. If each player i chooses his mixed strategy p^i, then the players jointly generate the profile $\mathbf{p} = (p^1, \ldots, p^n)$.

An *equilibrium* in a game Γ is a profile $\mathbf{p}^* = (p^{*1}, \ldots, p^{*n})$ composed of mixed strategies such that for every player i and all his mixed strategies p^i there is

$$F^i(p^{*1}, \ldots, p^{*i-1}, p^i, p^{*i+1}, \ldots, p^{*n}) \,.\, F^i(\mathbf{p}^*);$$

in other words, no player can increase his payoff by a unilateral change of his strategy.

THEOREM 5 (Nash [1950])**.** *Every finite game has an equilibrium in mixed strategies.* □

For references to this section see Aumann [1964, 1966], Balder [1995], Flåm and Wieczorek [2006], Rath [1994] and Vind [1964].

Appendix 2. Relations to games with continuum of players

A *game with a continuum of players* or, more properly, a game *with a measure space of players*, is given by a specification of the *players*, usually identified with elements of a normed measure space $(T, \mathcal{T}, \mu)$, the players´ nonempty *strategy sets* S^t, $t\in T$, assumed to be all included in some set S (usually for technical reasons equipped with a σ-field Σ) and the players′ *payoff functions*. To define the latter, we need the notion of a *strategy profile*:

it is a measurable function s: $T \rightarrow S$ such that $s(t) \in S^t$ for all $t \in T$. The payoff function of player t, $u^t(\sigma^t,s)$, depends on the player′s own choice of strategy $\sigma^t \in S^t$ and the strategy profile s. We assume that $u^t(\sigma^t,s) = u^t(\sigma^t,s')$ whenever the profiles s and s' are measure equivalent. Hence, a game with a measure space of players is identified with a system

$$\Gamma = ((T,\mathcal{T},\mu), (S^t|t\in T), (S,\Sigma), (u^t|t\in T)).$$

Measurable sets of players of measure zero are referred to as *negligible*.
A strategy profile s is said to form a *Cournot-Nash equilibrium* if the set of players t who can find a strategy $\sigma^t \in S^t$ such that $u^t(\sigma^t,s) > u^t(s(t), s)$ is negligible. We say that players are *of the same type* whenever they have the same strategy sets and payoff functions.

THEOREM 6 (equal treatment)**.** *At a Cournot-Nash equilibrium the payoff of almost all players of the same type is equal* (*even though they may use different strategies*). □

Suppose now that in a game Γ there are, possibly except for a negligible set of players, only finitely many types of players endowed with finite strategy sets.
We now define a game γ, called a game *corresponding* to Γ. For the initial game Γ, we label the types of players by 1, ..., n and denote by T^i the set of all players of type i (we assume that these sets are measurable and $\mu(T^i) > 0$). Let 1, …, k_i label elements in the strategy set S^i of players of type i. Neglecting the players in a negligible set we may then assume that S is finite (with Σ being the field of all subsets). For any strategy profile s any type i and any $j \in S^i$, we denote by

$$\kappa_s(i, j) = \mu(\{t \in T^i|s(t) = j\})/\mu(T^i)$$

the *frequency* of players of type i who use strategy j at profile s. The function κ_s will also be referred to as the *distribution* of s. We also set

$$K_s(i) := (\kappa_s(i,1), ..., \kappa_s(i,k_i)).$$

We shall say that a game with a measure space of players is *of finite type* whenever there are finitely many types of players, endowed with finite strategy sets and, for every player t of type i, his payoff function has the form

$$u^t(\sigma^t,s) := F^i(\sigma^t;K_s(1), ..., K_s(n)),$$

i.e. every player′s payoff only depends on his own choice of action and the distribution of the other players′ actions.
In that case we define a large game γ *corresponding* to Γ as

$$\gamma := (n; k_1, ..., k_n; F^1, ..., F^n),$$

where all numbers and functions above are those already considered for Γ.

THEOREM 7. *Let a large game γ correspond to a game Γ with a measure space of players, of finite type, and let s be any strategy profile for Γ. Then s is a Cournot-Nash equilibrium for Γ if and only if* $\mathbf{K}_s := (K_s(1), ..., K_s(n))$ *is an equilibrium for γ.* □

Games with a continuum of players were introduced by Schmeidler [1973] and then studied by many authors, among them Mas-Colell [1984]. The model of Schmeidler is intuitive and mathematically elegant but it hardly fits practical (computational) problems because of its high complexity.

Appendix 3. Relations to evolutionary game theory

Large games perfectly fit the framework of ”evolutionary games” which often deal with single populations, but the authors are forced to use two-person games to model. Large games allow to avoid this disadvantage.

For instance, an equilibrium $\mathbf{p} = (p^1, \ldots, p^n)$ for a large game $\Gamma = (G, (S^g | g \in G), (F^g | g \in G))$ is called *evolutionary stable* whenever, for every distribution $\mathbf{q} = (q^1, \ldots, q^n)$ different from p and such that $\text{supp } q^i \subseteq \text{supp } p^i$ for $i = 1, \ldots, n$, there is $\langle q^i; F^i(\mathbf{q})\rangle < \langle p^i; F^i(\mathbf{q})\rangle$ for some i.

(Note that $\langle q^i; F^i(\mathbf{q})\rangle = \Sigma_{j=1,\ldots,k}\, q^i{}_j \cdot F^i(j; \mathbf{q})$ and $\langle p^i; F^i(\mathbf{q})\rangle = \Sigma_{j=1,\ldots,k}\, p^i{}_j \cdot F^i(j; \mathbf{q})$.)

For references to this section see Cressman [1995], Maynard Smith [1982], Ritzberger and Weibull [1995], Taylor [1979] and Weibull [1995].

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