Anna WALASZEK-BABISZEWSKA[*], Wojciech MENDECKI [**]

FUZZINESS AND PROBABILITY FOR PORTFOLIO MANAGEMENT

Abstract

In the paper the portfolio of financial assets has been formulated using the fuzzy numbers idea and the statistical information concerning assets. The optimization task of the portfolio structure has been formulated for two types of fuzzy numbers. The data from Polish market have been used for exemplary calculations.

1. INTRODUCTION – PROBABILISTIC TOOLS IN FINANCIAL MARKET ANALYSIS

Financial markets have been investigated in probabilistic and stochastic categories by economists, mathematicians and researchers of natural sciences for over a hundred years. It is justified to mention French mathematician, Bachelier, who presented his doctoral thesis in 1900 at the Academy of Paris. His pioneering work dealt with the pricing of options and he formulated, the random walk process as a model of price changes. After years, this process has been more mathematically formalized and developed by Einstein and then by Wiener. Problems of the price changing have been modelling using many classes of models. The Black and Scholes option – pricing model (1973) provided a very important instrument for building a strategy for market investors [4].

Let $X(t)$ be the price of a chosen financial asset at time t. There are variables derived from prices of financial goods which are commonly considered by market researchers:

- price changes as differences

$$Y(t) = X(t + \Delta t) - X(t) , \qquad (1)$$

- discounted price changes

$$Y_D(t) = \left[X(t + \Delta t) - X(t) \right] D(t) , \qquad (2)$$

where D(t) means a discounting factor,

[*]Ph.D. Anna Walaszek-Babiszewska, prof. PO, Instytut Automatyki i Informatyki, Politechnika Opolska, ul. Mikołajczyka 5, 45-271 Opole, walaszek-babiszewska@po.opole.pl
[**]M.Sc. Wojciech Mendecki, Energoinstal S.A., Al. Rozdzieńskiego 188d, 40-203 Katowice, mendecki@o2.pl

- returns

$$R(t) = \frac{X(t + \Delta t) - X(t)}{X(t)},$$ (3)

- differences of the natural logarithm of prices

$$Z(t) = \ln X(t + \Delta t) - \ln X(t) = \ln \frac{X(t + \Delta t)}{X(t)}.$$ (4)

In this paper, returns of assets are expressed by (3).

Changes of assets depend on:

- macroeconomic and political situation of the country,
- fluctuations of the global market,
- situation of the company.

Stochastic modelling has been developed by many authors, for the price process $X(t)$ and also for its indicators, under different assumptions, market hypotheses. The problem of the distribution of price changes was connected with the stochastic models of stock prices. Authors proposed Gaussian distribution, log-normal distribution, power-low distribution or Levy distribution for understanding or to predict the behaviour of real price changes and for the risk prediction.

The portfolio theory proposed models which tried to find a riskless portfolio, or an optimum portfolio, characterized by the high portfolio return at a relatively low risk [1]. The portfolio return is a weighted sum of returns of component assets. In many approaches of portfolio creating, risk as a probabilistic category, is measured e.g. by a variance of the portfolio return. In such approach, the mean value of the portfolio return is a weighted sum of the expected values of component returns. The variance of the portfolio return is a variance of the random variable which is the weighted sum of component random variables (assets).

In this paper we discuss the methods of portfolio management which are based on the fuzzy set theory and the statistical analysis. The formal description uses also perception-based experts' information.

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2. FUZZY PORTFOLIO MANAGEMENT
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The idea of a fuzzy model of portfolio management has been formulated in [7]. The future values of returns, according to the market uncertainty, are estimated by financial investors rather as certain intervals than numbers. This is the justification that fuzzy numbers could represent an uncertainty of asset returns.

We define a fuzzy return R in a space of real numbers $\boldsymbol{R}$, by means of a membership function $\mu_R(r)$ that assigns the level of affiliation of any value $r \in \boldsymbol{R}$ to the fuzzy set R, as follows:

$$\mu_R(r) : \boldsymbol{R} \to [0,1]$$ (5)

$$\exists r_0 \in \boldsymbol{R} : \mu_R(r_0) = 1$$ (6)

and its α-cut R_α

$$R_\alpha := \{ r \in \mathbf{R} : \mu_R(r) \geq \alpha \}, \quad \forall \alpha \in [0,1] \tag{7}$$

is a closed interval in a space of real numbers $\mathbf{R}$ (according to [2, 3, 8, 9]).

In the classic portfolio analysis, the return R_P of the portfolio is a weighted sum of particular returns:

$$R_P = \sum_{i=1}^{I} c_i R_i \tag{8}$$

where $\{R_i\}$, $i=1,2,...,I$ is a return set of component assets, and c is a real vector, such that

$$c^T = [c_1...c_i...c_I], \qquad 0 \leq c_i \leq 1, \quad i=1,2,...,I; \quad \sum_{i=1}^{I} c_i = 1. \tag{9}$$

- Then the portfolio return (8) fulfills the relationship

$$Min(R_1,...R_i,...R_I) \leq R_P \leq Max(R_1,...R_i,...R_I). \tag{10}$$

The proposed in this paper fuzzy portfolio analysis is based on fuzzy numbers. The shape of a membership function and a spread representing the risk of the asset are very important parameters and should be accepted by experts of market studies. The parameters of the membership function one can choose based on the empirical mean value m_R and the variance s_R^2 of the historical data of returns, observed in a time period $t \in T$.

Let now $\{R_i\}$, $i=1,2,...,I$ is a set of fuzzy numbers representing returns of component assets, and c is a real vector fulfilling (9).

We consider an exemplary membership function, in a form of the triangular symmetrical function with parameters r_0 and s:

$$\mu_R(r) = \begin{cases} 1 - \dfrac{|r - r_0|}{s} & when \quad r_0 - s \leq r \leq r_0 + s \\ 0 & otherwise \end{cases} \tag{11}$$

where
$r_0 = m_R$
s - parameter representing risk of the asset,

$$s = s_R \quad or \quad s \leq \frac{1}{2} |R_{max} - R_{min}| \tag{12}$$

R_{min}, R_{max} – the observed lowest an highest values of the return.

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When membership functions $\mu_{R_i}(r), i = 1,2,...,I$ of the particular returns of the portfolio components are triangular as (11), then the membership function of the portfolio return as a linear function of fuzzy numbers is also triangular and has a form:

$$\mu_{R\,p}(r) = \begin{cases} 1 - \dfrac{\left| r - c^T r_0 \right|}{c^T s} & when \left| r - c^T r_0 \right| \le c^T s \text{ and } c \ne 0 \\ \qquad 0 & otherwise \end{cases} \tag{13}$$

where

$$r_0^T = \left[r_{0,1}...r_{0,i}...r_{0I} \right], \qquad s^T = \left[s_1...s_i...s_I \right] \tag{14}$$

and

$$r_P = c^T r_0, \quad s_P = c^T s \tag{15}$$

are parameters of the membership functions of the portfolio returns.

Investors are looking for such assets from the market data, which are characterized by high returns at a relatively low risk. It means that the preliminary criterion of the choice is:

$$\frac{s_i}{r_{0,i}} \to \min; \quad r_{0,i} > 0. \tag{16}$$

The optimization task of the portfolio defined by (11) – (16), can be formulated in a category of real numbers, as follows:

- choose the set of real numbers $c_i \ge 0$, $i=1,2,...,I$ which maximizes the return r_P or (and) minimizes the risk s_P of the portfolio

$$r_p = \sum_{i=1}^{I} c_i r_{0,i} \to \max \tag{17}$$

$$s_P = \sum_{i=1}^{I} c_i s_i \to \min \tag{18}$$

- under the condition

$$\sum_{i=1}^{I} c_i = 1. \tag{19}$$

More generally, for the any type of membership functions $\mu_{R_i}(r), i = 1,2,...,I$ of the particular assets, the return of the portfolio can be calculated using the extension principle, and the decomposition rule as follows:

$$(R_P)_\alpha = \sum_{i=1}^{I} c_i (R_i)_\alpha \tag{20}$$

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$$R_P = \bigcup_{\alpha \in [0,1]} \alpha (R_P)_\alpha \tag{21}$$

where Y denotes the standard fuzzy union [3] and R_α means α-cut of a fuzzy number.

3. EXEMPLARY OPTIMIZATION TASKS

3.1. Portfolio with triangular fuzzy numbers

Assume that the portfolio is composed of two assets A and B. Return R is a fuzzy number with the membership function of a triangular type, for the asset A, as follows:

$$\mu_A(R) = \begin{cases} \dfrac{R - R_{A\min}}{R_{Am} - R_{A\min}}, & for \qquad R \in (R_{A\min}, R_{Am}) \\[3mm] \dfrac{R_{A\max} - R}{R_{A\max} - R_{Am}}, & for \qquad R \in <R_{Am}, R_{A\max}) \\[3mm] 0 & for \quad (-\infty, R_{A\min}) \cup (R_{A\max}, \infty) \end{cases} \tag{22}$$

where $R_{A\min}$, $R_{A\max}$, R_{Am} are the lowest, highest and middle values of the return, observed in the historical data set or estimated according to the expert opinion.

Let c is a real number, $0 \le c \le 1$, which represents the share of the asset A in the portfolio. The portfolio return is a fuzzy number

$$R_P = cR_A + (1-c)R_B \tag{23}$$

where R_B is the return of the B asset, characterized by the membership function (22) with parameters $R_{B\min}$, $R_{B\max}$, R_{Bm}, respectively.

Portfolio return (23) as a linear combination of fuzzy components with triangular membership functions, has a form:

$$\mu_P(R) = \begin{cases} \dfrac{R - R_{P\min}}{R_{Pm} - R_{P\min}}, & for \qquad R \in (R_{P\min}, R_{Pm}) \\[3mm] \dfrac{R_{P\max} - R}{R_{P\max} - R_{Pm}}, & for \qquad R \in <R_{Pm}, R_{P\max}) \\[3mm] 0 & for \quad (-\infty, R_{P\min}) \cup (R_{P\max}, \infty) \end{cases} \tag{24}$$

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where parameters of the membership function (24) of the portfolio return can be express by parameters of the particular membership functions of assets:

$$R_{P\,m} = cR_{A\,m} + (1-c)R_{B\,m} \tag{25a}$$

$$R_{P\,\min} = cR_{A\,\min} + (1-c)R_{B\,\min} \tag{25b}$$

$$R_{P\,\max} = cR_{A\,\max} + (1-c)R_{B\,\max} \tag{25c}$$

It is very important for market investors to predict the reduction threat to expected value of returns. Denote the α-cut of the portfolio return as a closed interval of real numbers:

$$(R_P)_\alpha = [R_{P(-)}, R_{P(+)}] \tag{26}$$

Denote also the crisp interval Δ_α as a measure of the reduction threat (risk) to expected value of portfolio return with the level of the truth equal to $\alpha \in [0,1]$. It can be calculated using α-cut of the membership function of the portfolio return, as follows:

$$\Delta_\alpha = R_{P\,m} - R_{P(-)} = \\ = (1-\alpha)\big[c(R_{A\,m} - R_{A\,\min}) + (1-c)(R_{B\,m} - R_{B\,\min}\big] \tag{27}$$

Now, the task of the portfolio optimization has been formulated in the category of real numbers, not fuzzy numbers:
choose the value of a share c, $0 \leq c \leq 1$ which fulfills:

$$R_{P\,m} = cR_{A\,m} + (1-c)R_{B\,m} \to \max \tag{28}$$

under the condition

$$\Delta_\alpha = f(c,\alpha) \leq \Delta_{cr} \tag{29}$$

where Δ_{cr} is the critical value of risk, accepted by investors, $\alpha \in [0,1]$, for the set of tested assets.

Example 1.
Let values of six shares of Polish stock exchange are given, observed daily in the period: May 2005 – June 2005. The parameters of the shares have been calculated and presented in Table 1.
The calculated parameters concerning these six shares have been used for the simulation test of potential portfolios, which are composed of pairs of assets taken from Table 1.
The crisp interval

$$\Delta_\alpha = f(c,\alpha)$$

representing risk of the reduction of the mean values of portfolio returns can be calculated according to (27), in a form of the equation depended on parameters:

c - share of the first asset in the portfolio,

α - assumed level of the truth (possibility) of the occurrence the interval Δ_α.

Tab. 1. The parameters of the returns for six chosen shares of Polish stock exchange ([5])

	AMC	BPH	DUD	PKN	TPS	WWL
R_{min}	-2.90	-3.28	-3.29	-3.46	-2.26	-2.75
R_{max}	0.13	0.47	0.23	0.24	0.31	0.23
R_m	3.64	7.75	4.90	3.49	3.76	3.14
$R_m - R_{min}$	3.03	3.75	3.52	3.70	2.57	2.98

Tab. 2. The equations of the risk of portfolio returns for pairs of assets of Polish stock exchange (according to [5])

PAIRS OF ASSETS	$\Delta_\alpha = f(c, \alpha)$
AMC/BPH	$-0.71c(1-\alpha) + 3.75(1-\alpha)$
AMC/ DUD	$-0.49c(1-\alpha) + 3.52(1-\alpha)$
AMC/ PKN	$-0.67c(1-\alpha) + 3.70(1-\alpha)$
AMC/ TPS	$0.46c(1-\alpha) + 2.57(1-\alpha)$
AMC/ WWL	$0.06c(1-\alpha) + 2.98(1-\alpha)$
BPH/ DUD	$0.22c(1-\alpha) + 3.52(1-\alpha)$
BPH/ PKN	$0.04c(1-\alpha) + 3.70(1-\alpha)$
BPH/ TPS	$1.18c(1-\alpha) + 2.57(1-\alpha)$
BPH/ WWL	$0.77c(1-\alpha) + 2.98(1-\alpha)$
DUD/ PKN	$-0.18c(1-\alpha) + 3.70(1-\alpha)$
DUD/ TPS	$0.95c(1-\alpha) + 2.57(1-\alpha)$
DUD/ WWL	$0.55c(1-\alpha) + 2.98(1-\alpha)$
PKN/ TPS	$1.13c(1-\alpha) + 2.57(1-\alpha)$

PKN/ WWL	$0.73c(1-\alpha) + 2.98(1-\alpha)$
TPS / WWL	$-0.41c(1-\alpha) + 2.98(1-\alpha)$

Assuming $\alpha = 0.75$, $\Delta_{cr} = 0.7$ and using data and equations from Table 2. we derive only five values of c, fulfilling $0 \leq c \leq 1$. Table 3 shows the calculated shares c and mean values of portfolios R_{Pm}.

For the new value $\Delta_{cr} = 0.8$ and $\alpha = 0.75$ we derive as results of calculations nine portfolios, fulfilling conditions of c, $0 \leq c \leq 1$ (Table 4).

Tab. 3. The values of shares c and mean values of portfolios R_{Pm} calculated on the base of data from Table 2. and assumed values $\alpha = 0.75$, $\Delta_{cr} = 0.7$ (according to [5])

PAIRS OF ASSETS	$\Delta_{\alpha} = f(c,\alpha) = 0.7 \Rightarrow c, \quad R_{Pm}$
AMC/ TPS	$c = 0.50, \quad R_{Pm} = 0.22$
BPH/ TPS	$c = 0.20, \quad R_{Pm} = 0.34$
DUD/ TPS	$c = 0.24, \quad R_{Pm} = 0.29$
PKN/ TPS	$c = 0.20, \quad R_{Pm} = 0.30$
TPS / WWL	$c = 0.43, \quad R_{Pm} = 0.26$

The optimal portfolio is composed of 20% BPH assets and 80% TPS assets. The mean return is 0.34.

Tab. 4. The values of shares c and mean values of portfolios R_{Pm} calculated on the base of data from Table 2. and assumed values $\alpha = 0.75$, $\Delta_{cr} = 0.8$ (according to [5])

PAIRS OF ASSETS	$\Delta_{\alpha} = f(c,\alpha) = 0.8 \Rightarrow c, \quad R_{Pm}$
AMC/BPH	$c = 0.77, \quad R_{Pm} = 0.16$
AMC/ DUD	$c = 0.66, \quad R_{Pm} = 0.17$
AMC/ PKN	$c = 0.75, \quad R_{Pm} = 0.16$
BPH/ TPS	$c = 0.54, \quad R_{Pm} = 0.39$

BPH/ WWL	$c = 0.29, \quad R_{P\,m} = 0.3$
DUD/ TPS	$c = 0.66, \quad R_{P\,m} = 0.26$
DUD/ WWL	$c = 0.41, \quad R_{P\,m} = 0.23$
PKN/ TPS	$c = 0.56, \quad R_{P\,m} = 0.27$
PKN/ WWL	$c = 0.31, \quad R_{P\,m} = 0.23$

When the critical risk interval increases, then the set of portfolios increases too. We have the set of nine portfolios, the optimal portfolio is characterized by the mean value of return equal to 0.39, 54% BPH assets and 46% TPS assets.

Similarly we can calculate the interval of the chance of increasing the mean value of portfolio returns

$$
\begin{aligned}
\Delta_\alpha^+ &= R_{P(+)} - R_{P\,m} = \\
&= (1-\alpha)\left[c(R_{A\,\max} - R_{A\,m}) + (1-c)(R_{B\,\max} - R_{B\,m})\right]
\end{aligned}
\tag{30}
$$

for the assumed level of possibility α.

3.2. Portfolio optimization task with exponential fuzzy numbers

Assume, like in paragraph 3.1, that the portfolio is composed of two assets A and B. We define returns R of the particular components, using fuzzy number of the L-R type, in the exponential form:

$$
\mu(R) =
\begin{cases}
L\left(\dfrac{R_m - R}{s_L}\right), & R \le R_m \\[4mm]
P\left(\dfrac{R - R_m}{s_R}\right), & R \ge R_m
\end{cases}
\tag{31}
$$

$$
\mu(R) =
\begin{cases}
\exp\left[-\left(\dfrac{R_m - R}{s_L}\right)^2\right], & R \le R_m \\[4mm]
\exp\left[-\left(\dfrac{R - R_m}{s_R}\right)^2\right], & R \ge R_m
\end{cases}
\tag{32}
$$

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where s_L and s_R mean the standard deviation, calculated for left (R_m-R_{min}) and right (R_{max}-R_m) intervals of observed fuzzy data.

Generally, if fuzzy returns of assets A and B are characterized by membership functions with parameters (33), noted according to [2]:

$$A = (R_{Am}, s_{AL}, s_{AR}) , \quad B = (R_{Bm}, s_{BL}, s_{BR}) \tag{33}$$

then the portfolio return has parameters of the membership function

$$R_{Pm} = cR_{Am} + (1-c)R_{Bm} \tag{34a}$$

$$s_{PL} = cs_{AL} + (1-c)s_{BL} \tag{34b}$$

$$s_{PR} = cs_{AR} + (1-c)s_{BR} \tag{34c}$$

We are interested in calculating a measure of the reduction threat (risk) to expected value of portfolio return, for the assumed level of possibility $\alpha \in [0,1]$. Taking into account relations (32) - (34c), we received the real interval as a function of α and c:

$$\Delta_\alpha = (-\ln \alpha)^{1/2}[cs_{AL} + (1-c)s_{BL}] \tag{35}$$

The task of the portfolio optimization can be again formulated, in the meaning written by (28), (29) and using (34a) and (35).

Example2.

The set of assets presented in Table 1. has been considered as a set of potential components of portfolios. Parameters s_L of particular returns have been calculated according to daily observation in the period May 2005 – June 2005. Assuming the level of truth $\alpha = 0.75$, the critical value of risk $\Delta_{cr} = 0.7$ and using equation (35) we can calculate share values c of particular pairs of assets. Table 5. shows the calculated shares c, fulfilling $0 \le c \le 1$ and mean values of portfolio returns R_{Pm}.

Tab. 5. The values of shares c and mean values of portfolios R_{Pm} calculated on the base of exponential membership functions; $\alpha = 0.75$, $\Delta_{cr} = 0.7$ (according to [5])

PAIRS OF ASSETS	$\Delta_\alpha = f(c,\alpha) = 0.7 \Rightarrow c, \quad R_{Pm}$
AMC/ TPS	$c = 0.79, \quad R_{Pm} = 0.17$
AMC/ WWL	$c = 0.54, \quad R_{Pm} = 0.18$
BPH/ TPS	$c = 0.28, \quad R_{Pm} = 0.35$

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BPH/ WWL	$c = 0.11, \quad R_{P\,m} = 0.25$
DUD/ TPS	$c = 0.62, \quad R_{P\,m} = 0.26$
DUD/ WWL	$c = 0.33, \quad R_{P\,m} = 0.23$
PKN/ TPS	$c = 0.34, \quad R_{P\,m} = 0.29$
PKN/ WWL	$c = 0.14, \quad R_{P\,m} = 0.23$

The structure of the optimal portfolio calculated on the base of the exponential fuzzy numbers (28% BPH assets, 72% TPS assets, the mean return 0.35) is similar to the structure of the optimal portfolio calculated for the triangular fuzzy numbers, shown in Table 3. (20% BPH, 80% TPS, the mean return 0.34). The set of portfolios from Table 5. consists of eight pairs of assets but in the case of triangular membership functions the set of portfolios - only five pairs.

Both portfolios have been calculated for the critical risk interval $\Delta_{cr} = 0.7$ and the level of possibility $\alpha = 0.75$.

4. CONCLUSION

This paper discusses possibilities of utilising fuzzy set theory for making decision in market investigations. Fuzzy numbers represent the uncertainty of future values of returns. The type of fuzzy numbers and the shape of membership functions are very important elements of the optimization task of the portfolio. Experiences and knowledge of market experts are very helpful in such tasks.

References

[1]   JAJUGA K. T.: *Investments, financial instruments, financial risk, financial engineering,* (in Polish), PWN, Warszawa, 1998
[2]   KACPRZYK J.: *Fuzzy sets in system analysis,* (in Polish), PWN, Warszawa, 1986
[3]   KLIR G. J. YUAN BO: *Fuzzy sets and Fuzzy Logic. Theory and Applications*, Prentice Hall PTR, New Jersey, 1995
[4]   MANTEGNA R.N., STANLEY H.E.: *An Introduction to Econophysics. Correlations and Complexity in Finance,* Cambridge University Press, Cambridge, 2000
[5]   MENDECKI W.: *Fuzzy models of the creation and analysis of an assets portfolio*, (Master Degree Work – unpublished, in Polish), Silesian University of Technology, Gliwice, 2005
[6]   WALASZEK-BABISZEWSKA A.: *Fuzzy Probability for Modelling of Particle Preparation Processes*, The 4th Int. Conf. Intelligent Processing and Manufacturing of Materials, Sendai, Japan 2003; Intelligence in a Small Material World, Selected papers from IPMM-2003, Meech J. A.,

Kawazoe Y., Kumar V., Maguire J.F. (Eds.), DEStech Publications, Inc., Lancaster, Pensylwania, 2005, pp. 119 - 129

[7] WALASZEK-BABISZEWSKA A.: *Applications of Fuzzy and Probabilistic Methods in Financial Market Analyses*, First Warsaw International Seminar on Intelligent Systems (WISIS 2004), Polish Academy of Sciences; Issues in Intelligent Systems. Models and Techniques, Dramiński M. Grzegorzewski P., Trojanowski K., Zadrożny S. (Eds.), book series: Problemy Współczesnej Nauki, Informatyka, Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2005, pp. 263-271

[8] YAGER R. R., FILEV D. F.: *Essential of Fuzzy Modelling and Control*, (in Polish), WNT, Warszawa, 1995

[9] ZADEH L. A.: *Probability Measures of Fuzzy Events,* Journal of Math. Anal. Appl., 23/1968, pp. 421-427