POZNAN UNIVERSITY OF TECHNOLOGY ACADEMIC JOURNALS
No 76 Electrical Engineering 2013

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SCALING OF NUMBERS IN RESIDUE ARITHMETIC WITH THE FLEXIBLE SELECTION OF SCALING FACTOR

A scaling technique of numbers in residue arithmetic with the flexible selection of the scaling factor is presented. The required scaling factor can be selected from the set of moduli products of the Residue Number System (*RNS*) base. By permutation of moduli of the number system base it is possible to create many auxiliary Mixed-Radix Systems (*MRS*). They serve as the intermediate systems in the scaling process. All *MRS*'s are associated with the given *RNS* with respect to the base, but they have different sets of weights. For the scaling factor value resulting from the requirements of the given signal processing algorithm, the suitable *MRS* can be chosen that allows to obtain the scaling result in most simple manner.

1. INTRODUCTION

In many digital signal processing applications the multiplication of signal samples by a constant factor is a common task. Such applications may use various kinds of arithmetics. Typically the distributed arithmetic [1, 2] is used. An alternative can be the Residue Number System (*RNS*) [3-5], that allows to decompose operations on large integers into sets of operations carried out in small integer rings. This advantage pertains first of all to multiplication. When the *RNS* base is composed of relatively small moduli with the binary size of 5- 6- bits, multiplication by a constant can be carried out by memory look-up using look-up tables with 5-or 6-address or by the logic blocks with the respective number of variables. The *RNS* permits to attain high pipelining rates because the fine granulation of the circuit becomes possible. The main drawback when performing the multiplication in the *RNS* is the necessity to scale the product by a fixed constant after multiplication. This results from the fact that the *RNS* is an integer number system but in the signal processing algorithms coefficients are fractional or real numbers, therefore the algorithm coefficients have to transformed to integers by premultiplication by a suitable constant, termed the scaling factor, that will provide for the sufficiently accurate representation of the coefficients. The scaling factor may also include the compensation of the dynamic of the signal

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growth due to summations of terms in the given algorithm. The scaling algorithms have been considered since the pioneering work of Szabo and Tanaka [3] by many authors. The review of the scaling techniques and their properties was presented in [6, 10]. The recent works on scaling has been presented by Braden and Philips [7], Dasygienis et *al*. [8], Ma [9].

The scaling algorithms use the Chinese Remainder Theorem (*CRT*) [6, 10], the *MRS* [11] or the core function Burgess [12]. In this work the use of the *MRS* for scaling performed in the *FPGA* environment is considered. The *MRS* may be advantageous for these applications when scaling of signed numbers is needed. The *MRS* allows for relatively easy detection of the sign of the *RNS* number for certain *RNS* bases with the specific ordering of the moduli of the base. Moreover, certain orderings of the moduli of the base may allow to obtain the required scaling factor and may also lead to the simplification of the scaler structure.

The *RNS* is created for the system base $B=\{m_1,m_2,...,m_n\}$, where m_j , j=1,2,..., *n*, are the pairwise relatively prime. The number *X* from the range [0, *M*-1], where $M=\prod_{j=1}^{n} m_j$, is one-to-one represented by the vector $\{r_1,r_2,...,r_n\}$, where $r_j=|X|_{m_j}$ and $|X|_{m_j}$ is the nonnegative residue of division of *X* by m_j .

In the *MRS* associated with the *RNS* with respect to the system base *B,* the number X from the range [0, *M*-1] is calculated as

$$X=\sum_{j=1}^{n} E_j w_j = E_0+E_1 m_1+E_2 m_1 m_2+...+E_{n-1} m_1 m_2 \cdot ... \cdot m_{n-1} \quad (1)$$

where

$$E_j=\left|\left\lfloor \frac{X}{w_j} \right\rfloor\right|_{m_{j+1}}, \quad w_j=\prod_{k=0}^{j} m_j \text{ and } 0 \le E_j < m_{j+1}$$

The number range of the *MRS* is the same as in the *RNS*.

The *MRS* forward conversion can be easily performed using residue arithmetic. For this reason, the *MRS* is often used as the auxiliary weighted number system that supports scaling, sign detection, base extension and other residue arithmetic operations considered to be difficult.

3. SCALING IN THE RNS

The scaling result in the *RNS* for unsigned numbers can be calculated as:

$$Y=\frac{X-|X|_K}{K}, \quad (2)$$

where the scaling factor *K* is any natural number. The scaling operation can be easier when *K* is the product certain moduli from the *RNS* base which simultaneously form the weight of the *MRS*, w_k. After the conversion of the *RNS* number *X* to the *MRS*, the scaling operation can be performed as follows:

$$Y = \frac{X - |X|_{w_{kK}}}{w_k} = \frac{E_0 + E_1 w_1 + \ldots + E_k w_k + \ldots + E_{n-1} w_{n-1}}{w_K} = \frac{E_0}{w_k} + \frac{E_1 m_1}{w_k} + E_k + E_{k+1}\frac{w_{k+1}}{w_k} + \ldots + E_{n-1}\frac{w_{n-1}}{w_k} \quad (3)$$

where every products $\frac{w_j}{w_k}$, for $j > k$ is an integer. The scaling error is smaller than 1, because

$$\frac{|X|_{w_k}}{w_k} = \frac{E_0 + E_1 w_1 + \ldots + E_{k-1} w_{k-1}}{w_k} < 1 \quad (4)$$

The biggest drawback here is that the scaling factor has to be a product of sequential moduli of the number system base and on the other hand it has to provide the necessary accuracy of the representation of coefficients of the *DSP* algorithm.

4. EXTENSION OF THE SCALING FACTORS SET

It will be assumed that scaling is performed with the use of conversion to the *MRS*. The scaling factor *K* is the product of the moduli of the number system. If all possible permutations of the moduli from the base B are known, the number of the scaling coefficients can be computed.

The number of permutations of n-element set is $n!$, this determines the number of different *MRS*'s that can be created. Each permutation determines one *MRS*. Every *MRS* is associated with the *RNS* with respect to the *RNS* base. The *MRS*'s have the same the number range M, but different weights and digits.

Assume that the scaling factor is a product of $s < n$ moduli. We may consider two options:

Option 1: The scaling factor is a product of s moduli, where $s = 1,2,\ldots,n-1$. Then the number of different coefficients is equal to the number of the different *MRS*'s weights created for all possible permutations. For example, we can take the *RNS* with the base $B = \{2,3,5,7\}$. In an associated *MRS* we have $x = E_0 + E_1 \cdot 2 + E_2 \cdot 2 \cdot 3 + E_3 \cdot 2 \cdot 3 \cdot 5$, that shows that the nontrivial coefficients can be equal to the weights $2,6,30$. In fact, the number of the base permutation $n! = 4! = 24$ gives 24 *MRS*'s, which means that we obtain 14 different weights that can be used as the coefficients:

$$A = \{2,3,5,6,7,10,14,15,21,30,35,42,70,105\}.$$

The set of *MRS*'s required for the creation of A can be reduced based on the following permutations of the base

$$\{2,3,5,7\}, \{3,7,5,2\}, \{5,3,2,7\}, \{5,2,7,3\}, \{7,5,3,2\}, \{7,2,5,3\}$$

Option 2: The scaling factor is a product of s from n moduli, where $s = 2,3,\ldots,n-1$. In this case, the coefficient always can be created as a product of at most 2 moduli. The quantity of possible coefficients is lower than in Option 1. It shall be assumed that the real number of possible coefficients does not exceed $n = 8$ and $2 \le s \le 5$. The number of all possible coefficients can be calculated based on $C = \dfrac{n!}{(n-s)!s!}$. The results are presented in Table 1.

Table 1. The number of possible scaling coefficients for $n \le 8$ and $2 \le s \le 5$

	n				
K	4	5	6	7	8
1	2	3	4	5	6
2	6	10	15	21	21
3	4	10	20	35	35
4	1	5	15	35	35
5	-	1	6	21	21

5. SUMMARY

Scaling operation in the *RNS* can be based on the conversion to the *MRS*, subsequent scaling of individual terms of *X* and forward conversion to the *RNS*. The residue number can be scaled by a coefficient equal to one of the weights of *MRS*. The main drawback of the solution is small amount of different values that is equal to $n-1$. In this work a technique of extension of the set of scaling has been presented. The optimal coefficient can be selected from the set of different weights of *MRS*'s, created with the use of all possible permutations of the moduli.. This approach extends the possibility of selecting an appropriate scaling factor and demonstrates how it can be attained.

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