# ESSENTIAL NORM
# OF AN INTEGRAL-TYPE OPERATOR
# FROM $\omega$-BLOCH SPACES TO $\mu$-ZYGMUND SPACES
# ON THE UNIT BALL

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**Abstract.** In this paper, we give an estimate for the essential norm of an integral-type operator from $\omega$-Bloch spaces to $\mu$-Zygmund spaces on the unit ball.

## 1. INTRODUCTION

Let $\mu$ be a positive continuous function on $[0, 1)$. We say that $\mu$ is normal if there exist positive numbers $a$ and $b$, $0 < a < b$, and $\delta \in [0, 1)$ such that (see [19]),

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1-r)^a} = 0,$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$, $H(\mathbb{B})$ be the space of all holomorphic functions in $\mathbb{B}$. When $n = 1$, $\mathbb{B}$ is the open unit disk $\mathbb{D}$ of the complex plane and $H(\mathbb{D})$ is the holomorphic function space on $\mathbb{D}$.

For $f \in H(\mathbb{B})$, the radial derivative and complex gradient of $f$ at $z$ will be denoted by $\Re f(z)$ and $\nabla f(z)$, respectively, that is,

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \text{ and } \nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right).$$

Let $\omega$ be normal. An $f \in H(\mathbb{B})$ is said to belong to $\omega$-Bloch space (or Bloch type space), denoted by $\mathcal{B}_\omega = \mathcal{B}_\omega(\mathbb{B})$, if

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z \in \mathbb{B}} \omega(|z|)|\Re f(z)| < \infty.$$

The little $\omega$-Bloch space $\mathcal{B}_{\omega,0} = \mathcal{B}_{\omega,0}(\mathbb{B})$, consists of all $f \in \mathcal{B}_\omega$ such that

$$\lim_{|z| \to 1} \omega(|z|)|\Re f(z)| = 0.$$

$\mathcal{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $0 < \alpha < \infty$ and $\omega(t) = (1-t^2)^\alpha$, we get the $\alpha$-Bloch space (often also called Bloch type space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B})$. In particular, when $\omega(t) = 1 - t^2$, we get the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{B})$. See [32, 36] for more information of the Bloch space $\mathcal{B}$ and $\omega$-Bloch space $\mathcal{B}_\omega$ on the unit ball.

Suppose $\mu$ is normal on $[0, 1)$. We say that an $f \in H(\mathbb{B})$ belongs to $\mathscr{Z}_\mu = \mathscr{Z}_\mu(\mathbb{B})$ if

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + \sup_{z \in \mathbb{B}} \mu(|z|)|\Re^2 f(z)| < \infty,$$

where $\Re^2 f(z) = \Re(\Re f)(z)$. Under the above norm, $\mathscr{Z}_\mu$ becomes a Banach space. $\mathscr{Z}_\mu$ will be called the $\mu$-Zygmund space or the Zygmund type space. The little $\mu$-Zygmund space, denoted by $\mathscr{Z}_{\mu,0}$, is the space consisting of all $f \in \mathscr{Z}_\mu$ such that

$$\lim_{|z| \to 1} \mu(|z|)|\Re^2 f(z)| = 0.$$

When $\mu(r) = 1 - r^2$, the $\mu$-Zygmund space becomes the Zygmund space $\mathscr{Z}$. See [16, 24, 36] for more information of the Zygmund space on the unit ball. There has been some recent interest in studying of various concrete operators, including integral-type ones, from or to the Zygmund type spaces, see, for example, [2, 11, 14, 15, 17, 18, 24, 27]. Some generalizations of the Zygmund type spaces and operators on them, can be found in [29, 30].

Assume that $g \in H(\mathbb{B}), g(0) = 0$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$. In [20], Stević introduced the following integral-type operator $P_\varphi^g$ on $H(\mathbb{B})$:

$$P_\varphi^g f(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}.$$

Some results on the operator has been got in [7, 21–23, 25, 26, 28, 31, 38].

Here we consider a particular case of $P_\varphi^g$ when $\varphi(z) = z$, that is the operator

$$P_g f(z) = \int_0^1 f(tz)g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}.$$

For all $g \in H(\mathbb{B})$, $P_{\Re g}$ is just the extended Cesàro operator (or Riemann-Stieltjes operator), which was introduced in [3], and studied in [1, 3–5, 8–10, 12, 13, 16, 32, 34]. Some related integral-type operators can be found in [1, 17, 20–23, 37, 38].

In [17], Li and Stević studied the boundedness and compactness of the operator $P_g : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \to \mathscr{Z}_\mu(\mathscr{Z}_{\mu,0})$. Motivated by [17], in this paper we investigate the essential norm of $P_g : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \to \mathscr{Z}_\mu$.

Recall that the essential norm of $P_g : X \to Y$, denoted by $\|P_g\|_{e,X \to Y}$, is defined by

$$\|P_g\|_{e,X \to Y} = \inf\{\|P_g - K\|_{X \to Y}; K \text{ is a compact operator from } X \text{ to } Y\}.$$

In this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$. The symbol $A \approx B$ means $A \lesssim B \lesssim A$.

## 2. AUXILIARY RESULTS

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

**Lemma 2.1** ([6]). *Suppose $\omega$ is normal on $[0, 1)$. Then there exists $\omega_* \in H(\mathbb{D})$ such that*

(i) *$\omega_*(t)$ is positive and increasing on $[0, 1)$,*
(ii) *for all $z \in \mathbb{D}$, $|\omega_*(z)| \leq \omega_*(|z|)$,*
(iii) *$0 < \inf\limits_{0 < r < 1} \omega(r)\omega_*(r) \leq \sup\limits_{0 < r < 1} \omega(r)\omega_*(r) < \infty$.*

In the rest of the paper we will always use $\omega_*$ to denote the analytic function related to $\omega$ in Lemma 2.1.

To study the compactness, we need the following lemma, which can be get by Lemma 2.10 in [33].

**Lemma 2.2.** *Suppose $\omega$ and $\mu$ are normal. If $T : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \to \mathscr{Z}_\mu$ is bounded, then $T$ is compact if and only if whenever $\{f_k\}$ is bounded in $\mathcal{B}_\omega(\mathcal{B}_{\omega,0})$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$, $\lim\limits_{k \to \infty} \|Tf_k\|_{\mathscr{Z}_\mu} = 0$.*

**Lemma 2.3** ([21]). *Assume $g \in H(\mathbb{B})$ and $g(0) = 0$. Then, for every $f \in H(\mathbb{B})$,*

$$\Re P_g f(z) = f(z)g(z).$$

By some calculations, we have the following lemma.

**Lemma 2.4** ([2, Lemma 2]). *Suppose $\omega$ is normal. Then the following statements hold.*

(i) *There exists a $\delta \in (0, 1)$, such that $\omega$ is decreasing on $[\delta, 1)$ and $\lim\limits_{t \to 1} \omega(t) = 0$.*
(ii) *Fix $\alpha > 1, \beta \in (0, 1)$. When $t \in (0, 1), s \in (\beta, 1)$,*

$$\omega(t) \approx \omega(t^\alpha) \approx \frac{1}{\omega_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\omega(t)} dt \approx \int_0^s \frac{1}{\omega(t)} dt.$$

(iii) *For any $z \in \mathbb{D}$, $\left|\int_0^z \omega_*(\eta)d\eta\right| \lesssim \int_0^{|z|} \omega_*(t)dt$. If $|\eta| \leq |z|$, $\omega(|z|)|\omega_*(\eta)| < C$.*

**Lemma 2.5** ([35, Lemmas 2.2 and 2.4]). *Suppose $\omega$ is normal. Then the following statements hold.*

(i) *If $f \in \mathcal{B}_\omega$, then $|f(z)| \lesssim \left(1 + \int_0^{|z|} \frac{1}{\omega(t)dt}\right) \|f\|_{\mathcal{B}_\omega}$.*

(ii) *If $\int_0^1 \frac{1}{\omega(t)} dt = \infty$ and $f \in \mathcal{B}_{\omega,0}$, then $\lim_{|z| \to 1} \frac{|f(z)|}{\int_0^{|z|} \frac{1}{\omega(t)} dt} = 0$.*

(iii) *If $\int_0^1 \frac{1}{\omega(t)} dt < \infty$, $\{f_k\}$ is bounded in $\mathcal{B}_{\omega,0}$ and converges to 0 uniformly on compact subsets of $\mathbb{B}$, then $\lim_{k \to \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0$.*

The following lemma gives some folklore estimates, but we will prove it for the completeness.

**Lemma 2.6.** *Suppose $\omega$ is normal, $0 < r, s < 1$ and $f \in H(\mathbb{B})$. Then, for all $|z| \leq s$,*

$$|\Re f(z)| \leq \frac{2n}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)| \quad and \quad |f(z) - f(rz)| \leq \frac{2n(1-r)}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

*Proof.* Set $z = (z_1, z_2, \ldots, z_n) \in \mathbb{B}$ such that $|z| \leq s$. For $i = 1, 2, \ldots, n$, let

$$\Gamma_{z,i} = \left\{\eta \in \mathbb{D} : |\eta - z_i| = \frac{1-s}{2}\right\},$$

and

$$\lambda(z, i, \eta) = (z_1, \ldots, z_{i-1}, \eta, z_{i+1}, \ldots, z_n), \quad \eta \in \Gamma_{z,i}.$$

Since $f \in H(\mathbb{B})$, $\frac{\partial f}{\partial z_i} \in H(\mathbb{B})$. Taking $f$ as a one complex variable function about the $i$-th component of $z$, by Cauchy's integral formula, we have

$$\left|\frac{\partial f}{\partial z_i}(z)\right| = \frac{1}{2\pi} \left|\int_{\Gamma_{z,i}} \frac{f(\lambda(z, i, \eta))}{(\eta - z_i)^2} d\eta\right|$$

$$= \frac{1}{\pi(1-s)} \left|\int_0^{2\pi} f\left(\lambda\left(z, i, z_i + \frac{1-s}{2} e^{i\theta}\right)\right) e^{-i\theta} d\theta\right| \leq \frac{2}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

Here we use the change $\eta = z_i + \frac{1-s}{2} e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$. Then,

$$|\Re f(z)| \leq \frac{2n}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

When $|z| \leq s$,

$$|f(z) - f(rz)| = \left|\int_r^1 \frac{df(tz)}{dt} dt\right| = \left|\int_r^1 \langle(\nabla f)(tz), \overline{z}\rangle dt\right|$$

$$\leq (1-r) \sup_{|z| \leq s} |\nabla f(z)| \leq \frac{2n(1-r)}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 3. MAIN RESULTS AND PROOFS

**Theorem 3.1.** *Assume that $\mu$, $\omega$ are normal and $g \in H(\mathbb{B})$ such that $g(0) = 0$ and $\int_0^1 \frac{1}{\omega(t)} dt = \infty$. If $P_g : \mathcal{B}_\omega \to \mathscr{Z}_\mu$ is bounded, then*

$$\|P_g\|_{e,\mathcal{B}_\omega \to \mathscr{Z}_\mu} \approx \|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu}$$

$$\approx \limsup_{|z| \to 1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt + \limsup_{|z| \to 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|.$$

*Proof.* The following inequality is obvious.

$$\|P_g\|_{e,\mathcal{B}_\omega \to \mathscr{Z}_\mu} \geq \|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu}.$$

First, we find the lower estimate of $\|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu}$.

Suppose $K : \mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu$ is compact and $\{z_k\} \subset \mathbb{B}$ such that $|z_k| \geq \frac{1}{2}$ and $\lim_{k \to \infty} |z_k| = 1$. Let

$$f_k(z) = \frac{2p_k^3(z)}{p_k^2(z_k)} - \frac{3p_k^2(z)}{p_k(z_k)} \quad \text{and} \quad h_k(z) = \frac{p_k^3(z)}{p_k^2(z_k)} - \frac{p_k^2(z)}{p_k(z_k)},$$

where $p_k(z) = \int_0^{\langle z, z_k \rangle} \omega_*(t) dt$.

By Lemmas 2.1 and 2.4, we have

$$|p_k(z)| \leq \int_0^{|\langle z, z_k \rangle|} \omega_*(t) dt \leq \int_0^{|z||z_k|} \omega_*(t) dt,$$

and

$$|\Re p_k(z)| = |\langle z, z_k \rangle \omega_*(\langle z, z_k \rangle)| \leq \omega_*(|z||z_k|).$$

When $j = 2, 3$, $\Re p_k^j(z) = j p_k^{j-1}(z) \Re p_k(z)$. Since $\int_0^1 \frac{1}{\omega(t)} dt = \infty$, by Lemmas 2.1 and 2.4, we have the following statements.

(i) $f_k \in \mathcal{B}_{\omega,0}$, $\|f_k\|_{\mathcal{B}_\omega} \lesssim 1$, $\Re f_k(z_k) = 0$, $|f_k(z_k)| \approx \int_0^{|z_k|} \frac{1}{\omega(t)} dt$.

(ii) $h_k \in \mathcal{B}_{\omega,0}$, $\|h_k\|_{\mathcal{B}_\omega} \lesssim 1$, $h_k(z_k) = 0$, $|\Re h_k(z_k)| \approx \frac{1}{|\omega(z_k)|}$.

(iii) Both $\{f_k\}$ and $\{h_k\}$ converge to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$.

By Lemma 2.3,

$$\begin{aligned}
\|P_g - K\|_{\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu} &\gtrsim \|(P_g - K)f_k\|_{\mathscr{Z}_\mu} \\
&\geq \mu(|z_k|)|\Re g(z_k)||f_k(z_k)| - \mu(|z_k|)|g(z_k)||\Re f_k(z_k)| - \|K f_k\|_{\mathscr{Z}_\mu} \\
&\approx \mu(|z_k|)|\Re g(z_k)| \int_0^{|z_k|} \frac{1}{\omega(t)} dt - \|K f_k\|_{\mathscr{Z}_\mu}.
\end{aligned}$$

(3.1)

Letting $k \to \infty$, by Lemma 2.2, we have

$$\|P_g - K\|_{\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu} \gtrsim \limsup_{k \to \infty} \mu(|z_k|)|\Re g(z_k)| \int_0^{|z_k|} \frac{1}{\omega(t)} dt.$$

Since $K$ and $\{z_k\}$ are arbitrary, we obtain

$$\|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu} \gtrsim \limsup_{|z| \to 1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt.$$

Replacing $f_k$ by $h_k$ in (3.1), we similarly obtain that

$$\|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu} \gtrsim \limsup_{|z| \to 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|.$$

Therefore

$$\|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu} \gtrsim \limsup_{|z| \to 1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt + \limsup_{|z| \to 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|.$$

Next, we find the upper estimate of $\|P_g\|_{e,\mathcal{B}_\omega \to \mathscr{Z}_\mu}$.

For $0 < r < 1$ and $f \in \mathcal{B}_\omega$, let $f_r(z) = f(rz)$ and $T_{g,r}f = P_g f_r$. Since $\|f_r\|_{\mathcal{B}_\omega} \lesssim \|f\|_{\mathcal{B}_\omega}$, we see that

$$\|T_{g,r}f\|_{\mathscr{Z}_\mu} = \|P_g f_r\|_{\mathscr{Z}_\mu} \lesssim \|P_g\|_{\mathcal{B}_\omega \to \mathscr{Z}_\mu} \|f\|_{\mathcal{B}_\omega}.$$

So $T_{g,r} : \mathcal{B}_\omega \to \mathscr{Z}_\mu$ is bounded.

Using the test functions $h(z) = 1$ and $q(z) = z_j$ $(j = 1, 2, \ldots, n)$, we have

$$M_1 := \sup_{z \in \mathbb{B}} \mu(|z|)|\Re g(z)| < \infty \text{ and } M_2 := \sup_{z \in \mathbb{B}} \mu(|z|)|g(z)| < \infty. \tag{3.2}$$

By Lemma 2.3,

$$\|T_{g,r}f\|_{\mathscr{Z}_\mu} = \sup_{z \in \mathbb{B}} \mu(|z|)|\Re^2 T_{g,r}f(z)| \leq M_1 \sup_{z \in \mathbb{B}} |f(rz)| + M_2 \sup_{z \in \mathbb{B}} |(\Re f)(rz)|.$$

By Lemmas 2.2 and 2.6, $T_{g,r}$ is compact.

Assume $\|f\|_{\mathcal{B}_\omega} \leq 1$ and $s \in (\frac{1}{2}, 1)$. By Lemma 2.3,

$$\begin{aligned}
\|P_g f - T_{g,r}f\|_{\mathscr{Z}_\mu} &= \|P_g(f - f_r)\|_{\mathscr{Z}_\mu} \\
&\leq \sup_{|z| \leq s} \mu(|z|)|g(z)||\Re(f - f_r)(z)| \\
&\quad + \sup_{|z| \leq s} \mu(|z|)|\Re g(z)||f(z) - f(rz)| \\
&\quad + \sup_{s < |z| < 1} \mu(|z|)|g(z)||\Re(f - f_r)(z)| \\
&\quad + \sup_{s < |z| < 1} \mu(|z|)|\Re g(z)||f(z) - f(rz)|.
\end{aligned} \tag{3.3}$$

Since $\int_0^1 \frac{1}{\omega(t)} dt = \infty$, by Lemmas 2.5 and 2.6, we have

$$\sup_{|z|\leq s} |f(z) - f(rz)| \leq \frac{2n(1-r)}{1-s} \max_{|z|\leq\frac{1+s}{2}} |f(z)| \lesssim \frac{2n(1-r)}{1-s} \int_0^{\frac{1+s}{2}} \frac{1}{\omega(t)} dt, \qquad (3.4)$$

and

$$\sup_{|z|\leq s} |\Re(f - f_r)(z)| = \sup_{|z|\leq s} |(\Re f)(z) - (\Re f)(rz)| \leq \frac{2n(1-r)}{1-s} \max_{z\leq\frac{1+s}{2}} \frac{1}{\omega(|z|)}. \qquad (3.5)$$

Since $f \in \mathcal{B}_\omega$ and by Lemma 2.5, we get

$$\sup_{s<|z|<1} \mu(|z|)|g(z)||\Re(f - f_r)(z)| \lesssim \sup_{s<|z|<1} \frac{\mu(|z|)|g(z)|}{\omega(|z|)} \qquad (3.6)$$

and

$$\sup_{s<|z|<1} \mu(|z|)|\Re g(z)||f(z) - f(rz)| \lesssim \sup_{s<|z|<1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{dt}{\omega(t)}. \qquad (3.7)$$

By (3.2)–(3.7), we have

$$\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu} \leq \limsup_{r\to1} \|P_g - T_{g,r}\|_{\mathcal{B}_\omega\to\mathscr{Z}_\mu}$$

$$\lesssim \sup_{s<|z|<1} \frac{\mu(|z|)|g(z)|}{\omega(|z|)} + \sup_{s<|z|<1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{dt}{\omega(t)}.$$

Letting $s \to 1$, we get

$$\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu} \lesssim \limsup_{|z|\to1} \mu(|z|)|\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} + \limsup_{|z|\to1} \frac{\mu(|z|)}{\omega(|z|)}|g(z)|, \qquad (3.8)$$

as desired. The proof is complete. $\qquad\square$

**Theorem 3.2.** *Assume that $\mu, \omega$ are normal and $g \in H(\mathbb{B})$ such that $g(0) = 0$ and $\int_0^1 \frac{1}{\omega(t)} dt < \infty$. If $P_g : \mathcal{B}_\omega \to \mathscr{Z}_\mu$ is bounded, then*

$$\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu} \approx \|P_g\|_{e,\mathcal{B}_{\omega,0}\to\mathscr{Z}_\mu} \approx \limsup_{|z|\to1} \frac{\mu(|z|)}{\omega(|z|)}|g(z)|. \qquad (3.9)$$

*Proof.* We begin with finding the lower estimate of $\|P_g\|_{e,\mathcal{B}_{\omega,0}\to\mathscr{Z}_\mu}$.

Suppose $K : \mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu$ is compact and $\{z_k\} \subset \mathbb{B}$ such that $\lim_{k\to\infty} |z_k| = 1$ and $|z_k| > \frac{1}{2}$. Let

$$q_k(z) = \omega(|z_k|) \int_0^{\langle z,z_k\rangle} \omega_*^2(\eta) d\eta.$$

By Lemmas 2.1 and 2.4, $\{q_k\}$ are bounded in $\mathcal{B}_{\omega,0}$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemmas 2.2 and 2.5, we see that

$$\lim_{k\to\infty} \|Kq_k\|_{\mathscr{Z}_\mu} = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup_{z\in\mathbb{B}} |q_k(z)| = 0. \tag{3.10}$$

$$\begin{aligned}
\|P_g - K\|_{\mathcal{B}_{\omega,0}\to\mathscr{Z}_\mu} &\gtrsim \|(P_g - K)q_k\|_{\mathscr{Z}_\mu} \\
&\geq \mu(|z_k|)|\Re^2(P_g q_k)(z_k)| - \|Kq_k\|_{\mathscr{Z}_\mu} \\
&\geq \mu(|z_k|)|g(z_k)||\Re q_k(z_k)| - \mu(|z_k|)|\Re g(z_k)||q_k(z_k)| \\
&\quad - \|Kq_k\|_{\mathscr{Z}_\mu}.
\end{aligned} \tag{3.11}$$

Since $\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu$ is bounded and $f(z) = 1 \in \mathcal{B}_{\omega,0}$, we get

$$\sup_{z\in\mathbb{D}} \mu(|z|)|\Re g(z)| < \infty. \tag{3.12}$$

Letting $k \to \infty$ in (3.11), by (3.10) and (3.12), we obtain

$$\|P_g - K\|_{\mathcal{B}_{\omega,0}\to\mathscr{Z}_\mu} \gtrsim \limsup_{k\to\infty} \frac{\mu(|z_k|)}{\omega(|z_k|)}|g(z_k)|.$$

Since $K$ and $\{z_k\}$ are arbitrary, we see that

$$\|P_g\|_{e,\mathcal{B}_{\omega,0}\to\mathscr{Z}_\mu} \gtrsim \limsup_{|z|\to 1} \frac{\mu(|z|)}{\omega(|z|)}|g(z)|. \tag{3.13}$$

Next, we find the upper estimate of $\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu}$.

Since $\int_0^1 \frac{1}{\omega(t)}dt < \infty$, for any given $\varepsilon > 0$, there is a $\xi \in (\frac{1}{2}, 1)$, when $\xi < r < 1$, we have $\int_r^1 \frac{1}{\omega(t)}dt < \varepsilon$.

Fix $r, s \in (\sqrt{\xi}, 1)$. For all $f \in \mathcal{B}_\omega$ with $\|f\|_{\mathcal{B}_\omega} \leq 1$, when $|z| > s$, we have

$$|f(z) - f(rz)| = \left| \int_r^1 \frac{df(tz)}{dt} dt \right| = \left| \int_r^1 \frac{(\Re f)(tz)}{t} dt \right| \leq \int_r^1 \frac{1}{\omega(t|z|)t} dt \lesssim \varepsilon.$$

Thus

$$\sup_{s<|z|<1} \mu(|z|)|\Re g(z)||f(z) - f(rz)| \lesssim \varepsilon \sup_{z\in\mathbb{B}} \mu(|z|)|\Re g(z)|. \tag{3.14}$$

Then similarly to have (3.8), just instead (3.7) with (3.14), we have

$$\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu} \lesssim \limsup_{|z|\to 1} \frac{\mu(|z|)}{\omega(|z|)}|g(z)| + \varepsilon \sup_{z\in\mathbb{B}} \mu(|z|)|\Re g(z)|.$$

Since $\varepsilon$ is arbitrary, by (3.12),

$$\|P_g\|_{e,\mathcal{B}_\omega\to\mathscr{Z}_\mu} \lesssim \limsup_{|z|\to 1} \frac{\mu(|z|)}{\omega(|z|)}|g(z)|. \tag{3.15}$$

Obviously, we have

$$\|P_g\|_{e,\mathcal{B}_\omega \to \mathscr{Z}_\mu} \gtrsim \|P_g\|_{e,\mathcal{B}_{\omega,0} \to \mathscr{Z}_\mu}. \tag{3.16}$$

Then the result follows from (3.13), (3.15) and (3.16). The proof is complete. $\qquad\square$

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