Symulacja w Badaniach i Rozwoju
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Roman BOGACZ, Włodzimierz KURNIK
Politechnika Warszawska
Wydział Samochodów i Maszyn Roboczych
E-mail: rbogacz@ippt.gov.pl, wku@simr.pw.edu.pl

On Some Dynamical Effects in Discrete and Continuous Columns with Supports Subjected to Circulatory Load

1 Introduction

The analysis of beams and columns under compressive axial forces has a long history, cf. [1, 2]. In the recent years the effect of non-conservative forces has found a lot of attention. In particular, the cases of additional lateral forces depending on the transverse displacement have been widely discussed. The well known examples are Beck's and Reut's columns shown in Fig.1 [2]. In Beck's case, the lateral force Q at the tip of the column is negative linearly dependent on the angle of inclination at the end point. Such a force belongs to the category of *follower forces*, and particularly is a circulatory load (see [2] for more references).

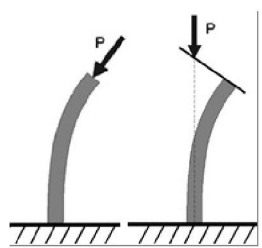

Fig. 1. Beck's and Reut's columns

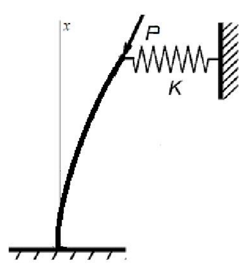

Fig. 2. Beck's column with an elastic support

The additional spring at the tip of column enables one to transfer to a column simply supported on both ends which is a divergence-type structure. The supporting spring can stabilize or destabilize the column, what means that the critical follower force can be lower or grater than in the critical force of original Reut's, Beck's and Ziegler's problems [3], [4].

1.1 Continuous formulation

The appearance of a force or moment directed back towards the undeformed configuration leads to a considerable increase of the critical force, i.e. the value of the axial load, for which for the first time certain modes of harmonic oscillations have a growing amplitude. However, there is a new phenomenon: while in the case of a pure axial loading, so-called *dead loading*, the displacement at each point of the column grows monotonously (divergence), Beck's column loses stability in an oscillatory way. In the post-critical zone the displacement at each point is a product of an exponential and a sine time functions. This behaviour is called *flatter*, as opposed to the classical *divergent loss of stability* for Euler's column.

There have been many approaches to increasing the maximum critical force [2]. There are various kinds of lateral supports, specific material properties, e.g. viscous internal damping, use of piezo-electric effects, and changes of the column geometry. In all cases, the overall cost of the column, including material, assembly and maintenance, have to be taken into account [2]. One may either try to maximize the carrying capacity under cost limitations or reduce cost on a lower boundary of the critical force. The classic case is a redistribution of a given total mass identified with the column cost, so that the critical force as an objective becomes a maximum. In dimensionless quantities, the classical Euler's column has a very low critical force of about 2.46 in comparison with the critical follower force of Beck's column reaching the value of 20.05. While the optimum found so far is about 140 (see ref. [5]). The maximal critical force of uniform Beck's column with the support located at the top (Fig.1) is almost two times higher as critical force of the uniform Beck's column. For the case of stiffer support Beck's column losses its stability by divergence, then becomes stable again in a small range of force and finally losses stability by flutter.

The above numbers show the high potential of research towards column shape and supports stiffness optimisation. However, there are some limitations to the approaches studied so far. First, a basic element of the solution is the discussion of characteristic equations for linear models. The discretization of the continuous beam (column) is difficult, because it not always leads to a satisfactory approximate solution. To explain this problem let us compare the Ziegler and Beck models of a column elastically supported on the top.

1.2 Discrete formulation

The classic Ziegler column (left-hand side of Fig. 3) is supplemented with an additional elastictic support at the top of the column as shown in the right-hand-side of Fig. 3. The equations of motion are given in ref. [3].

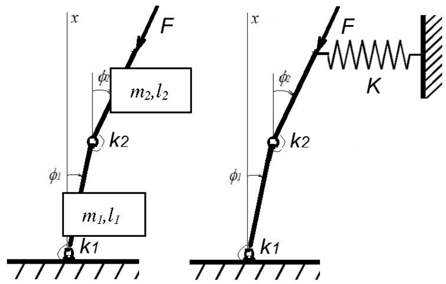

Fig. 3. Ziegler column and column with an elastic support

The typical shape of characteristic curves on the (P,ω) plane for various values of the support stiffness K is shown in Fig. 8 according to ref. [3].

2 Modelling of Beck's column

The equation of motion for the column modelled as the Bernoulli-Euler beam with a constant value of axial load P takes the form:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+P\frac{\partial^2 y}{\partial x^2}+\rho A\frac{\partial^2 y}{\partial t^2}=0\,, \tag{1}$$

where the notations are: EI – column bending stiffness, y – transverse displacement, ρ, A - column mass density and cross-section area, respectively and t,x – time and space variables.

The boundary conditions for the cantilever column with the follower force at the top tangential to the deflected axis of the column are as follows:

$$\begin{aligned}&x=0: \quad y(0,t)=0, \quad \frac{\partial y(x,t)}{\partial x}=0,\\ &x=L: \quad \frac{\partial^2}{\partial x^2}y(x,t)=0, \quad \frac{\partial}{\partial x}\left[EJ\frac{\partial^2 y(x,t)}{\partial x^2}\right]=0\end{aligned} \tag{2}$$

In that case the exact form of harmonic solutions can be expected in the form:

$$y(x,t)=w(x)\,e^{i\omega t}\,, \tag{3}$$

where w and ω are eigen-form and frequency, respectively.

The characteristic equation – it determines suitable values of ω, cf. ref. [2] – becomes in the case of a homogeneous column:

$$EIk^4+Pk^2-\omega^2\rho A=0\,. \tag{4}$$

The general solution for $w(x)$ for constant mass and stiffness distributions along the column has the well known form:

$$w(x) = A_1 \sinh k_1 x + A_2 \cosh k_1 x + A_3 \sin k_2 x + A_4 \cos k_2 x \ , \tag{5}$$

where

$$k_{1/2} = \sqrt{\pm \frac{P}{EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{\rho A \omega^2}{EI}}} \ . \tag{6}$$

Introducing the following notations: M - bending moment, Q - shear force, φ - angle of cross-section rotation, we can define the beam segment state vector **G***i*as follows:

$$\mathbf{G}_i = [w, \varphi, M, Q]^T = [w, w', EIw'', EIw''']^T \ . \tag{7}$$

For the generalized beam segment (segment of beam or support, [2]) the elementary transfer matrix $\mathbf{T_i}$ provides a relation between the states vectors for its both boundaries. For a single we have:

$$\mathbf{G}_{i+1}^0 = \mathbf{T}_i \mathbf{G}_i^0 \ . \tag{8}$$

In the case of a uniform column supported by the elastic support of stiffness K at the position x_i the partial matrix takes the form:

$$\mathbf{T}_K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ K & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

The successive left-hand-side multiplication of partial matrices results in the global transfer matrix $\mathbf{T}$ for the complete structure in the general case:

$$\mathbf{G}_{n+1}^0 = \mathbf{T} \, \mathbf{G}_n^0 = \mathbf{T}_n \mathbf{T}_{n-1} ... \mathbf{T}_2 \mathbf{T}_1 \mathbf{G}_n^0 \ . \tag{10}$$

Fulfilling boundary conditions (2) yields the following characteristic equation:

$$\begin{vmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{vmatrix} = \Phi(P, \omega) = 0 \ , \tag{11}$$

where T_{ij} are elements of matrix $\mathbf{T}$.

The classic Beck-Reut's problem was formulated and solved more than 60 years ago and the characteristic curves are shown in Fig. 4a acording to ref. [6]. However, there was a lack of explanation of variation of natural frequency versus external load for various kinds of column. In ref. [6] one can find a classification of columns into three different types of structures: flutter type, divergence type and divergence-pseudo-flutter type which are shown in Fig. 4 a,b and c, respectively.

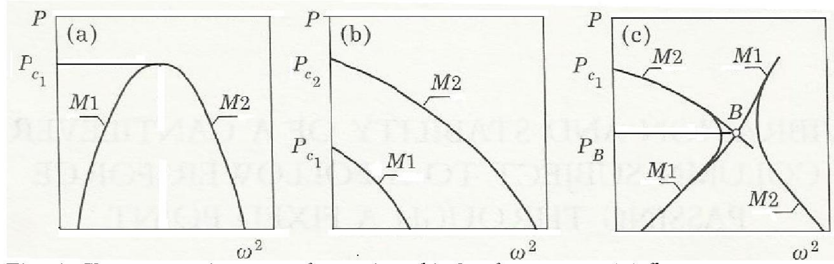

Fig. 4. Characterestic curves for various kinds of structure: (a) flutter type, (b) divergence type, (c) divergence-pseudo-flutter type

The third type of the characteristic curves shown in Fig. 4c constitutes the first step for understanding the relation between eigen-frequencies and eigen-modes of the system. However, the first type of eigen-modes configuration should be corrected. The same concerns many other papers like [1] and [7]. It was difficult to find the boundary between the first and second modes. One can state that at the critical value of the load double frequency occurs for which the corresponding eigen-form is the same. It means that the transfer point from the first to the second eigen-form ought to be below the critical value of load. It is interesting that in the case of the column elastically supported on the top all characteristic curves $\Phi(P,\omega)=0$ intersect each other in one point. This point in the case of a uniform continuous column has coordinates $P=16$ and $\omega=7$, as shown in Fig. 5. A similar singular point exists in case of the Ziegler column supported on the top (for chosen set of parameters at the value of force equal to almost 20), see Fig. 7.

3 Numerical analysis of Beck's column

The typical shape of characteristic curves on the (P,ω) plane for various values of support stiffness K is shown in Fig. 5.

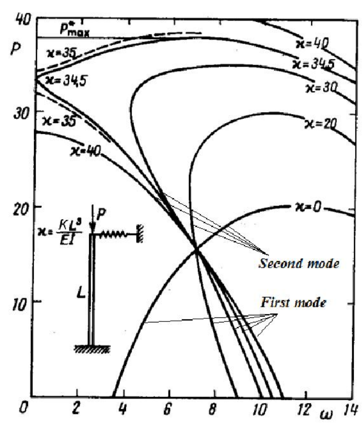

Fig. 5. Shape of characteristic curves for chosen values of support stiffness crossing singular point $P=\lambda^*=16$

One can see that for the nondimensional force $P<16$ we have classic configuration of characteristic curves (the first eigen-frequency corresponds to the first eigenform). The situation changes for the force $P>16$. The double knot existing at $x=0$ for $P=16$ splits into two knots. One of them shifts along the x axis, see Fig. 6.

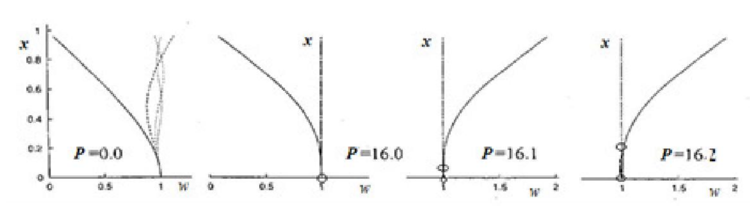

Fig. 6. Shapes of eigenfunctions at $P=\lambda^*=0$ and in the neighborhood of singular value of force $P=\lambda^*=16$

This is the way of transition of the first eigen-mode into the second one. For $P>16$ the first eigen-form does not exist and we have the second eigen-form only (with two knots). Thus, there is a difference between determination of modes in Figs.4a and 5, as explained above.

4 Elastically supported Ziegler column

Consider two rigid rods of mass and length m_1, m_2 and l_1, l_2, respectively, linked by an elastic joint and supported and loaded by a follower force, as shown in the left-hand side of Fig. 3. With two linear angular springs of stiffnesses k_1 and k_2 and absence of gravity and damping the rods constitute a two-degrees-of-freedom elastic system known as Ziegler's column. It is well known that the static equilibrium of the straight-line column loses stability at a certain value of the compressive follower force and periodic flutter vibration occurs if an existing nonlinearity allows to close a limit cycle. In a classic stability analysis the amplitudes of limit cycles are not studied although the near-critical vibrations are also interesting especially from the point of view of possible soft or hard self-excitation (see [8] where a nonlinear rotating Leipholz column with active stabilization is studied). The mathematical tool for the near-critical column behaviour one can find in ref. [9].

The original Ziegler's column is generalized by including an additional spring of stiffness K at the top, as shown in the right-hand side of Fig. 3. The nonlinear dynamic equations of motion are derived as the Lagrange equations for the generalized coordinates being rotation angles φ_1, φ_2 from the straight vertical line of the column. Full nonlinear equations of motion are as follows [3]:

$$\left(\frac{1}{3}m_1+m_2\right)l_1^2\ddot{\varphi}_1+\frac{1}{2}m_2l_1l_2\cos(\varphi_2-\varphi_1)\ddot{\varphi}_2-\frac{1}{2}m_2l_1l_2\sin(\varphi_2-\varphi_1)\dot{\varphi}_2^2+k_1\varphi_1+k_2(\varphi_1-\varphi_2)+$$
$$+K(l_1\sin\varphi_1+l_2\sin\varphi_2)l_1\cos\varphi_1+Fl_1\sin(\varphi_2-\varphi_1)=0,$$

$$\tag{12}$$

$$\frac{1}{3}m_2l_2^2\ddot{\varphi}_2+\frac{1}{2}m_2l_1l_2\cos(\varphi_2-\varphi_1)\ddot{\varphi}_1+\frac{1}{2}m_2l_1l_2\sin(\varphi_2-\varphi_1)\dot{\varphi}_1^2+k_2(\varphi_2-\varphi_1)+$$
$$+K(l_1\sin\varphi_1+l_2\sin\varphi_2)l_2\cos\varphi_2=0.$$

Equations (12) can be transformed into a single linearized matrix equation of the first order

$$\dot{\mathbf{u}}=\mathbf{A}(F,K)\mathbf{u}\ , \tag{13}$$

where $\mathbf{u}=[\varphi_1,\dot{\varphi}_1,\varphi_2,\dot{\varphi}_2]$ and

$$\mathbf{A}=\begin{bmatrix} \mathbf{0} & 1 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

The elements of matrix $\mathbf{A}$ are as follows

$$\begin{aligned} A_{21}&=(b_{12}k_{21}-b_{22}k_{11})/D & A_{41}&=(b_{21}k_{11}-b_{11}k_{21})/D \\ A_{22}&=(b_{12}c_{21}-b_{22}c_{11})/D & A_{42}&=(b_{21}c_{11}-b_{11}c_{21})/D \\ A_{23}&=(b_{12}k_{22}-b_{22}k_{12})/D & A_{43}&=(b_{21}k_{12}-b_{11}k_{22})/D \\ A_{24}&=(b_{12}k_{22}-b_{22}k_{12})/D & A_{44}&=(b_{21}c_{12}-b_{11}c_{22})/D \end{aligned} \tag{14}$$

where $D=b_{11}b_{22}-b_{12}b_{21}$ and

$$b_{11}=\left(\frac{1}{3}m_1+m_2\right)l_1^2,\qquad b_{12}=\frac{1}{2}m_2l_1l_2,$$
$$b_{21}=\frac{1}{2}m_2l_1l_2,\qquad b_{22}=\frac{1}{3}m_2l_2^2,$$

$$k_{11}=k_1+k_2-Fl_1+Kl_1^2,\qquad k_{12}=-k_2+Fl+Kl_1l_2,$$
$$k_{21}=-k_2+Kl_1l_2,\qquad k_{22}=k_2+Kl_2^2.$$

All the dynamic properties of the considered system including stability of the equilibrium, free vibrations of the stable column and transient motions in cases of instability are determined by the eigen-values and eigen-vectors of matrix $\mathbf{A}$.

The frequency versus value of the follower force F for the six chosen values of support stiffness K [N/m] are shown in Fig. 7 for an exemplary set of data in the dimensional form. The system parameters and the set support stiffness is as follows: $m_1=m_2=3.0$ Ns^2/m, $l_1=l_2=1$ m, $k_1=k_2=9$ Nm/rad, $K=0$; 2.0; 3.24; 3.7; 3,75; 5.0. In this case we can also see that for tangential force almost equal to F=20 N there exists a similar singular point as for the case of Beck-Reut's column as well as for continuous column subjected to conservative load shown in Fig. 4c. The singular point is also connected with a change of eigen-forms and is typical for pure second eigen-form of the Ziegler column. This particular case is shown in Fig. 8.

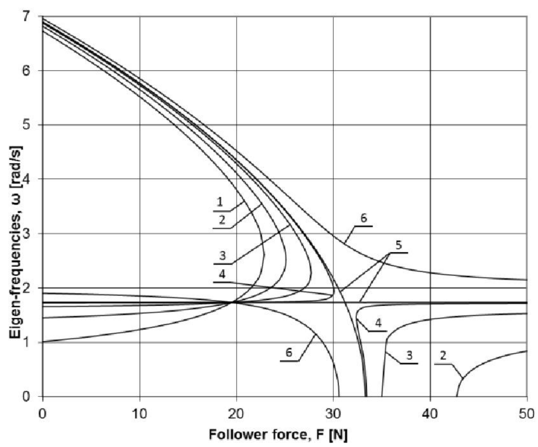

Fig. 7. Scheme of the supported Ziegler column and eigenfrequencies versus follower force F for six values of support stiffness K = 0; 2.0; 3.24; 3.7; 3,75; 5.0

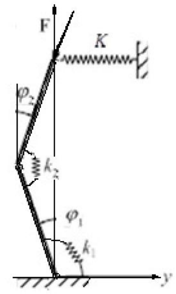

Fig. 8. The Ziegler column with pure second eigenform

5 Conclusions

The present generalization of the classic Beck's column [10]-[12] and Ziegler's column [3] by introducing an additional elastic suport enlights a rich variety of possible eigen-frequency configurations and eigen-value behavior corresponding to important configuration of the force-frequency characteristics affecting stability boundary and eigen-forms of investigated columns. The additional elastic support at the top of column enables one to transit to a column simply supported on both ends which is a second eigenform of the Ziegler column. The "stabilizing" spring can destabilize the column what means that the critical tangential follower force can be lower than that of the classic Ziegler's problem.

The generalized Beck and Ziegler columns are idealized elastic models of a variety of more realistic continuous and discrete systems and constitute a theoretical "skeleton" enlighting their basic properties and dynamic phenomena related to stability and eigen-forms of columns. The results concerning the eigen-forms of columns explain also some phenomena of experimental investigations obtained in referreces [6] and [13].

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Summary

The paper is devoted to the analysis of influene of configuration of charakteristic curves on the eigen-forms and stability of continuous and discrete columns subjected to the circulatory load. Special attention is paid to Beck's and Ziegler's columns elastically supported on the top of column. It is pointed out that in both cases there exsist some singular points where transition from the first to the second eigen-form occurs.

Keywords: circulatory load, stability, eigen-forms

O pewnych efektach dynamicznych dyskretnych i ciągłych kolumn z podporami poddanych obciążeniu cyrkulacyjnemu

Streszczenie

Pracę poświęcono analizie wpływu krzywych charakterystycznych na postaci własne i stabilność rozważanych kolumn ciągłych i dyskretnych poddanych obciążeniu cyrkulacyjnemu. Szczególną uwagę zwrócono na kolumny Becka i Zieglera sprężyście podpartych na końcach. Wykazano istnienie szczególnych punktów osobliwych na płaszczyźnie siła-częstość, w których następuje zmiana z pierwszej na drugą postać własną.

Słowa kluczowe: obciążenie cyrkulujące, stateczność, postaci własne