DOMAINS AND SOLUTIONS OF THE BRAESS PARADOX

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Abstract

The Braess paradox in road planning presents a case, where adding a new connection in a road network may lead to delayed arrival because of violation of the balance in the traffic flow. The paper discusses a generalization of this paradox. The initial, the asymmetrical, and the Pareto optimal domain are identified. Administrative solution with the participation of a controller is introduced, which aims to minimize the time of arrival, and thus has an environmental aspect. The preferences of the groups of passengers in the vehicles are modeled by an analytical arctan-approximated utility. Nash arbitration is employed to find an optimal solution that maximizes the Nash utility criterion. It is performed over the optimal Pareto domain that is outlined in four stages. A numerical example with 40 vehicles and five types of preferences of the passengers demonstrates the ideas.

Keywords: Nash arbitration, bargaining set, absolute Pareto domain, arctan-approximated utilities, disagreement point controversy, environmentally-friendly solutions

1 Introduction

The literature on urban planning offers example of cases, in which adding a new connection in a road network disturbs the balanced of streams and leads to higher time loss in travelling. In the same fashion, eliminating a road network connection could facilitate the traffic [Kolata, 1990; Knödel, 1969]. In 1968, Braess formulated these observations into the so called Braess Paradox [Braess, 2005], whose generalization is the scope of this work.

Figure 1a shows a one-way road network, leading from *Start* to *Finish*. A flow of N vehicles (N is an even number, and it is at least 4), the i-th of which contains m_i passengers, is driving from *Start* to *Finish*. Assume that x cars have chosen the route *Start*-*A*. The time needed to go on the route *Start*-*A* and on *B*-*Finish* is the number of vehicles traveling on the respective route, multiplied by $40/N$. The time to go on routes *Start*-*B* and *A*-*Finish* is always 45 minutes regardless of the traffic load. Since both routes are symmetrical in terms of time, then x is approximately $N/2$. Then each route takes $(40N/2)/N+45=20+45=65$ minutes. Let's follow the scheme on fig. 1b and construct a new one-way connection between points A and B, where the time to travel is very short and tends to zero. Assume that y vehicles have chosen the route *A*-*Finish*. Even if all vehicles have selected the route *Start*-*A*-*B*-*Finish*, the time to travel on the sections *Start*-*A* and *B*-*Finish* would be $(40N)/N=40$ minutes, whereas the alternative sections *Start*-*B* and *A*-*Finish* would take 45 minutes to travel. For that reason, in the newly established situation, each rational driver would select the route *Start*-*A*-*B*-*Finish*. As a result, the time to travel for each vehicle would be 40+0+40=80 minutes, which is more than the on-average 65-minute trip in case the new route *A*-*B* did not exist.

Assume that the police are regulating points *Start* and *A* and in fact chooses the pair (x; y) by directing the drivers to the respective routes. The objective is to find such a pair (x_{opt}; y_{opt}) that corresponds to a given type of rationality. This is a typical group decision, because for example the pair ($N/2$; $N/2$) is to be preferred over the individually "rational" pair (N; 0) that causes a 15-minute delay to all. The police have to compromise on the individual choices of the drivers. Initial results on solving this task have been proposed in [Nikolova, et al., 2012].

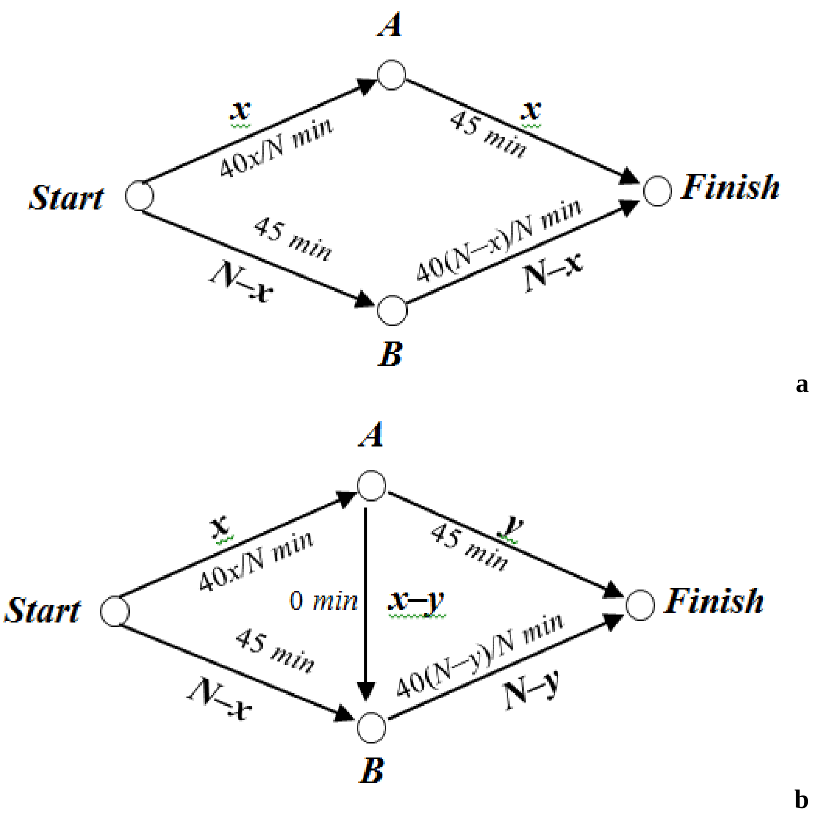

Figure 1. Time and number of vehicles on the route network with (A) and without (B) the additional connection *AB*

2 Domains for the Braess Paradox

There are three ways to go from *Start* to *Finish*: I) *Start-B-Finish*; II) *Start-A-Finish*; III) *Start-A-B-Finish*. For a fixed pair (x; y) the vehicles may be divided into three groups, containing respectively n_I, n_{II} and n_{III} vehicles, which arrive respectively in T_I, T_{II}, and T_{III} minutes:

$$T_I = 85 - 40y / N \; ; \; n_I = N - x \quad (1)$$

$$T_{II} = 40x / N + 45 \; ; \; n_{II} = y \quad (2)$$

$$T_{III} = 40(x - y) / N + 40 \; ; \; n_{III} = x - y \quad (3)$$

Each vehicle would randomly be assigned by the police to either of the groups I, II or III, and would have probability respectively n_I/N, n_{II}/N and n_{III}/N to travel respectively T_I, T_{II} or T_{III} minutes. Therefore, a discrete random variable $T(x; y)$ corresponds to the fixed pair $(x; y)$. Regardless of whether this is a group decision of all the passengers or an individual decision of the police, the task is to rank the discrete random variables according to preference in a given domain $R \notin EN$ of pairs $(x; y)$. Since x is N at most, and y does not exceed x, the decision belongs to the triangle ΔFCD from fig. 2 defined by the conditions:

$$\begin{cases} x \in 0,1,2,\ldots,N \\ y \in 0,1,2,\ldots,x \end{cases} \tag{4}$$

The formula (4) can be called *initial domain*. Let the point $(N/2; N/2)$ be denoted as E. Let the point R $(x_R; y_R)$ be an arbitrary point from ΔEDF, where $R \notin EN$. Let's define S $(x_s; y_s)=(N–y_R; N–x_R,)$. It is easy to prove that the line segment RS is perpendicular to the line segment RN. The midpoint of the line segment RS is in $M(x_M; y_M)=((N–y_R+x_R)/2; N–x_M)$ and belongs to the line segment ED. It follows that point S is symmetrical to point R according to the line segment ED.

If the times and the number of people in each group for the point $S(x_s; y_s)$ are calculated and compared to those in point $R(x_R; y_R)$, it would turn out that $T_1^S = T_{II}^R, n_I^S = n_{II}^R$, $T_{II}^S = T_I^R$, $n_{II}^S = n_I^R$, $T_{III}^S = T_{III}^R$, and $n_{III}^S = n_{III}^R$. The result is that points R and S are symmetrical according to time, i.e. similar results are acquired, and only groups I and II switch places. Following this conclusion, it is possible to limit the domain of analyzed points $(x; y)$ in ΔCDE, where:

$$\begin{cases} x \in y, y+1,\ldots,N-y \\ y \in 0,1,2,\ldots,N/2 \end{cases} \tag{5}$$

The formula (5) represents the so-called *asymmetrical domain*.

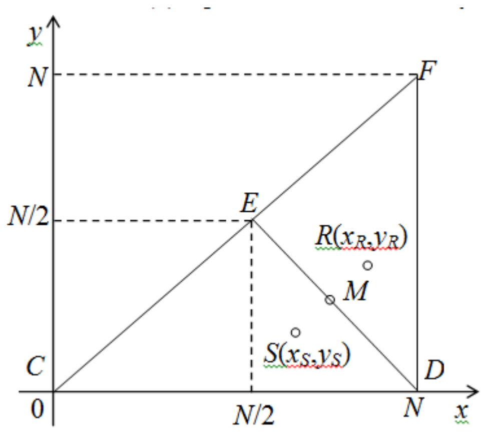

Figure 2. Domains of (x; y) in the Braess paradox

The domain (5) may be additionally tightened to an *absolute Pareto domain*. The only assumption to the preferences of the passengers is that all wish to travel as fast as possible. Let's assume that at some distribution of the vehicles (x; y) some of those reduce their time, but no one extends its travel in comparison with another distribution of the vehicles (x^*; y^*). As long as each distribution of the vehicles corresponds to a discrete random variable "time for arrival of an arbitrary vehicle (in minutes)", it is evident that the cumulative distribution function (CDF) corresponding to (x; y) would lie to the left of the CDF* corresponding to (x^*; y^*), as it is shown on fig. 3. The last distribution is called *dominated*, and its corresponding point (x^*; y^*) does not belong to the absolute Pareto domain, because the optimal decision (x_{opt}; y_{opt}) would always be different from (x^*; y^*) under any individual utility functions, as long as they are decreasing on time. As a result the absolute Pareto domain contains only the points from (5), which are not dominated. The adjective "absolute" is assigned because the domain does not depend on the particular form of the utility functions as long as they are decreasing functions of time.

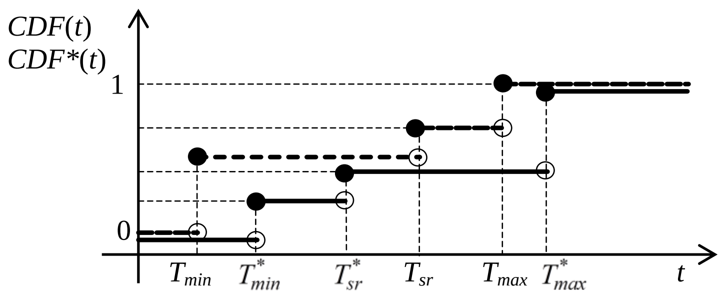

Figure 3. CDF of the dominating point (dashed line) and CDF* of the dominated point (solid line)

For example, the point (N/2; N/2) is dominating when compared with the point (0; 0), and the resulting CDFs are shown on fig. 4.

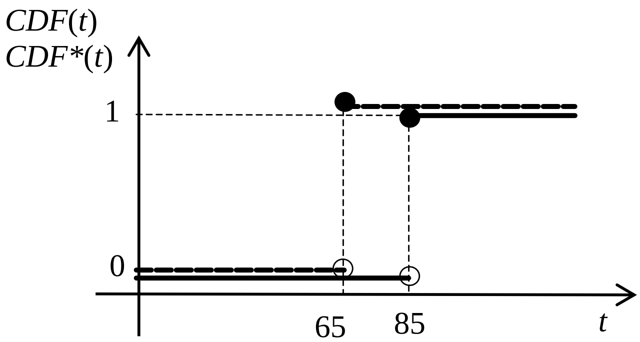

Figure 4. CDF of the dominant point (N/2; N/2) (dashed line) and of the dominated point (0; 0) (solid line)

The Appendix to the paper proves consecutively the following dominances:

- $(N/2; N/2) \succ (0; 0)$
- $(N/2; N/2) \succ (N; 0)$
- $(N/2; N/2) \succ (N-1; 0)$
- $(N-1; 1) \succ (N-2; 0)$
- $(x^*+1; 1) \succ (x^*; 0)$ for $x^*=\{1, 2, \ldots, N-3\}$
- $(2x^*; x^*) \succ (x^*; x^*)$ for $x^*=\{1, 2, \ldots, \lceil N/3 \rceil - 1\}$
- $(2N/3; 2N/3) \succ (N/3; N/3)$ if 6 divides N
- $(N/2; N/2) \succ (x^*; N-x^*)$ for $x^*=\{\lceil 13N/16 \rceil, \lceil 13N/16 \rceil+1, \ldots, N-1\}$
- $(N/2;N/2) \succ (x^*;y^*)$ for $y^*=\{1,2,\ldots, \lceil 3N/16 - 1/2 \rceil\}$, and $x^*=\{y^*+\lceil 5N/8 \rceil, y^*+\lceil 5N/8 \rceil+1,\ldots, N-y^*-1\}$

where $\succ$ stands for the binary relation "more preferred than". The notation $\lfloor a \rfloor$ stands for the greatest integer that is less or equal to a, whereas $\lceil a \rceil$ stands for the smallest integer that is greater or equal to a, where a is a real number.

The work [Tenekedjieva, 2012a] proved that the non-dominated set (absolute Pareto domain) of the generalized Braess paradox with even number of vehicles $N \geq 4$ is:

$$y \in \{1,2,\cdots,N/2\},$$

$$x \in \begin{cases} y+1;\ y+2;\cdots;\ y+\lceil 5N/8 \rceil -1 & \text{for } y \le \lfloor 3N/16-1/2 \rfloor \\ y+1;\ y+2;\cdots;\ N-y-1 & \text{for } \begin{cases} y > \lfloor 3N/16-1/2 \rfloor \\ y \le N-\lceil 13N/16 \rceil \end{cases} \\ y+1;\ y+2;\cdots;N-y & \text{for } \begin{cases} y > N-\lceil 13N/16 \rceil \\ y \le \lfloor N/3 \rfloor \end{cases} \\ y;\ y+1;\cdots;N-y & \text{for } y > \lfloor N/3 \rfloor \end{cases} \tag{6}$$

The absolute Pareto domain for N=40 is shown in dots on fig. 5.

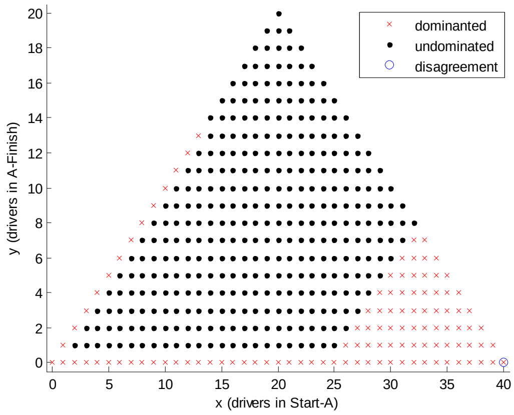

Figure 5. Absolute Pareto domain for N=40 vehicles depicted with dots, whereas the dominated points are given with crosses, and the disagreement point is circled

3 Administrative Solutions of the Braess Paradox

The choice of the pair (x_{opt}; y_{opt}) in the Braess paradox can be done administratively by the controller – in this case the police – without taking into account the individual preferences of the passengers in the vehicles. An obvious solution is to choose such an allocation of vehicles so that the total travelling time is minimized [Tenekedjieva, 2012b]. In other words, (x_{opt}; y_{opt}) =arg{min(K_1(x; y))}, where

$$K_1(x,y) = n_I.T_I + n_{II}.T_{II} + n_{III}.T_{III} = (N-x).\left(85-\frac{40}{N}.y\right)+ \\ +y.\left(\frac{40}{N}.x+45\right)+(x-y).\left(\frac{40}{N}.(x-y)+40\right) \tag{7}$$

The optimization is carried out in the absolute Pareto domain (6). This solution has an environmental aspect, since all other being equal, the longer the vehicles are running the higher the emissions are. The task has many other solutions.

Let's have a function $\varphi(v)$, which is the average fuel consumption of the vehicles per unit time at a given speed *v*. Then $(x_{opt}; y_{opt}) = \arg\{\min(K_2(x; y))\}$, where K_2 is the total fuel consumption of the vehicles from *Start* to *Finish*.

$$K_2(x, y) = (N - x).\left[45.\varphi\left(\frac{SB}{45}\right) + \frac{(N-y).40}{N}.\varphi\left(\frac{BF.N}{(N-y).40}\right)\right] + y.\left[\frac{40.x}{N}.\varphi\left(\frac{SA.N}{40x}\right) + 45.\varphi\left(\frac{AF}{45}\right)\right] + (x - y).\left[\frac{40.x}{N}.\varphi\left(\frac{SA.N}{40x}\right) + \frac{(N-y).40}{N}.\varphi\left(\frac{BF.N}{(N-y).40}\right)\right] \quad (8)$$

Here, *SB*, *BF*, *SA*, *AF* are respectively the length of the section *Start-B*, *B-Finish*, *Start-A*, *A-Finish*. Since we are not sure about the length of the routes, in this case the optimization is performed in the domain (4).

Let $\psi(v)$ be the average quantity of emissions per unit time at a given speed. Then if $\varphi(v)$ is replaced by $\psi(v)$ in (8), the resulting criterion $K_3(x; y)$ would be the total emissions for the whole vehicle flow. Similarly, the optimization is performed in the domain (4).

4 Arbitrage Decisions of the Braess Paradox

The controlling party (the police) may (and perhaps should) search for such a decision $(x_{opt}; y_{opt})$ that best corresponds to the preferences of the controlled individuals. The process of finding such a decision shall be called *arbitration*.

4.1. Describing preferences by utilities

The preferences of individuals over uncertain alternatives are described by utility functions [von Neumann, Morgenstern, 1947]. The work [Nikolova, 2007] justified the possibilities and advantages of using arctan-approximated one-dimensional decreasing utility functions over time *t* in the interval $[T_{min}; T_{max}]$, which are present in the Braess case:

$$u(t) = \frac{\text{arctg}[a(T_{max} - t_0)] - \text{arctg}[a(t - t_0)]}{\text{arctg}[a(T_{max} - t_0)] - \text{arctg}[a(T_{min} - t_0)]} \quad (9)$$

The resulting local risk aversion function shows that the arctan-approximated utility models the typical risk attitudes of people.

4.2. Decisions with equal utility functions

Let $u_i(T)$ be a utility function, defined over the time to travel T of the passengers in the i-th vehicle. It is assumed that $u_i(T)$ is a decreasing function for each i. Then, according to the passengers in the i-th vehicle $(x_{opt}; y_{opt})=\arg\{\max(E_i(u|x, y))\}$, where $E_i(u|x, y)$ is the expected utility of the i-th vehicle under distribution of the vehicles $(x; y)$.

$$E_i\ u\,|\,x, y\ = \frac{n_I}{N}.u_i\ \ T_I\ \ + \frac{n_{II}}{N}.u_i\ \ T_{II}\ \ + \frac{n_{III}}{N}.u_i\ \ T_{III} \quad (10)$$

Such a decision would apply in case the controlling party assumed that all passengers in all vehicles have equal utilities, coinciding with that of the i-th one, or when $u_i(.)$ is the utility function of the controlling party, or when $u_i(.)$ is interpreted as the average function for all passengers. The optimization is performed in the absolute Pareto domain (6). The pair that optimizes (10) could be called $(x_{opt,i}; y_{opt,i})$, because this is the desired solution by the passengers in the i-th vehicle.

4.3. Nash arbitration under different utility functions

The assumptions made in the end of section 4.2 do not correspond to reality. Therefore a decision is needed that finds the best point $(x_{opt}; y_{opt})$ from the point of view of the time for arrival, taking into account the different preferences of the passengers in the vehicles. Such a decision has been proposed by Nash, who introduced procedures for fair and rapid arbitration solution between players (conflicting parties) that need to allocate a given resource [Osborne, Rubenstein, 1994; Nash, 1950].

Assume there are M players with different utility functions. The players allocate a resource, which in this case is the time for arrival and is defined by the pair $(x; y)$. In this case, $M = \sum_{i=1}^{N} m_i$ is the total number of passengers. For the sake of simplicity a reasonable assumption is adopted that the passengers in the i-th vehicle have common utility function $u_i(.)$. In fact they can be viewed as a super-player with relative negotiation power m_i/M. Each super-player would like to have such an allocation of the resource that corresponds to her value system and risk attitude. When the preferences of all super-players are taken into consideration, for any arbitrary decision (x; y), which belongs to the asymmetric domain (5), the expected utilities (10) of the super-players would form a point in the N-dimensional space of expected utilities:

$$(E_1(u|x, y), E_2(u|x, y), \ldots, E_N(u|\, x, y)) \qquad (11)$$

Let all the feasible points (11) form an N-dimensional region called *super-feasibility region* G_s. In fact, the information in (11) can be organized in an M-dimensional space, where the i-th coordinate is copied m_i times, and each new coordinate represents the expected utility of a passenger. The resulting space is called a *feasibility region* G. As long as G_s and G contain equal information, and provided that m_i (i=1, 2, …, N) are known, it is possible to use (11) because it is of lower dimension ($N<M$). There is one-to-one mapping between the points in the two-dimensional asymmetric domain, the N-dimensional super-feasibility region G_s and the M-dimensional feasibility region G. It is assumed that in the case of disagreement, the distribution of the resource is known and is called *disagreement point*. In this case, this point is (x_{no}; y_{no})=(N; 0) (which means that all vehicles go on the route *Start-A-B-Finish*), which has the following form in the N-dimensional space of expected utilities:

$$(E_1(u|x_{no}, y_{no}), E_2(u \mid x_{no}, y_{no}), \ldots, E_N(u \mid x_{no}, y_{no})) \qquad (12)$$

Nash formulated four assumptions for rational allocation of the resource between the players as a result of arbitrating. In this case the solution called *agreement point* (x_{opt}, y_{opt}) in the asymmetric domain is represented in G_s as:

$$(E_1(u|x_{opt}, y_{opt}), E_2(u \mid x_{opt}, y_{opt}), \ldots, E_N(u \mid x_{opt}, y_{opt})) \qquad (13)$$

The first Nash assumption is that (13) belongs to the so-called *bargaining set* B_S, which is part of the Pareto efficient set PES. The PES is the part of G_s that remains once all dominated points are extracted. In other words, the dominated point corresponds to an expected utility value that is not higher than the one in the dominated point for each player, whereas for at least one player the dominant point has a higher expected utility compared to that in the dominated one. That is why in the case of two players, PES is defined as the north-eastern border of G_s. In this case, PES would be part of the absolute Pareto domain (6). Furthermore, Nash assumed that the points in B_S would have coordinates that are not less than the corresponding coordinates of (12) (for each super-player). The expectation is that in order to find the agreement point, the arbiter would focus on this part of possible actions on G that *dominates* the disagreement point. Finally, the first Nash assumption is that the agreement point belongs to B_S, which is the part of PES that dominates (12). It is needless to say that B_S can be transformed into G and into the asymmetric domain. This assumption is called *Pareto optimality*.

The second Nash assumption is that if G (but not G_s) is a symmetric M-dimensional area on all its arguments and the disagreement point (12) is with M equal arguments in G, then the agreement point (13) would also be with M equal arguments. This is obvious since if the players have equal value systems and equal negotiation powers, then the established compromise would be symmetrical. This assumption is called *symmetricity*.

The third Nash assumption is that if the utility functions of players are subjected to positive affine transformations, then the distribution of resources that corresponds to the agreement point, would not change. This is in fact a uniqueness theorem for the utility function [French, 1993], which in this particular case means that for an arbitrary real $c_i>0$ and b_i, the function $w_i(t)=c_iu_i(t)+b_i$ equivalently describes the value system and risk attitude of the passengers in the i-th vehicle. This assumption is called *invariance to affine transformations*.

The fourth Nash assumption is that if the arbitrating was conducted in G', which is part of G, and G' contained the disagreement point and the agreement point when arbitrating in G, then the agreement points in G and G' would the equal. This assumption allows widening the area of feasibility region until it became symmetrical on all its arguments. This assumption is called *independence of irrelevant alternatives*.

On the basis of these assumptions, Nash proved that the agreement point should maximize the Nash utility criterion, which is the product of the M differences between expected utilities of the players in a given point and the expected utilities of the players in the disagreement point. In this case, the optimal distribution of the vehicles would be $(x_{opt}; y_{opt})$ = arg{max($dEu(x, y)$)}, where the Nash utility criterion is:

$$dEu(x,y)=\prod_{i=1}^{N}\left(E_i(u\,|\,x,y)-u_i(T_{no})\right)^{m_i}=$$
$$=\prod_{i=1}^{N}\left(E_i(u\,|\,x,y)-u_i(80)\right)^{m_i} \tag{14}$$

The optimization of (14) should be performed in the bargaining set. However, finding it is a task much more complicated that the optimization itself. Therefore, following the second Nash assumption, the optimization of (14) can be conducted in the part of the absolute Pareto domain (6), where $E_i(u\,|\,x,y)\ge u_i(T_{no})$, for i=1, 2, …, N. The dependence in (14) exploits the fact that $E_i(u\,|\,x_{no},y_{no})=u_i(T_{no})$, since in the disagreement point, the time of arrival is not a random variable, but a constant. The assumptions in (14) are that the passengers in a given vehicle have equal utilities, and the four Nash assumptions hold.

5 An Illustrative Example

Let the number of vehicles be N=40. The number of passengers in each vehicle is given in column 4 of table 1. The minimal time T_{min}=41 min for an arbitrary vehicle is reached in the pair (x; y) = (1; 0), where 39 vehicles would travel for 85 min in group 1 on the route *Start-B-Finish* and one vehicle would travel 41 min in group 3 on the route *Start-A-B-Finish*. The maximal time T_{max}=85 min for an arbitrary vehicle is reached in the pair (x; y) = (0; 0), where 40 vehicles would travel for 85 min in group 1 on the route *Start-B-Finish*. The disagreement point (in the case of refusal of the police to direct the vehicle flow) is (x_{no}; y_{no})=(40; 0), where 40 vehicles would travel for 80 min in group 3 on the route *Start-A-B-Finish*.

The utility function of the passengers in each vehicle is arctan-approximated in the interval [41; 85], with parameters a and t_0, given in column 2 and 3 of table 1. The utility functions of the passengers in vehicles from 1 to 10 are risk prone (fig. 6). Those people prefer an uncertain arrival in time that equals to the random variable $T(x, y)$ – "time of arrival at given x and y" to a certain arrival for the expected value of $T(x, y)$: $E(T|x, y)=(n_I.T_I + n_{II}.T_{II} + n_{III}.T_{III})/N$. The utility functions of the passenger groups in vehicles from 11 to 20 are risk averse (fig. 7). They prefer a certain arrival for the expected value of T (x, y) to an uncertain arrival with time that equals to the random variable $T(x, y)$. The utility functions of the passenger groups in vehicles from 21 to 30 have an inflex point $(z_0)_i$ in the interval [T_{min}; T_{max}], thus they are risk prone in the interval [T_{min}; $(t_0)_i$] and risk averse in the interval [$(t_0)_i$; T_{max}] (fig. 8). The utility functions of the passenger groups in vehicles from 31 to 35 show they are only interested whether they would arrive before $(z_0)_i$ or after (fig. 9). Such a behavior might be explained by the lack of delay tolerance due to other arrangements (a flight, a business meeting, etc.). The utility functions of the passenger groups in vehicles from 36 to 40 are linear (fig. 10). They are risk neutral and they are indifferent on the time of arrival.

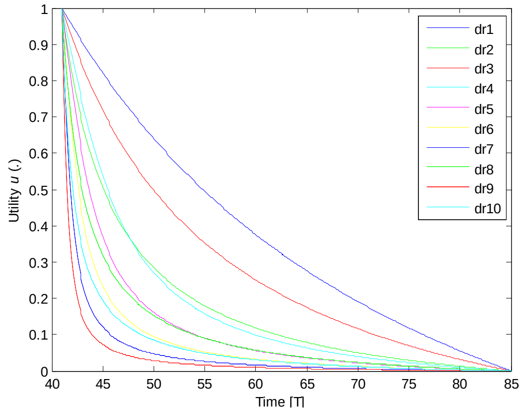

Figure. 6. Arctan-approximated utility functions of the passengers in vehicles from 1 to 10

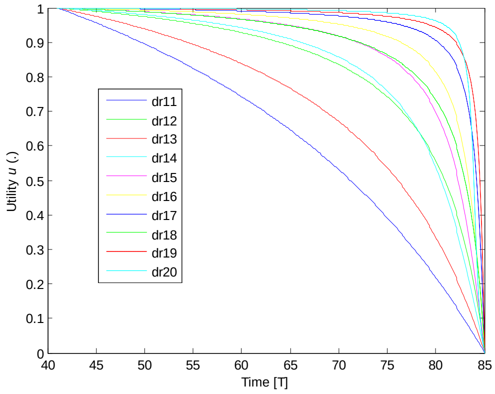

Figure 7. Arctan-approximated utility functions of the passengers in vehicles from 11 to 20

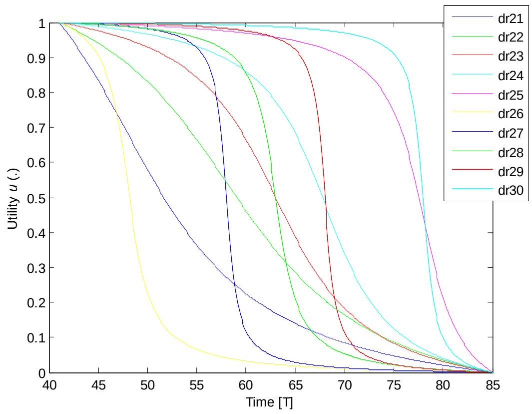

Figure 8. Arctan-approximated utility functions of the passengers in vehicles from 21 to 30

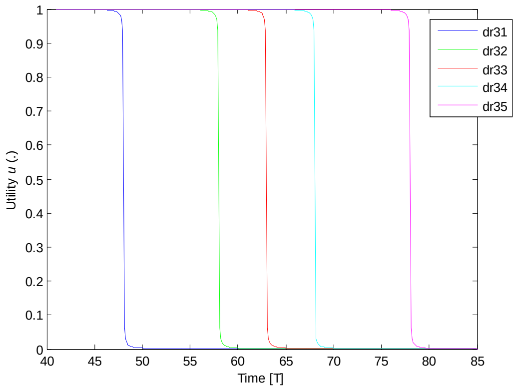

Figure 9. Arctan-approximated utility functions of the passengers in vehicles from 31 to 35

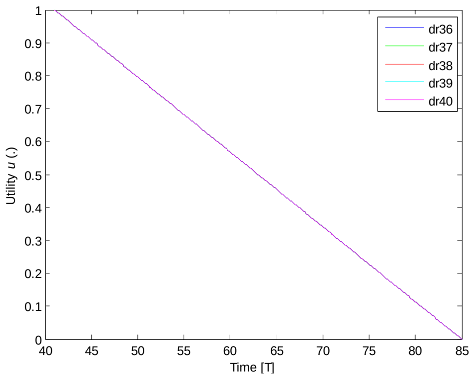

Figure 10. Arctan-approximated utility functions of the passengers in vehicles from 36 to 40

Table 1. Utilities and number of the passengers in the vehicles, expected utilities in arbitrage and in disagreement, as well as individual solutions

No.	a_i	$(t_0)_i$	m_i	$E_i(u\|x_{opt}, y_{opt})$	$E_i(u\|x_{no}, y_{no})$	$x_{opt,i}$	$y_{opt,i}$	$E_i(u\| x_{opt,i}, y_{opt,i})$
1	0.1	6	1	2.894e-1	5.207e-2	23	16	2.969e-1
2	10	36	2	3.696e-2	2.851e-3	20	19	4.200e-2
3	20	26	3	1.816e-1	2.500e-2	23	17	1.865e-1
4	0.2	41	4	1.193e-1	8.270e-3	23	17	1.243e-1
5	0.35	41	5	9.610e-2	4.141e-3	22	18	9.619e-2
6	0.6	41	1	8.776e-2	2.196e-3	22	18	8.781e-2
7	1.2	41	1	8.845e-2	1.020e-3	22	18	8.847e-2
8	1.2	39	1	2.963e-2	9.828e-4	21	19	5.021e-2
9	2	41	2	9.176e-2	5.942e-4	22	18	9.177e-2
10	2.2	40	2	5.494e-2	5.166e-4	21	18	7.088e-2
11	0.1	120	1	6.330e-1	2.187e-1	20	20	6.460e-1
12	10	90	3	8.798e-1	5.568e-1	20	20	8.909e-1
13	20	1	4	7.507e-1	3.352e-1	20	20	7.662e-1
14	0.2	85	1	8.982e-1	5.388e-1	20	20	9.096e-1
15	0.35	85	3	9.421e-1	6.983e-1	20	20	9.488e-1
16	0.6	85	1	9.665e-1	8.148e-1	20	20	9.705e-1
17	1.2	85	1	9.834e-1	9.058e-1	20	20	9.854e-1
18	1.2	87	1	9.413e-1	7.335e-1	20	20	9.476e-1
19	2	85	2	9.901e-1	9.434e-1	20	20	9.913e-1
20	2.2	84	2	9.943e-1	9.621e-1	20	20	9.951e-1
21	0.1	48	1	2.026e-1	2.145e-2	24	15	2.294e-1
22	0.09	58	3	3.370e-1	4.050e-2	26	14	3.616e-1
23	0.15	63	1	4.336e-1	3.785e-2	20	20	4.900e-1
24	0.2	68	4	6.795e-1	5.356e-2	20	20	7.680e-1
25	0.35	78	2	9.462e-1	4.125e-1	20	20	9.549e-1
26	0.6	48	1	1.125e-1	2.719e-3	24	16	1.758e-1
27	1.2	58	1	1.266e-1	2.716e-3	29	11	4.032e-1
28	0.6	63	1	3.509e-1	9.205e-3	20	20	4.977e-1
29	1.2	68	2	8.731e-1	9.041e-3	20	20	9.554e-1
30	1.1	78	3	9.848e-1	4.734e-1	20	20	9.873e-1
31	50	48	1	1.001e-1	3.034e-5	24	15	2.238e-1
32	50	58	2	1.006e-1	6.370e-5	29	10	4.722e-1
33	50	63	3	3.006e-1	1.062e-4	32	8	5.964e-1
34	50	68	2	9.965e-1	2.123e-4	20	20	9.989e-1
35	50	78	4	9.997e-1	4.994e-1	20	20	9.997e-1
36	0.10	63	1	4.568e-1	1.136e-1	22	18	4.614e-1
37	0.10	63	1	4.568e-1	1.136e-1	22	18	4.614e-1
38	0.10	63	1	4.568e-1	1.136e-1	22	18	4.614e-1
39	0.10	63	2	4.568e-1	1.136e-1	22	18	4.614e-1
40	0.10	63	5	4.568e-1	1.136e-1	22	18	4.614e-1

The Nash utility criterion (14) is calculated for that part of the absolute Pareto domain (6), which dominates the disagreement point. The results in a logarithmic scale are shown on fig. 11. The maximum value of the Nash utility criterion is 9.907e-56, under the optimal solution (x_{opt}; y_{opt})=(20;16). In that case, 20 vehicles would travel for 69 min in group 1 on the route *Start-B-Finish*, 16 vehicles would travel for 65 min in group 2 on the route *Start-A-Finish* and 4 vehicles would travel for 44 min in group 3 on the route *Start-A-B-Finish*.

The expected utilities $E_i(u|x_{opt}, y_{opt})$ for the 40 groups of passengers on the arbitrage solution (x_{opt}; y_{opt})=(20; 16) are given in column 5 of table 1. They show substantial improvement compared to the expected utilities $E_i(u|x_{no}, y_{no})$ for the respective passenger groups in the disagreement point (x_{no}; y_{no})=(40; 0), which are given in column 6 of table 1. The desired individual solutions (x; y), which maximize (10) for each of the 40 groups of passengers, are given in columns 7 and 8 of table 1. Their expected utilities $E_i(u|x_{opt,i}, y_{opt,i})$ are calculated and given in column 9 of table 1.

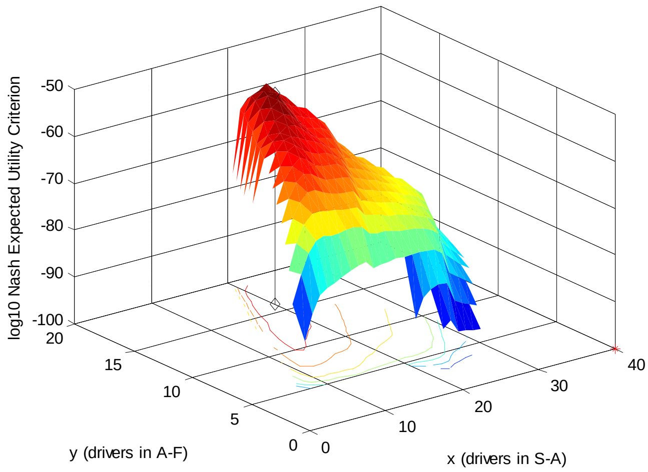

Figure 11. Nash utility criterion for the absolute Pareto domain in a Braess case with N=40

As expected, the individual solutions are better than those reached by the Nash criterion. The difference in the expected utilities in columns 9 and 5 may be interpreted as the cost of cooperation between the groups of passengers, whereas the difference between columns 5 and 6 is the benefit of cooperation. The optimal solution, the disagreement point, the absolute Pareto domain and the bargaining set are shown on fig. 12. The bargaining set is calculated by comparing the expected utilities of all 40 groups of passengers for each pair of points from the absolute Pareto domain. After the comparisons, all points, whose 40 expected utilities are less or equal than the corresponding expected utilities of the other point in the pair, are excluded from the bargaining set. Since the bargaining set depends on the 40 individual utility functions, it cannot be preliminarily calculated, unlike the absolute Pareto domain.

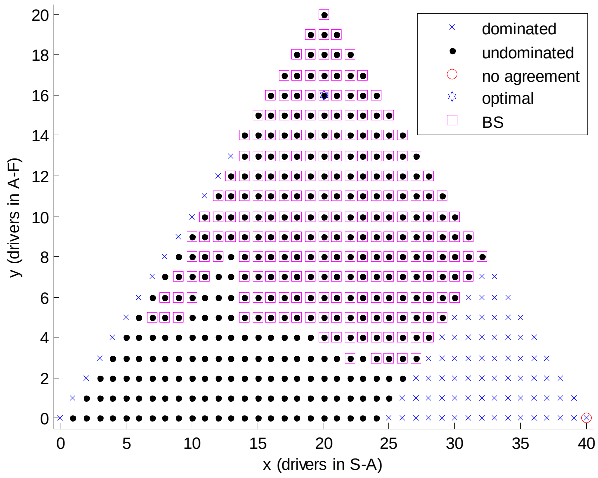

Figure 12. Bargaining set of the solution in the illustrative example

6 Conclusions

The transport network planning problem, known as the Braess paradox has been discussed in this paper. It represents the case when adding a new branch of a road network leads to increased time of arrival. Its generalization for even number of vehicles was presented. The possible time for arrival was interpreted as a discrete random variable that corresponds to a given distribution of vehicles, and those random variables had to be ranked according to preference. The optimal Pareto domain was identified by limiting the initial domain in several steps. Administrative and arbitrating solutions were described, the latter assuming equal preferences of all passengers. However, as those last assumptions are quite unlikely in a real setup, the true preferences of the passengers should be measured. Here, this was performed by using an analytical arctan-approximation of the utility function, and five different types of preferences were described. The Nash procedure of finding arbitration solutions was employed to find the optimal distribution of vehicles (the agreement point) by maximizing the Nash expected utility criterion. All calculation and visualization procedures were performed by original software programs in MATLAB 2012a environment, and available free of charge upon request from the authors.

The classical Nash theory applies to the case where several individually rational players have to allocate a given resource. In case they define their utilities over the distribution of the resource, as well as their disagreement action, then the Nash theory is in position to conclude upon the agreement that the players should and would reach. The disagreement point has the meaning of the action each player should take in case they do not reach an allocation solution.

In the course of application of the theory, two different setups appear. In the first setup, referred to as Nash bargaining, the theory is considered descriptive. The players save time by directly reaching an allocation solution among themselves by declaring their utilities and disagreement actions. Here, the players being obliged to execute the disagreement action if solution is not reached are likely to be as realistic as possible when defining that action. But in the bargaining case, each player is in position to influence the solution by altering her declared utilities over the distribution of the resource so that to reach resource allocation as best as possible for her. Declaring true utilities is considered irrational, and since they are all rational following the assumptions of Nash, they are all going to lie, thus the Nash theory is inapplicable.

In the second setup, called Nash arbitration, the theory is considered normative. The players have an arbiter who reaches the fair distribution of the resource, and to whom they declare their utilities and disagreement actions. Here, the problem of altered utilities is not crucial, as the arbiter is in position to correct them in case she thinks they are falsified. This is not the case, however, with the disagreement point. In this case, arbitrating is obligatory, i.e. a solution would always be reached. Therefore the disagreement point has no physical meaning as the players are never supposed to actually do it. Yet, the disagreement point is part of the Nash equilibrium as shown in (14), therefore players are in position to influence the final resource allocation by declaring a false disagreement action without any fear of having to execute it. Therefore, a player declaring her real disagreement action is actually considered irrational, which implies that Nash arbitration is inapplicable.

In the case discussed, however, the disagreement point has a clear physical meaning – if the passengers are not pleased with the decision of the police, they would ask for reduction of the regulation, thus arbitration would not exist. In that case, the individual rational decisions as shown in section 1, are (N; 0), i.e. they would all choose the route *Start-A-B-Finish*.

References

1. Braess, D., Nagurney, A.,Wakolbinge T., 2005, *On a Paradox of Traffic Planning*, Transportation Science, Vol. 39, No. 4, pp. 446-450.
2. French, S, 1993, *Decision theory: an Introduction to the Mathematics of Rationality*, Elis Horwood.

3. Knödel, W., 1969, *Graphentheoretische Methoden und ihre Anwendungen*, Springer-Verlag. pp. 57–9.
4. Kolata, G., 1990, *What if They Closed 42d Street and Nobody Noticed?*, New York Times, Retrieved 2008-11-16.
5. Nash, J., 1950, *The Bargaining Problem*, Econometrica, Volume 18, No. 2, pp. 155–162.
6. Nikolova, N.D., 2007, *Arctg-approximation of Monotonically Decreasing Utility Functions*, Computer Science and Technology, Volume 1, pp. 60-69
7. Nikolova, N.D., Armenski, I., Tenekedjieva, L-T., Toneva-Zheynova, D., 2012, *Multi-dimensional Nash arbitration in the Braess Paradox*, Proc. 13th International IFAC Symposium on Control in Transportation Systems CTS'2012, September 2012, Sofia, Bulgaria (in print)
8. Osborne, M.J., Rubenstein A., 1994, *A Course in Game Theory*, The Maple-Vail Book Manufacturing Group
9. Tenekedjieva, L-T., 2012a, *Finding the Pareto domain in the Braess paradox for even number of vehicles,* 12th Student Conference of the Institute of Mathematics and Informatics, Gabrovo, Bulgaria (in Bulgarian).
10. Tenekedjieva, L., 2012, *Administrative and Arbitrary Decisions in Braess's Paradox, XXXXI Spring Conference of the Union of the Mathematicians in Bulgaria*, Student Section, Borovetz, 9-12 April 2012, Bulgaria (in Bulgarian).
11. Von Neumann, J., O. Morgenstern, 1947, *Theory of Games and Economic Behavior*. Second Edition. Princeton University Press.

Appendix: Finding the absolute Pareto domain

A1. Notations and types of points in the asymmetric domain

This Appendix proves consecutively that different parts of the domain (5) are dominated. Discussed are the points from the asymmetrical domain ΔCDE from Figure 2. The purpose is to limit ΔCDE to the absolute Pareto domain by identifying the dominating points, in accordance with the comments from section 2. The random time of arrival that corresponds to an arbitrary point from ΔCDE is a discrete random variable with either one, or two, or three possible values. The following dependencies hold:

$$T_{III}<T_{II}<T_I \tag{A1}$$

$$n_I \geq n_{II} \tag{A2}$$

It is convenient to introduce T_{min} as the smallest time, for which the respective number of vehicles is greater than zero, T_{max} as the largest time, for which the respective number of vehicles is greater than zero, and eventually T_{sr} as the average time, in case the number of vehicles in each group is greater than zero, and $T_I \neq T_{II}$. Let n_{min} be the number of vehicles, for which the time of arrival is less or equal to T_{min}. Also, let n_{sr} be the number of vehicles, for which the time of arrival is less or equal to T_{sr}, and n_{max} be the number of vehicles, for which the time of arrival is less or equal to T_{max}. It is obvious that n_{max} is always equal to N. There are three types of points that may be identified in ΔCDE, which shall be analyzed consecutively:

1) Points at the vertices of ΔCDE.

These are vertices C, D, and E. According to formulae (1)-(3), the time of arrival is a discrete random variable with a single value, i.e. it is a constant value. Then:

a) for the vertex C, x=y=0. Then

$$T_{min}=T_{max}=T_I=85 \quad \text{(A3)}$$

$$n_{min}=n_{max}=n_I=N \quad \text{(A4)}$$

$$n_{II}=n_{III}=0. \quad \text{(A5)}$$

b) for the vertex D, x=N, and y=0. Then

$$T_{min}=T_{max}=T_{III}=80 \quad \text{(A6)}$$

$$n_{min}=n_{max}=n_{III}=N \quad \text{(A7)}$$

$$n_I=n_{II}=0. \quad \text{(A8)}$$

c) for the vertex E, x=y=N/2. Then

$$T_{min}=T_{max}=T_I=T_{II}=65 \quad \text{(A9)}$$

$$n_{min}=n_{max}=n_I+n_{II}=N \quad \text{(A10)}$$

$$n_{III}=0. \quad \text{(A11)}$$

2) Points on the sides CD, CE, and DE of ΔCDE;

According to formulae (1)-(3), the time of arrival is a discrete random variable with two possible values. Then:

a) for the points on the side CD, x=1, 2, …, N–1, y=0. Then

$$T_{min}=T_{III}=40x/N+40 \quad \text{(A12)}$$

$$T_{max}=85 \quad \text{(A13)}$$

$$n_{min}=n_{III}=x \quad \text{(A14)}$$

$$n_{max}=n_I+n_{III}=N. \quad \text{(A15)}$$

b) for the points on the side CE, x=1, 2, …, N/2 – 1, y=x. Then

$$T_{min}=T_{II}=40x/N+45 \quad \text{(A16)}$$

$$T_{max}=T_I=85-40x/N \quad \text{(A17)}$$

$$n_{min}=n_{II}=y=x \quad \text{(A18)}$$

$$n_{max}=n_I+n_{II}=N. \quad \text{(A19)}$$

c) for the points on the side DE, x=N/2+1, N/2+2, …, N–1, y= =N–x. Here, T_I coincides with T_{II}, i.e. they combine into a single value. Then

$T_{min}= T_{III}=40(2x–N)/N+40=80x/N$ (A20)
$T_{max}= T_{I}= T_{II}= 40x/N+45$ (A21)
$n_{min}=n_{III}=2x – N$ (A22)
$n_{max}=n_{I} +n_{II}+n_{III} =N.$ (A23)

3) Inner points for ΔCDE

According to formulae (1)-(3), the time of arrival is a discrete random variable with three possible values. Here, $x=y+1, y+2, …, N–y–1$, and $y=1, 2, …, N/2–1$. Then:

$T_{min}= T_{III}=40(x–y)/N+40$ (A24)
$T_{sr}= T_{II}=40x/N+45$ (A25)
$T_{max}= T_{I}=85–40y/N$ (A26)
$n_{min}=n_{III}=x – y$ (A27)
$n_{sr}=n_{II}+n_{III} =x$ (A28)
$n_{max}= n_{I} +n_{II}+n_{III} =N.$ (A29)

For the sake of unity, in the following proofs the dominating point shall be denoted (x; y), whereas the dominated would be (x*; y*). The corresponding times and vehicle numbers of the latter are denoted $T_I^*, T_{II}^*, T_{III}^*, T_{min}^*, T_{max}^*, T_{sr}^*, n_I^*, n_{II}^*, n_{III}^*, n_{min}^*, n_{max}^*, n_{sr}^*$.

A2. Identifying dominated points from the asymmetric domain

A2.1. The point (0; 0)

The point (x*; y*)=(0; 0) is the vertex C. It shall be proven that vertex C is dominated by vertex E, which is the point (x; y)=(N/2; N/2). According to (A3) and (A9), it follows that T_{min}^* =80>65=T_{max}, which proves that

(N/2; N/2) $\succ$ (0; 0).

The distributions of both points are shown on Figure 4.

A2.2. The point (N; 0)

The point (x*; y*)=(N; 0) is the vertex D. It shall be proven that vertex D is dominated by vertex E, which is the point (x; y)=(N/2; N/2). According to (A6) and (A9), it follows that T_{min}^* =80>65=T_{max}, which proves that

(N/2; N/2) $\succ$ (N; 0).

The distributions of both points are shown on Figure A1.

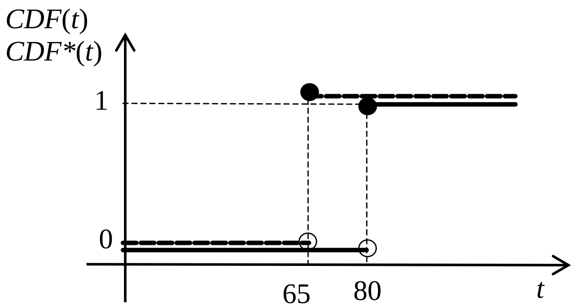

Figure A1. Dominance on the point (N; 0), given in solid line, by the point (N/2; N/2), given in thick dotted line

A2.3. The point (x*; 0) for x*={1, 2, …, N – 1}

The points (x*; y*)=(x*; 0) for x*={1, 2, …, N – 1} are inner points for the side CD. Their dominated status is proven by dividing into three subtypes.

A2.3.1. The point (N–1; 0)

It shall be proven that the point (x*; y*)=(N–1; 0) is dominated by vertex E, which is the point (x; y)=(N/2; N/2). According to (A12), T^*_{min} =80–40/N. As long as N is at least 4, then $T^*_{min}\geq 70$. According to (A9), $T^*_{min}\geq$ 70>65=T_{max}, which proves that

(N/2; N/2) $\succ$ (N–1; 0).

The distributions of both points are shown on Figure A2.

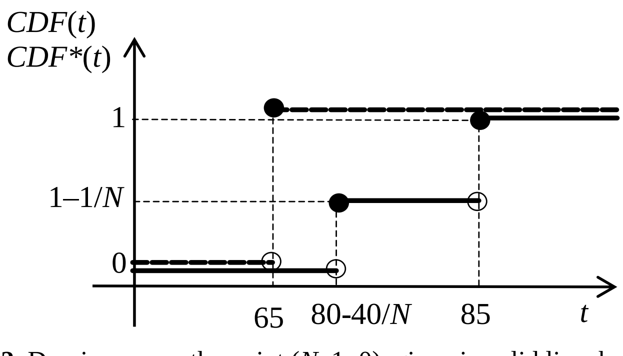

Figure A2. Dominance on the point (N–1; 0), given in solid line, by the point (N/2; N/2), given in thick dotted line

A2.3.2. The point (N–2; 0)

It shall be proven that the point (x^*; y^*)=(N–2; 0) is dominated by the point (x; y)=(N–1; 1), which is inner to the side DE. According to (A12)-(A15), T^*_{min} =80–80/N, $n^*_{min}/N = x^*/N = 1-2/N$, T^*_{max} =85. According to (A20)-(A23), T_{min}=80–80/N=T^*_{min}, n_{min}/N=1–2/N=n^*_{min}/N, T_{max}=85–40/N<85=T^*_{max}. It follows that

(N–1; 1)$\succ$ (N–2; 0).

The distributions of both points are shown on Figure A3.

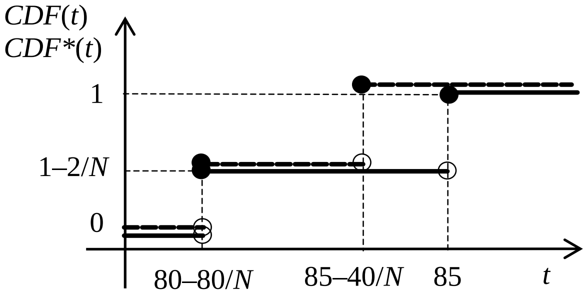

Figure A3. Dominance on the point (N–2; 0), given in solid line, by the point (N–1; 1), given in thick dotted line

A2.3.3. The points (x^*; 0) for x^*={1, 2, …, N – 3}

It shall be proven that each of the points (x^*; y^*)=(x^*; 0) for x^*={1, 2, …, N – 3} is dominated by the point (x; y)=(x^*+1; 1), which is inner for to the triangle ΔCDE . According to (A12)-(A15), T^*_{min} = 40x^*/N+40, n^*_{min}/N =x^*/N, T^*_{max} =85. According to (A24)-(A29), T_{min}= 40x^*/N+40=T^*_{min}, n_{min}/N = x^*/N= n^*_{min}/N , T_{sr}=45+40x^*/N+40/N, n_{sr}/N =x^*/N+2/N, T_{max}=85–40/N<85=T^*_{max}. It follows that

(x^*+1; 1)$\succ$ (x^*; 0) for x^*={1, 2, …, N – 3}.

The distributions of both points are shown on Figure A4.

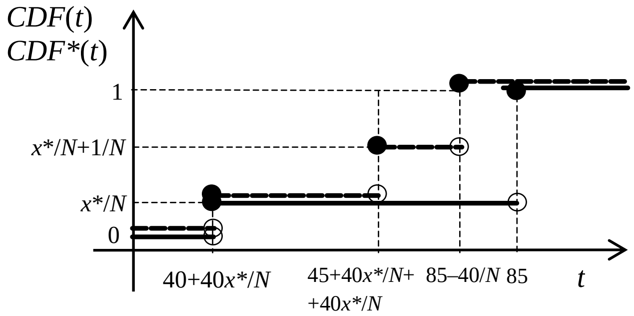

Figure A4. Dominance on the point (x^*; 0), given in solid line, by the points (x^*+1; 1) for $x^*=\{1, 2, \ldots, N-3\}$, given in thick dotted line

A2.4. The point (x^*; x^*) for $x^*=\{1, 2, \ldots, \lfloor N/3 \rfloor\}$

The points (x^*; y^*)=(x^*; x^*) for $x^*=\{1, 2, \ldots, \lfloor N/3 \rfloor\}$ are inner points for the side CE. Their dominated status is proven by dividing into two subtypes.

A2.4.1. The points (x^*; x^*) for $x^*=\{1, 2, \ldots, \lceil N/3 \rceil - 1\}$

It shall be proven that the point (x^*; y^*)=(x^*; x^*) is dominated by the point (x; y)=($2x^*$; x^*) that is inner for the triangle ΔCDE. According to (A16)-(A19), $T^*_{min}=40x^*/N+45$, $n^*_{min}/N=x^*/N$, $T^*_{max}=85-40x^*/N$. According to (A24)-(A29), $T_{min}=40x^*/N+40<T^*_{min}$, $n_{min}/N=x^*/N=n^*_{min}/N$, $T_{sr}=45+80x^*/N$, $n_{sr}/N=2x^*/N$, $T_{max}=85-40x^*/N=T^*_{max}$. It follows that

($2x^*$; x^*) $\succ$ (x^*; x^*) for $x^*=\{1, 2, \ldots, \lceil N/3 \rceil - 1\}$.

The distributions of both points are shown on Figure A5.

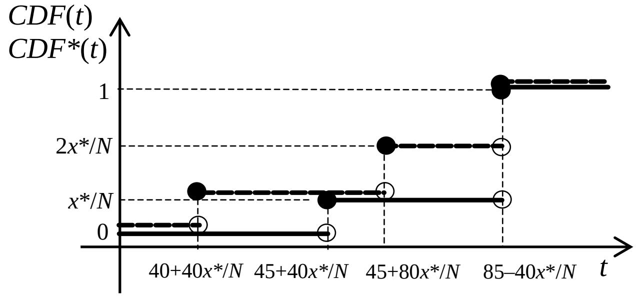

Figure A5. Dominance on the point (x^*; x^*), given in solid line, by the points ($2x^*$; x^*) for $x^*=\{1, 2, \ldots, \lceil N/3 \rceil - 1\}$, given in thick dotted line

A2.4.2. The point (N/3; N/3) if 6 divides N

It shall be proven that the point (x*; y*)=(N/3; N/3) is dominated by the point (x; y)=(2N/3; 2N/3) that is inner for the side DE. Of course, if those two points are to exist, N should be divisible to 6. According to (A16)-(A19), T^*_{min} =175/3, n^*_{min}/N =1/3, T^*_{max} =225/3. According to (A20)-(A23), T_{min}=160/3<175/3=T^*_{min} , n_{min}/N=1/3=n^*_{min}/N , T_{max}=225/3=T^*_{max} . It follows that

(2N/3; 2N/3) $\succ$ (N/3; N/3) if 6 divides N.

The distributions of both points are shown on Figure A6.

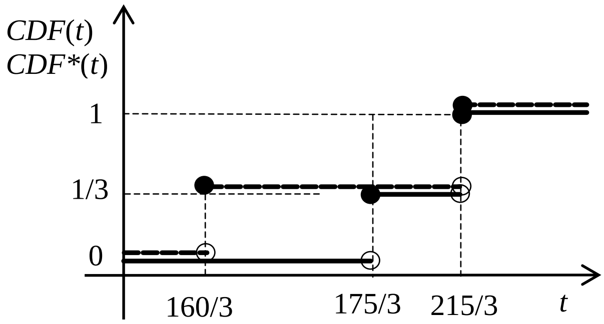

Figure A6. Dominance on the point (N/3; N/3), given in solid line, by the points (2N/3; 2N/3) if 6 divides N, given in thick dotted line

A2.5. The points (x*; N-x*) for x*={$\lceil 13N/16 \rceil$, $\lceil 13N/16 \rceil$ +1, …, N–1}.

It shall be proven that the points (x*; y*)=(x*; N-x*) for x*={$\lceil 13N/16 \rceil$ **,** $\lceil 13N/16 \rceil$ +1, …, N–1}, which are inner points for the side DE, are dominated by the point (x; y)=(N/2; N/2), which is the vertex E. According to (A20), T^*_{min} =80x*/N. According to (A9), T_{max}=65. It shall be proven that $T_{max} \leq T^*_{min}$. Let’s solve the inequality:

Domain: x*={$\lceil 13N/16 \rceil$ **,** $\lceil 13N/16 \rceil$ +1, …, N–1}

$T_{max} \leq T^*_{min}$

65 ≤ 80x*/N |*(N/80)>0

x* ≥ 13N/16

$\Rightarrow$ $x^* \in \left[13N/16; \infty\right)$ does not belong to the Domain and is not a solution

⇒ $x^*=\{\lceil 13N/16 \rceil, \lceil 13N/16 \rceil+1, \ldots, N-1\}$ does belong to the Domain and is a solution

It follows that
$(N/2;N/2) \succ (x^*;N-x^*)$ for $x^*=\{\lceil 13N/16 \rceil, \lceil 13N/16 \rceil+1,\ldots,N-1\}$.
The distributions of both points are shown on Fig. A7.

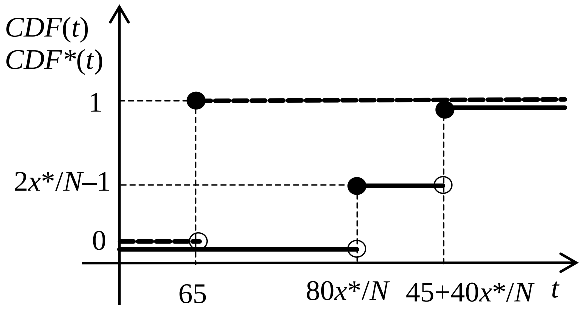

Figure A7. Dominance on the points $(x^*;N-x^*)$ for $x^*=\{\lceil 13N/16 \rceil, \lceil 13N/16 \rceil+1,\ldots,N-1\}$, given in solid line, by the points $(N/2; N/2)$, given in thick dotted line

A2.6. The points (x^*; y^*) for $y^*=\{1,2,\ldots, \lceil 3N/16-1/2 \rceil\}$, and $x^*=\{y^*+\lceil 5N/8 \rceil, y^*+\lceil 5N/8 \rceil+1,\ldots, N-y^*-1\}$.

It shall be proven that the points (x^*; y^*) for $y^*=\{1,2,\ldots, \lceil 3N/16-1/2 \rceil\}$, and $x^*=\{y^*+\lceil 5N/8 \rceil, y^*+\lceil 5N/8 \rceil+1,\ldots, N-y^*-1\}$, which are inner points for the triangle ΔCDE are dominated by the point $(x; y)=(N/2; N/2)$, which is the vertex E. According to (A24), $T^*_{min}=40(x^*-y^*)/N+40$. According to (A9), $T_{max}=65$. It shall be proven that $T_{max} \le T^*_{min}$. Let's solve the inequality, where x^* is unknown, and y* is a parameter:

Domain: $x^*=\{y^*+\lceil 5N/8 \rceil, y^*+\lceil 5N/8 \rceil+1,\ldots, N-y^*-1\}$

Parameter values: $y^*=\{1, 2,\ldots, \lceil 3N/16-1/2 \rceil\}$

$T_{max} \le T^*_{min}$

$65 \le 40(x^*-y^*)/N+40$ $\quad\quad |*(N/40)>0$

$x^* \ge 5N/8+y^* \Rightarrow x^* \in [5N/8+y^*;\infty$ does not belong to the Domain and is not a solution

$\Rightarrow$ $x^*=\{y^*+\lceil 5N/8 \rceil,\ y^*+\lceil 5N/8 \rceil+1,\ldots,\ N-y^*-1\}$ does belong to the Domain and is a solution.

This statement uses the fact that $y^*+5N/8 \leq N-y^*-1 \ \Leftrightarrow\ y^* \leq 3N/16-1/2$, which is true because

$y^* \leq \lfloor 3N/16-1/2 \rfloor \leq 3N/16-1/2$.

It follows that

$(N/2;N/2) \succ (x^*;y^*)$ for $y^*=\{1,2,\ldots,\lceil 3N/16-1/2 \rceil\}$, and

$x^*=\{y^*+\lceil 5N/8 \rceil,\ y^*+\lceil 5N/8 \rceil+1,\ldots,\ N-y^*-1\}$

The distributions of both points are shown on Fig. A8.

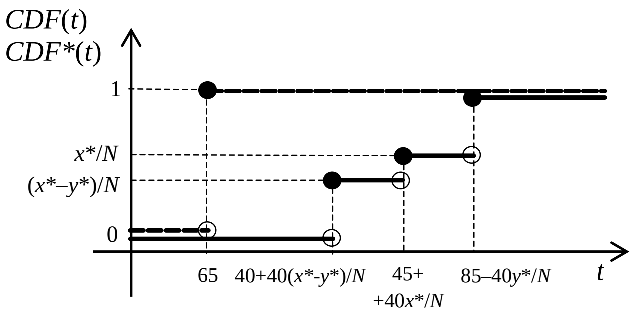

Figure A8. Dominance on the points $(x^*;\ y^*)$ for $y^*=\{1,2,\ldots,\lceil 3N/16-1/2 \rceil\}$, and $x^*=\{y^*+\lceil 5N/8 \rceil,\ y^*+\lceil 5N/8 \rceil+1,\ldots,\ N-y^*-1\}$, given in solid line, by the points $(N/2;\ N/2)$, given in thick dotted line.