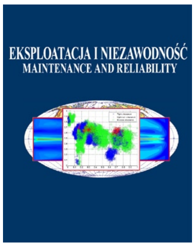

Eksploatacja i Niezawodnosc – Maintenance and Reliability

Volume 24 (2022), Issue 3

journal homepage: http://www.ein.org.pl

Article citation info:
Shang L, Zou A, Qiu Q, Du Y. A random maintenance last model with preventive maintenance for the product under a random warranty. Eksploatacja i Niezawodnosc – Maintenance and Reliability 2022; 24 (3): 544–553, http://doi.org/10.17531/ein.2022.3.15.

Indexed by: Web of Science Group

A random maintenance last model with preventive maintenance for the product under a random warranty

Lijun Shang[a], Anquan Zou[a], Qingan Qiu[b], Yongjun Du[c]

[a]Foshan University, School of Quality Management and Standardization, Foshan 528225, China
[b]Beijing Institute of Technology, School of Management & Economics, Beijing 100081, China
[c]Lanzhou University of Technology, School of Economics and Management, Lanzhou 730050, China

Highlights

- A novel random warranty is proposed by incorporating a refund and job cycles.
- The refund is used to guarantee users' fairness.
- A random periodic replacement last with PM (RPRL with PM) is modeled.
- RPRL with PM can further lengthen the service span after the expiry of the warranty.

Abstract

Although renewing pro-rate replacement warranty (RPRW) can help producers obtain some compensation from users, there seldom exists a two-dimensional random RPRW with a refund (2D-RRPRW with R) where a refund can guarantee the fairness of users. In addition, although random periodic replacement last (RPRL) can extend the service span after the expiry of the warranty, RPRL considering preventive maintenance (PM) has been seldom modeled to further lengthen the service span after the expiry of the warranty. In view of these, a 2D-RRPRW with R is devised to guarantee the fairness of users by integrating the limited job cycles and a refund into RPRW. Under the case where 2D-RRPRW with R warrants products with job cycles, a RPRL with PM is modeled to further lengthen the service span after the expiry of the warranty and reduce the failure frequency. It shows that to shorten the warranty period can makes the warranty cost of 2D-RRPRW with R to be less than the warranty cost of classic RPRW; and the performance of RPRL with PM outperforms the performance of classic RPRL.

Keywords

random RPRW, refund, replacement last, PM, failure frequency.

1. Introduction

From the viewpoint of marketing, a warranty is a type of advertising tool. Due to this, if producers (manufactures) offer each product an attractive warranty, then the corresponding warranty becomes a powerful promotion tool. From the users' perspective, warranty is an assurance of the product's reliability. In view of this, if a product under an attractive warranty is purchased, the product's reliability before the expiry of the warranty (i.e., the product's reliability in the warranty period) can be strongly ensured at no cost or at partial cost. Therefore, producers and users (consumers) can benefit from warranties.

Warranties (i.e., warranty policies, similarly hereinafter) have been extensively researched in recent years. The research direction on warranties can be summarized as three directions. The first direction aims to devise condition-based warranty by using a stochastic degradation process in [16, 22] to characterize the deterioration of the product. For example, the deterioration of items is modeled by an inverse-Gaussian (IG) process and an optimal condition-based renewable warranty was suggested in [2]; the deterioration of the product was modeled by an IG process and condition-based renewing free replacement warranty was optimized in [17]. The second direction focuses on the design of lifetime-based warranty by setting the lifetime of the product to follow a distribution function. For example, [31] modeled the first failure time as a distribution function and designed warranty menu for a two-dimensional warranty; [25] modeled the failure time as a distribution function and studied a warranty which is a combination of base and extended warranties; [15] set the lifetime of the product to obey a distribution function and optimized a three-dimensional warranty; other warranty research can be founded in [3, 5, 7, 26, 29]. The third direction designs performance-based warranty by modeling simultaneously the deterioration of the product as a stochastic degradation process and a distribution function. For example, [27] modeled the deterioration of the product as a stochastic degradation process and a distribution function, and studied a performance-based warranty for products, which includes competing hard and soft failures.

[17] divided warranties into one-dimensional warranties, two-dimensional warranties, simple warranties and combination warranties. Among all warranties, renewing (renewable) warranty (RW) is one of

the most attractive warranties. As a kind of special RW, renewing free replacement warranty (RFRW) [8, 28] requires that the failed product before the expiry of the warranty is replaced free of charge with a new identical product under RFRW. If the product's reliability is higher and the production cost is lower, then RFRW is obviously an ideal warranty. For the product with a lower reliability and a higher production cost, RFRW can make producers incur the higher warranty costs. In this case, RFRW is no longer an ideal warranty. For the product with a lower reliability and a higher production cost, one of its ideal warranties is renewing pro-rate replacement warranty (RPRW) [6, 15]. This is because RPRW can make producers to obtain some compensation from users, in the form of charging a pro-rate fee to perform a replacement triggered by a failed product.

Lately, warranties with refunds [9, 12] were studied and focused in academia and industry. Because this kind of warranty requires that producers offer a partial or full refund to users when the product's performance before the expiry of the warranty can't meet requirements, it is also one of the most attractive warranties. If a refund is integrated into an original RW, then RW will become a most attractive warranty, compared with original RW and original warranties with refunds [14]. However, few original RW has been redesigned by integrating a refund.

From the viewpoint of the reliability management, the reliability management in the life cycle of the product includes two types. The first type is that the reliability management before the expiry of the warranty (i.e., the reliability management in the warranty period), which is in the charge of the manufacturer; the second type is that the reliability management after the expiry of the warranty (i.e., the reliability management in the post-warranty period), which is not in the charge of the producer. In view of this, users tend to focus on how the product's reliability after the expiry of the warranty is guaranteed. From the user's viewpoint, some maintenance policies have been modeled to guarantee the product's reliability after the expiry of the warranty. For example, by using a stochastic degradation process in [30, 32, 35, 36] to characterize the deterioration of the product, [17] investigated an optimal condition-based maintenance to ensure the product's reliability after the expiry of the warranty. In addition, by modeling the lifetime of the product as a distribution function, some maintenance policies have been developed to guarantee the product's reliability after the expiry of the warranty. For example, [4] defined that the lifetimes of components follows some distribution functions and investigated optimization maintenance models for warranted coherent systems that are composed of n components ($n \geq 1$), which were used to guarantee the product's reliability after the expiry of the warranty; the product's lifetime was modeled as a distribution function and then the replacement strategy with minimal repairs was adopted to ensure the product's reliability after the expiry of the warranty in [10, 19] modeled a maintenance–replacement policy to guarantee the product's reliability after the expiry of the warranty, under the case where the product's lifetime was set to follow a distribution function.

Recently, wireless monitoring system (WMS) integrated with the advanced technologies (such as in-situ sensors, industrial networks, control technologies) are being applied extensively in the prognostic and health management of the product. In addition, WMS can monitor job cycles of the product which does successive missions at job cycles. In other words, WMS can help in real time measure the total working time of the product. This signals that the product's reliability can be ensured by means of WMS. By defining each working time as an independent random variable (i.e., random working time) and integrating it into some classic maintenance policies, [13] has been modeled some maintenance policies to ensure the product's reliability, which are called random maintenance policies in [13]. Besides, [33] proposed replacement first and last, under the case at which all working cycles are modeled as independent and identically distributed random variables; [20] proposed general age replacement, replacement first, replacement last and replacement next, which are modeled by integrating random working time; [34] studied age and periodic replacement last models with continuous and discrete policies, under the case where all working cycles were defined as variable working cycles.

Similarly, by introducing job/working cycles to the reliability management before the expiry of the warranty, some random warranties can be designed from the producer's perspective; by introducing job/working cycles to the reliability management after the expiry of the warranty, some random maintenance policies to guarantee the product's reliability after the expiry of the warranty can be modeled from the user's perspective. Considering these, by incorporating random working cycles into the life cycle of the product, [21] earlier proposed two random warranties to guarantee the product' s reliability before the expiry of the warranty and two random maintenance policies to guarantee the product' s reliability after the expiry of the warranty. Although two random warranties have been proposed earlier in the above literature, authors have neglected a fact that a refund can remove the unfairness of users whose warranty expires before the warranty period at the accomplishment of the limited random working cycles. In addition, to extend the service span after the expiry of the warranty, a random periodic replacement last to guarantee the product's reliability after the expiry of the warranty has been modeled earlier in the above literature, authors did not consider a reality that preventive maintenance (PM) can improve the product's reliability, which can further lengthen the service span after the expiry of the warranty and further reduce the failure frequency.

In this paper, we incorporating both the limited job cycles and a refund into original RPRW and devise a two-dimensional random RPRW with a refund (2D-RRPRW with R) from the producer' perspective. The devised 2D-RRPRW with R stipulates that ① if the failure does not happen until the warranty period or until the accomplishment of the limited job cycles, whichever comes first, then the producer charges the user a pro-rata fee to replace the failed product with a new identical product under the same warranty; otherwise, 2D-RRPRW with R will expire at the warranty period or at the accomplishment of the limited job cycles, whichever comes first; ② if 2D-RRPRW with R expires before the warranty period at the accomplishment of the limited job cycles, then the producer will provide users a partial refund to guarantee the fairness of users whose warranty expires before the warranty period at the accomplishment of the limited random working cycles. Extending 2D-RRPRW with R to the reliability management after the expiry of the warranty, a users' random maintenance policy is modeled to ensure the product's reliability after the expiry of the warranty. The random maintenance policy of the user satisfies that ① if the first job cycle (i.e., the first job cycle after the expiry of the warranty, hereinafter similarly) is accomplished after the planned time, then the product through 2D-RRPRW with R is replaced after the planned time at the accomplishment of the first job cycle; otherwise the product through 2D-RRPRW with R is replaced after the accomplishment of the first job cycle at the planned time; ② if the first job cycle still is not accomplished until the planned time, then PM is done at the planned time to improve the product's reliability, which can further lengthen the service span after the expiry of the warranty and further reduce the failure frequency; ③ minimal repair removes all failures before replacement. Obviously, the random maintenance policy mentioned above includes a random periodic replacement last (which is composed of ① as well as ③) and a PM, thus such random maintenance policy can be referred to as a random periodic replacement last with PM (RPRL with PM) policy.

The contribution of this paper is listed as three key aspects: (1) by introducing the limited job cycles and a refund to an original RPRW, a novel warranty is proposed to ensure the product's reliability before the expiry of the warranty; (2) a refund is used to guarantee the fairness of users whose warranty expiration occurs before the warranty period at the accomplishment of the limited job cycles; (3) by incorporating the limited job cycles and a PM into the service span after the expiry of the warranty, RPRL with PM is modeled to further lengthen the service span after the expiry of the warranty and to further reduce the failure frequency.

The paper structure is listed as follows. Section 2 provides model assumptions, devises the producer's random warranty with a refund, and evaluates the warranty cost. In Section 3, RPRL with PM is presented, and the cost rate model is derived. In Section4, numerical experiments to perform sensitivity analysis and to illustrate the proposed approaches are offered. Section 5 draws conclusions.

2. Random warranty of producers

2.1. Model assumption

The used assumptions are listed as:

- The product is repairable.
- The product can be continuously monitored and performs successive missions at job cycles.
- The time to replacement, PM or minimal repair is disregarded.
- Each warranty claim is accepted.
- Each abbreviation for warranty represents a corresponding warranty policy.
- All information is symmetric, which means that the producer and customer know all parameters and cost items.
- All job cycles are independent random variables with an identical distribution function which is memory-less.

2.2. Warranty devising

As far as the warranty term is concerned, if removing the pro-rata fee in RPRW, then RPRW is reduced to RFRW. This means that RFRW is a special case of RPRW and that RPRW is a generalized warranty. By incorporating the random working cycle into classic maintenances policies, [13] designed and modeled some new maintenances policies, which were called random maintenance policies in [13]. Without exception, by incorporating the limited job cycles into RPRW and considering 'whichever comes first', we can devise a two-dimensional random RPRW (2D-RRPRW) at which the warranty period w and the number m ($0<m<\infty$) of job cycles are two warranty limits. When such 2D-RRPRW warrants the product, the warranty service of the user with a higher product job frequency and a shorter product job cycle expires very easily before the warranty period w at the accomplishment of the m^{th} job cycle. This type of users may be aware that they are not treated as equal as other users whose warranty expiration occurs before the accomplishment of the m^{th} job cycle at the warranty period w. From the viewpoint of the warranty theory, the refund is one of methods to guarantee the fairness of users with a higher product job frequency and a shorter product job cycle.

Based on the above analysis, by integrating the limited job cycles and a refund into RPRW, we devise a two-dimensional random RPRW with a refund from the producer's perspective, as shown below.

1) If the failure does not happen until the warranty period w or until the accomplishment of the m^{th} job cycle, whichever comes first, then the warranty expires meanwhile the product goes through warranty.
2) If the failure occurs before the warranty period w or before the accomplishment of the m^{th} job cycle, whichever comes first, then the product does not go through warranty meanwhile a new identical product warranted by the present warranty will be adopted to replace the failed product.
3) The producer charges users a pro-rata fee to perform each replacement triggered by the failure.
4) If the warranty expiration occurs before the warranty period w at the accomplishment of the m^{th} job cycle, then the producer will provide users a partial refund to make users have the sense of fairness.

Note that ① obviously, w and m are two types of warranty limits, thus the warranty region of the devised warranty is given by $(0,w]\times(0,m]$; ② the replacement is done in the form of charging a pro-rata fee; ③ for users with a higher product job frequency and a shorter product job cycle, their warranty expiration happens very easily before the warranty period w at the accomplishment of the m^{th} job cycle, therefore a partial refund is offered to them in order that they are treated as equal as other users whose warranty expiration happens before the accomplishment of the m^{th} job cycle at the warranty period w. In view of these, similar to the name of random maintenances in [13], thus it is reasonable that such warranty is regarded as a two-dimensional random RPRW with a refund (2D-RRPRW with R).

2.3. Warranty cost evaluation

Let X be the time of the product first failure. Then, the distribution and reliability function of X are denoted by $F(x)$ and $\bar{F}(x)$, where $F(x)+\bar{F}(x)=1$. And let $G(y)$ be an identical distribution function of all job cycles Y_i ($i=1,2,\cdots,$). According to the reliability theory, then the distribution and reliability function of the working time S_m (where $S_m=\sum_{i=0}^{m}Y_i$) are denoted by a Stieltjes convolution $G^{(m)}(s)=\int_0^s G^{(m-1)}(s-u)\mathrm{d}G(u)$ and a Stieltjes convolution $\bar{G}^{(m)}(s)=1-G^{(m)}(s)$, respectively. Obviously, 2D-RRPRW with R expires before the accomplishment of the m^{th} job cycle at the warranty period w or before the warranty period w at the accomplishment of the m^{th} job cycle. By summing probabilities, then the probability q that 2D-RRPRW with R expires can be computed as:

$$q=\Pr\{w<S_m,w<X\}+\Pr\{S_m<w,S_m<X\}=\bar{G}^{(m)}(w)\bar{F}(w)+\int_0^w\bar{F}(u)\mathrm{d}G^{(m)}(u)=1-\int_0^w\bar{G}^{(m)}(u)\mathrm{d}F(u) \tag{1}$$

By 2D-RRPRW with R, if the failure happens before the warranty period w or before the accomplishment of the m^{th} job cycle, whichever comes first, then 2D-RRPRW with R is renewed rather than expiring. This event can be characterized as an inequality $X<\min(w,S_m)$. Thus, the probability p that 2D-RRPRW with R does not expire can be computed as:

$$p=\Pr\{X<\min(w,S_m)\}=\int_0^w\bar{G}^{(m)}(u)\mathrm{d}F(u) \tag{2}$$

Clearly, $q+p=1$.

$j-1$ failures have occurred in the warranty region $(0,w]\times(0,m]$ until the j^{th} product ($j=1,2,\ldots$) goes through 2D-RRPRW with R. According to the probability theory, the number $j-1$ satisfies the geometric distribution $p^{j-1}q$. Furthermore, until the first product goes through 2D-RRPRW with R, the expected value $E[j-1]$ of the replacements triggered by all failures can be modeled as:

$$E[j-1]=\sum_{j=1}^{\infty}p^{j-1}q\cdot(j-1)=\frac{p}{q}=\frac{\int_0^w\bar{G}^{(m)}(u)\mathrm{d}F(u)}{1-\int_0^w\bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{3}$$

Let X_k ($k=0,1,2,\cdots$) be the lifetime of the k^{th} product which fails in the warranty region $(0,w]\times(0,m]$. Obviously, This event can be characterized as an inequality $X_k\le\min(w,S_m)$. Then, the lifetimes X_k ($k=0,1,2,\cdots$) are independent and follow an identical distribution function $H(x)$:

$$H(x)=\Pr\{X_k<x\mid X_k\le\min(w,S_m)\}=\frac{\int_0^x\bar{G}^{(m)}(u)\mathrm{d}F(u)}{\int_0^w\bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{4}$$

where $X_0 = 0$ and $0 < x < w$.

Assume that the pro-rata fee that the producer charges is only affected by the working time t ($0 < t \le w$), then the pro-rata fee $P(t)$ at the working time t is modeled as:

$$P(t) = at^b \tag{5}$$

where $0 < a < 1$, $0 < b < \log_w(C_R / a)$ and C_R is the producer's unit replacement cost.

For the k^{th} product which failed in the warranty region $(0, w] \times (0, m]$, its working time is equal to its lifetime X_k. Then, its pro-rata fee is given by $P(X_k)$. Furthermore, after the k^{th} failed product is replaced, the producer suffers a net replacement cost $C_R - P(X_k)$. Therefore, after the $(j-1)^{\text{th}}$ failed product is replaced, the producer's total replacement cost TRC_{j-1} is given by:

$$TRC_{j-1} = \sum_{k=0}^{j-1} \left(C_R - P(X_k)\right) = \sum_{k=0}^{j-1} \left(C_R - a(X_k)^b\right) \tag{6}$$

As mentioned above, X_k follows $H(x)$ in (4) and $j-1$ satisfies $p^{j-1}q$. Therefore, the expected value $E[TRC]$ of the producer's total replacement cost TRC_{j-1} can be represented by:

$$E[TRC] = E[\sum_{j=1}^{\infty} p^{j-1}q \cdot TRC_{j-1}] = E[j-1] \cdot E[\left(C_R - a(X_k)^b\right)] = \frac{\int_0^w \left(C_R - ax^b\right)\bar{G}^{(m)}(x)\mathrm{d}F(x)}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{7}$$

where $E[j-1]$ has been offered by (3) and $E[\left(C_R - a(X_k)^b\right)] = \int_0^w \left(C_R - ax^b\right)\bar{G}^{(m)}(x)\mathrm{d}F(x) / \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)$.

By the definition of 2D-RRPRW with R, when 2D-RRPRW with R expires before the warranty period w at the accomplishment of the m^{th} job cycle, the producer provides the user a partial refund. Let $R(S_m)$ be a refund at the working time S_m, then the refund $R(S_m)$ is represented by:

$$R(S_m) = \begin{cases} C_R - \alpha(S_m)^{\beta}, & for\ S_m < w \\ 0, & for\ S_m \ge w \end{cases} \tag{8}$$

where $0 < \alpha < 1$ and $0 < \beta < \log_w(C_R / \alpha)$.

When 2D-RRPRW with R expires before the warranty period w at the accomplishment of the m^{th} job cycle, the working time of the product is equal to S_m. In this case, the distribution function $K(s)$ of the working time S_m can be derived as:

$$K(s) = \frac{\Pr\{S_m < s, S_m < X\}}{\Pr\{S_m < w, S_m < X\}} = \frac{\int_0^s \bar{F}(u)\mathrm{d}G^{(m)}(u)}{\int_0^w \bar{F}(u)\mathrm{d}G^{(m)}(u)} \tag{9}$$

where $0 < s < w$.

As mentioned above, the case that 2D-RRPRW with R expires can be divided two cases. The first case is that 2D-RRPRW with R expires before the accomplishment of the m^{th} job cycle at the warranty period w. The second case is that 2D-RRPRW with R expires before the warranty period w at the accomplishment of the m^{th} job cycle. Let q_w be the probability that 2D-RRPRW with R expires before the accomplishment of the m^{th} job cycle at the warranty period w, then q_w is represented by $q_w = \Pr\{w < S_m, w < X\} = \bar{G}^{(m)}(w)\bar{F}(w)$. The probability Q_w that the first product survives 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w can be computed as:

$$Q_w = \sum_{j=1}^{\infty} p^{j-1}q_w = \frac{q_w}{1-p} = \frac{q_w}{q} = \frac{\bar{G}^{(m)}(w)\bar{F}(w)}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{10}$$

Let q_m be the probability that 2D-RRPRW with R expires before the warranty period w at the accomplishment of the m^{th} job cycle, then q_m is represented by $q_m = \Pr\{S_m < w, S_m < X\} = \int_0^w \bar{F}(u)\mathrm{d}G^{(m)}(u)$. Furthermore, the probability Q_m that the first product goes through 2D-RRPRW with R before the warranty period w at the accomplishment of the m^{th} job cycle can be computed as:

$$Q_m = \sum_{j=1}^{\infty} p^{j-1}q_m = \frac{q_m}{1-p} = \frac{q_m}{q} = \frac{\int_0^w \bar{F}(u)\mathrm{d}G^{(m)}(u)}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{11}$$

Since $q_w + q_m = q$, the equation $Q_w + Q_m = 1$ holds.

The probability Q_w that the first product goes through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w has been offered in (10) and the probability Q_m that the first product goes through 2D-RRPRW with R before the warranty period w at the accomplishment of the m^{th} job cycle has been offered in (11). Since the working time S_m follows $K(s)$ in (9), the expected value R of the refund $R(S_m)$ can be represented by:

$$R = Q_w \cdot 0 + Q_m \cdot E[R(S_m)] = Q_m \cdot \int_0^w R(s)\mathrm{d}K(s) = \frac{\int_0^w (C_R - \alpha s^{\beta})\bar{F}(s)\mathrm{d}G^{(m)}(s)}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{12}$$

According to the definition of 2D-RRPRW with R, the warranty cost of 2D-RRPRW with R includes the expected value $E[TRC]$ of the total replacement cost TRC_{i-1} and the expected value R of the refund $R(S_m)$. By summing (7) and (12), the warranty cost WC of 2D-RRPRW with R is evaluated as:

$$WC = E[TRC] + R = \frac{\int_0^w \left(C_R - ax^b\right)\bar{G}^{(m)}(x)\mathrm{d}F(x) + \int_0^w (C_R - \alpha s^{\beta})\bar{F}(s)\mathrm{d}G^{(m)}(s)}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{13}$$

When $m \to \infty$, $\bar{G}^{(m)}(\bullet) \to 1$. This means that when $m \to \infty$, the warranty limit m fails and 2D-RRPRW with R is reduced to renewing pro-rata replacement warranty (RPRW) [6, 15]. Therefore, when $m \to \infty$, the warranty cost WC of 2D-RRPRW with R can be simplified as a warranty cost of RPRW, i.e.:

$$\lim_{m \to \infty} WC = \frac{\int_0^w \left(C_R - ax^b\right)\mathrm{d}F(x)}{\bar{F}(w)} \tag{14}$$

Besides, $ax^b = 0$ (i.e., $P(x) = 0$) means that the pro-rata fee is removed, and $m \to \infty$ means that the warranty limit m fails. Therefore, when $ax^b = 0$ (i.e., $P(x) = 0$) and $m \to \infty$, the warranty cost WC of 2D-RRPRW with R can be simplified as a warranty cost of renewing free replacement warranty (RFRW) [8, 28], i.e.:

$$\lim_{\substack{ax^b \to 0 \\ m \to \infty}} WC = \frac{C_R F(w)}{\bar{F}(w)} \tag{15}$$

3. The users' random maintenance model

As mentioned above, the user's focus is how the product's reliability after the expiry of the warranty is ensured. In this section, by defining that 2D-RRPRW with R warrants the product with job cycles, a random maintenance policy of users is planned to guarantee the product's reliability after the expiry of the warranty, as shown below.

It is planned that the product's reliability after the expiry of the warranty is ensured by a random periodic replacement last with PM (RPRL with PM) policy satisfying ① the product through 2D-RRPRW with R is replaced at the accomplishment of the first job cycle or at the planned time T , whichever comes last; ② if the first job cycle is not accomplished until the planned time T , then PM will be done at the planned time T to lengthen the service span after the expiry of the warranty and to reduce the failure frequency; ③ all failures before replacement undergo minimal repair. Obviously, if PM is removed from RRL with PM, then RPRL with PM is simplified as a random periodic replacement last (RPRL) [21] at which the replacement is carried out at the accomplishment of the first job cycle or at the planned time T , whichever comes last, and all failures before replacement undergo minimal repair.

Obviously, according to the definition of RPRL with PM, RPRL with PM includes two kinds of replacement. The first kind is to replace the product through 2D-RRPRW with R after the planned time T at the accomplishment of the first job cycle; the second kind is to replace the product through 2D-RRPRW with R after the accomplishment of the first job cycle at the planned time T .

For convenience, define that the product's life cycle begins in installation completing and ends in replacement occurring at the expense of users [10], which is a sum of the time span of 2D-RRPRW with R and the service span of RPRL with PM). By means of this definition, the cost rate model related to RPRL with PM can be derived, as shown below.

3.1. Life cycle cost computation

Hereinabove, it has been assumed that the distribution function $G(y)$ is memory-less. This assumption signals that the remaining-time-to-accomplishment of a job and all job cycles after the expiry of the warranty are independent random variables with the distribution function $G(y)$. Therefore, let Y_f be the first job cycle after the expiry of the warranty, then Y_f follows $G(y)$. For the product through 2D-RRPRW with R, when it is replaced in the form of the first kind of replacement, the first job cycle Y_f satisfies $Y_f > T$. When this case occurs, the first job cycle Y_f follows the distribution function $H(y)$:

$$H(y) = \Pr\{Y_f < y \mid Y_f > T\} = \frac{G(y)}{\bar{G}(T)} \tag{16}$$

where $y > T$.

Furthermore, for the product through 2D-RRPRW with R, when it is replaced in the form of the second kind of replacement, the first job cycle Y_f satisfies $Y_f < T$. When this case occurs, the first job cycle Y_f follows the distribution function $I(y)$:

$$I(y) = \Pr\{Y_f < y \mid Y_f < T\} = \frac{G(y)}{G(T)} \tag{17}$$

where $y < T$.

If the first job cycle still is not completed until the planned time T , the product through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w will undergo PM at the planned time T . Similar to [11, 23], age reduction is used to characterize the reliability improvement. Denote $\varphi \cdot (w+T)$ ($0 < \varphi < 1$) by a virtual age after MP at the planned time T , then the age reduction resulting from PM is given by $(1-\varphi)\cdot(w+T)$. Furthermore, PM cost $C_{pm}(T)$ is modeled as an increasing function with respect to the age reduction $(1-\varphi)\cdot(w+T)$, i.e.:

$$C_{pm}(T) = \eta c_I \left((1-\varphi)\cdot(w+T)\right)^{\kappa} \tag{18}$$

where η, $\kappa > 0$ and c_I ($c_I > 0$) is a cost coefficient on the reliability improvement.

For the product through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w , when it is replaced after the planned time T at the accomplishment of the first job cycle, the user's total minimal repair cost $C^w_{pw_1}(T;Y_f)$ after the expiry of the warranty can be represented by:

$$C^w_{pw_1}(T;Y_f) = c_m \int_0^T r(w+u)\mathrm{d}u + c_m \int_0^{Y_f} r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u \tag{19}$$

where $r(w+u)$ and $r\left(\varphi\cdot(w+T)+u\right)$ are the failure rate function at the warranty period w and at the virtual age $\varphi\cdot(w+T)$.

Since Y_f in (19) follows $H(y)$ in (16), the expected value $C^w_1(T)$ of the user's total minimal repair cost $C^w_{pw_1}(T;Y_f)$ can be rewritten as:

$$C^w_1(T) = \int_T^{\infty} C^w_{pw_1}(T;y)\,\mathrm{d}H(y) = \frac{c_m \int_T^{\infty}\left(\int_0^T r(w+u)\mathrm{d}u + \int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)}{\bar{G}(T)} \tag{20}$$

For the product through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w , when its replacement is carried out after the accomplishment of the first job cycle at the planned time T , the user's total minimal repair cost $C^w_2(T)$ after the expiry of the warranty can be given by:

$$C^w_2(T) = c_m \int_0^T r(w+u)\mathrm{d}u \tag{21}$$

The probability that the first kind of replacement occurs is given by $\bar{G}(T) = \Pr\{Y_f > T\}$ and the probability that the second kind of replacement occurs is given by $G(T) = \Pr\{Y_f < T\}$. Thus, for the product through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w , when RPRL with PM ensures its reliability after the expiry of the warranty, the expected total minimal repair cost $C^w(T)$ of RPRL with PM is:

$$\begin{aligned} C^w(T) &= \bar{G}(T)\cdot C^w_1(T) + G(T)\cdot C^w_2(T) \\ &= c_m\left(\bar{G}(T)\cdot\int_0^T r(w+u)\mathrm{d}u + \int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y) + G(T)\int_0^T r(w+u)\mathrm{d}u\right) \\ &= c_m\left(\int_0^T r(w+u)\mathrm{d}u + \int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right) \end{aligned} \tag{22}$$

For the product through 2D-RRPRW with R before the accomplishment of the first job cycle at the warranty period w , the expected total cost $TC^w(T)$ of RPRL with PM is represented by:

$$TC^w(T) = C_{pm}(T) + C^w(T) + C_r = \eta c_I\left((1-\varphi)\cdot(w+T)\right)^{\kappa} + c_m\left(\int_0^T r(w+u)\mathrm{d}u + \int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right) + C_r \tag{23}$$

where C_r represents the user's unit replacement cost.

For the product through 2D-RRPRW with R before the warranty period w at the accomplishment of the first job cycle, by replacing w in $TC^{w}(T)$ with S_m, the expected total cost $TC^{m}(T;S_m)$ of RPRL with PM is expressed as:

$$TC^{m}(T;S_m)=\eta c_I\left((1-\varphi)\cdot\left(S_m+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(S_m+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(S_m+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)+C_r \tag{24}$$

Since S_m in (24) follows $K(s)$ in (9), the expected total cost $TC^{m}(T;S_m)$ can be rewritten as:

$$TC^{m}(T)=\int_0^w TC^{m}(T;s)\mathrm{d}K(s)$$
$$=\frac{\int_0^w\left(\eta c_I\left((1-\varphi)\cdot\left(s+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(s+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(s+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\right)\bar{F}(s)\mathrm{d}G^{(m)}(s)}{\int_0^w \bar{F}(u)\mathrm{d}G^{(m)}(u)}+C_r \tag{25}$$

The probability Q_w that the first product goes through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w has been offered in (10) and the probability Q_m that the first product goes through 2D-RRPRW with R before the warranty period w at the accomplishment of the m^{th} job cycle has been offered in (11). Then, under the condition that the product is warranted by 2D-RRPRW with R, the expected total cost $E[C(T)]$ of RPRL with PM is computed as:

$$E[C(T)]=Q_w\cdot TC^{w}(T)+Q_m\cdot TC^{m}(T)$$
$$=\frac{\begin{array}{l}\bar{G}^{(m)}(w)\bar{F}(w)[\eta c_I\left((1-\varphi)\cdot\left(w+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(w+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)]\\+\int_0^w\left(\eta c_I\left((1-\varphi)\cdot\left(s+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(s+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(s+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\right)\bar{F}(s)\mathrm{d}G^{(m)}(s)\end{array}}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)}+C_r$$
$$=\frac{\begin{array}{l}\int_0^w \bar{G}^{(m)}(s)\mathrm{d}[\bar{F}(s)\left(\eta c_I\left((1-\varphi)\cdot\left(s+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(s+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(s+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\right)]\\+\eta c_I\left((1-\varphi)\cdot T\right)^{\kappa}+c_m\left(\int_0^T r(u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot T+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\end{array}}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)}+C_r \tag{26}$$

Each of all failures in the warranty region $(0,w]\times(0,m]$ makes the user to incur a failure cost c_f and meanwhile the user provides a pro-rata fee $P(X_j)$ to the producer. Thus, the j^{th} failed product in the warranty region $(0,w]\times(0,m]$ makes the user incur a total cost $c_f+P(X_j)$. Furthermore, by replacing $C_R-P(x)$ in WC with $c_f+P(x)$, the expected value $E[C_{fc}]$ of the total failure cost resulting from all failures is represented by:

$$E[C_{fc}]=\frac{\int_0^w (c_f+P(x))\bar{G}^{(m)}(x)\mathrm{d}F(x)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)}=\frac{\int_0^w (c_f+ax^b)\bar{G}^{(m)}(x)\mathrm{d}F(x)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{27}$$

As mentioned above, the life cycle is a sum of the warranty service period (i.e., a time span) of 2D-RRPRW with R and the time span of RPRL with PM. This indicates that the total cost in the life cycle is composed of the total failure cost $E[C_{fc}]$ of 2D-RRPRW with R, the expected value R of the refund $R(S_m)$ and the expected cost $E[C(T)]$ of RPRL with PM. Therefore, by algebraic operation, the expected total cost $LCC(T)$ in the life cycle is able to be computed as:

$$LCC(T)=E[C_{fc}]-R+E[C(T)]$$
$$=\xi+\frac{\begin{array}{l}\int_0^w \bar{G}^{(m)}(s)\mathrm{d}[\bar{F}(s)\left(\eta c_I\left((1-\varphi)\cdot\left(s+T\right)\right)^{\kappa}+c_m\left(\int_0^T r(s+u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(s+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\right)]\\+\eta c_I\left((1-\varphi)\cdot T\right)^{\kappa}+c_m\left(\int_0^T r(u)\mathrm{d}u+\int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot T+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\end{array}}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{28}$$

where

$\xi=E[C_{fc}]-R+C_r=\left(\int_0^w (c_f+ax^b)\bar{G}^{(m)}(x)\mathrm{d}F(x)-\int_0^w (C_R-\alpha s^{\beta})\bar{F}(s)\mathrm{d}G^{(m)}(s)\right)/\left(1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)\right)+C_r$.

3.2. Life cycle length calculation

Until the j^{th} product survives 2D-RRPRW with R, $j-1$ failures have occurred in the warranty region $(0,w]\times(0,m]$. When this case occurs, the total warranty service period resulting from $j-1$ failures is represented by $\sum_{k=0}^{j-1} X_k$, here $X_0=0$. Since the number $j-1$ satisfies $p^{j-1}q$, the expected value W_f of $\sum_{k=0}^{j-1} X_k$ can be expressed as:

$$W_f=\sum_{j=1}^{\infty}\left(p^{j-1}q\cdot E[\sum_{k=0}^{j-1} X_k]\right)=E[j-1]\cdot E[X_k]=\frac{\int_0^w x\bar{G}^{(m)}(x)\mathrm{d}F(x)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{29}$$

where $E[j-1]$ has been offered by (3) and $E[X_k]=\int_0^w x\mathrm{d}H(x)=\int_0^w x\bar{G}^{(m)}(x)\mathrm{d}F(x)/\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)$.

When the product goes through 2D-RRPRW with R before the accomplishment of the m^{th} job cycle at the warranty period w, the warranty service period of the product equates to w. When the product goes through 2D-RRPRW with R before the warranty period w at the accomplishment of the m^{th} job cycle, the warranty service period of the product is equal to S_m. Thus, for the product through 2D-RRPRW with R, the expected value W_g of the warranty service period can be expressed as:

$$W_g=Q_w\cdot w+Q_m\cdot E[S_m]=\frac{\bar{G}^{(m)}(w)\bar{F}(w)w+\int_0^w s\bar{F}(s)\mathrm{d}G^{(m)}(s)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{30}$$

where Q_w and Q_m have been offered in (10) and (11); $E[S_m]=\int_0^w s\mathrm{d}K(s)=\int_0^w s\bar{F}(s)\mathrm{d}G^{(m)}(s)/\int_0^w \bar{F}(u)\mathrm{d}G^{(m)}(u)$.

By summing (29) and (30), the warranty service period W produced by 2D-RRPRW with R is obtained as:

$$W=W_f+W_g=\frac{\int_0^w x\bar{G}^{(m)}(x)\mathrm{d}F(x)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)}+\frac{\bar{G}^{(m)}(w)\bar{F}(w)w+\int_0^w s\bar{F}(s)\mathrm{d}G^{(m)}(s)}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)}=\frac{\int_0^w \bar{G}^{(m)}(u)\bar{F}(u)\mathrm{d}u}{1-\int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} \tag{31}$$

When the product through 2D-RRPRW with R is replaced in the form of the first kind of replacement, the service span $L_{pw_1}(T)$ of RPRL with PM can be given by:

$$L_{pw_1}(T)=\int_T^{\infty} y\mathrm{d}H(y)=\frac{\int_T^{\infty} y\mathrm{d}G(y)}{\bar{G}(T)} \tag{32}$$

When the product through 2D-RRPRW with R is replaced in the form of the second kind of replacement, the service span $L_{pw_2}(T)$ of RPRL with PM can be represented by:

$$L_{pw_2}(T)=T \tag{33}$$

The probability that the first kind of replacement occurs is given by $\bar{G}(T)$ and the probability that the second kind of replacement occurs is given by $G(T)$. Thus, when RPRL with PM ensures the product's reliability after the expiry of the warranty, the service span $L_{pw}(T)$ of RPRL with PM can be given by:

$$L_{pw}(T) = \bar{G}(T) \cdot L_{pw_1}^{w}(T) + G(T) \cdot L_{pw_2}(T) = \int_T^{\infty} y \mathrm{d}G(y) + G(T) \cdot T = T + \int_T^{\infty} \bar{G}(y)\mathrm{d}y \tag{34}$$

By summing (31) and (34), the life cycle length $L(T)$ can be calculated as:

$$L(T) = W + L_{pw}(T) = \frac{\int_0^w \bar{G}^{(m)}(u)\bar{F}(u)\mathrm{d}u}{1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)} + T + \int_T^{\infty} \bar{G}(y)\mathrm{d}y \tag{35}$$

3.3. Cost rate derivation

Based on the renewal rewarded theorem in [1], the expected cost rate $CR(T)$ can be derived as:

$$CR(T) = \frac{LCC(T)}{L(T)} = \frac{\begin{gathered}\xi\left(1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)\right) + \\ \int_0^w \bar{G}^{(m)}(s)\mathrm{d}[\bar{F}(s)\left(\eta c_I\left((1-\varphi)\cdot(s+T)\right)^{\kappa} + c_m\left(\int_0^T r(s+u)\mathrm{d}u + \int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot(s+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\right)] \\ +\eta c_I\left((1-\varphi)\cdot T\right)^{\kappa} + c_m\left(\int_0^T r(u)\mathrm{d}u + \int_T^{\infty}\left(\int_0^y r\left(\varphi\cdot T+u\right)\mathrm{d}u\right)\mathrm{d}G(y)\right)\end{gathered}}{\int_0^w \bar{G}^{(m)}(u)\bar{F}(u)\mathrm{d}u + T + \int_T^{\infty}\bar{G}(y)\mathrm{d}y} \tag{36}$$

3.4. Model simplification

The expected cost rate $CR(T)$ in (36) is derived under the condition that the product is warranted by 2D-RRPRW with R and the product's reliability after the expiry of the warranty is ensured by RPRL with PM. By setting parameter values, some special models can be simplified from $CR(T)$, which can represent special cases, as shown below.

Case 1: the model in (36) when $m \to \infty$ is simplified as:

$$\lim_{m\to\infty} CR(T) = \frac{\int_0^w (c_f + ax^b)\mathrm{d}F(x)/\bar{F}(w) + C_r + \eta c_I\left((1-\varphi)\cdot(w+T)\right)^{\kappa} + c_m\int_T^{\infty}\left(\int_0^T r(w+u)\mathrm{d}u + \int_0^y r\left(\varphi\cdot(w+T)+u\right)\mathrm{d}u\right)\mathrm{d}G(y)}{\int_0^w \bar{F}(u)\mathrm{d}u/\bar{F}(w) + T + \int_T^{\infty}\bar{G}(y)\mathrm{d}y} \tag{37}$$

When $m \to \infty$, $\bar{G}^{(m)}(\bullet) \to 1$. Hereinbefore, $m \to \infty$ meant that the warranty limit m was removed and 2D-RRPRW with R was simplified as RPRW in [15]. Therefore, the model in (37) is a cost rate model at which RPRW is adopted to warrant the product and RPRL with PM is planned to ensure the product' s reliability after the expiry of the warranty.

Case 2: the model in (36) when $\varphi = 1$ can be simplified as:

$$\lim_{\varphi\to 1} CR(T) = \frac{\xi\left(1 - \int_0^w \bar{G}^{(m)}(u)\mathrm{d}F(u)\right) + c_m\left(\int_0^w \bar{G}^{(m)}(s)\mathrm{d}[\bar{F}(s)\left(\int_0^T r(s+u)\mathrm{d}u + \int_T^{\infty}\bar{G}(u)r(s+u)\mathrm{d}u\right)] + \left(\int_0^T r(u)\mathrm{d}u + \int_T^{\infty}\bar{G}(u)r(u)\mathrm{d}u\right)\right)}{\int_0^w \bar{G}^{(m)}(u)\bar{F}(u)\mathrm{d}u + T + \int_T^{\infty}\bar{G}(y)\mathrm{d}y} \tag{38}$$

$\varphi(\cdot) = 1$ means that PM at the planned time T is removed and RPRL with PM is simplified as RPRL. Therefore, the model in (38) is a cost rate model where 2D-RRPRW with R warrants the product and RPRL in [21] ensures the product's reliability after the expiry of the warranty.

Case 3: the model in (36) when $m \to \infty$ and $\varphi = 1$ can be simplified as:

$$\lim_{\substack{m\to\infty \\ \varphi\to 1}} CR(T) = \frac{\int_0^w (c_f + ax^b)\mathrm{d}F(x)/\bar{F}(w) + C_r + c_m\left(\int_0^T r(w+u)\mathrm{d}u + \int_T^{\infty}\bar{G}(u)r(w+u)\mathrm{d}u\right)}{\int_0^w \bar{F}(u)\mathrm{d}u/\bar{F}(w) + T + \int_T^{\infty}\bar{G}(y)\mathrm{d}y} \tag{39}$$

By **Case 1** and **Case 2**, the model in (39) a cost rate model in which RPRW in [15] warrants the product and RPRL in [21] ensures the product's reliability after the expiry of the warranty.

4. Numerical experiments

Assume that the time of the product first failure follows a distribution function $F(x) = 1 - \exp(-\int_0^x r(u)\mathrm{d}u)$ where $r(u) = \gamma u^{\varepsilon}$, $\gamma > 0$ and $\varepsilon > 0$; suppose that the distribution function of all job cycles is given by $G(y) = 1 - \exp(-\lambda y)$; suppose that cost and time are quoted in the dollar and year, respectively; and some constant parameters are offered in Table 1. Other parameters (not including the decision variable T) not to be mentioned in Table 1 are assigned when used. Next, the approaches investigated will be illustrated by means of parameters in Table 1.

Table 1. Parameter values

ε	b	η	c_I	κ	β
2	1	1	2	1	1

4.1. The proposed warranty's sensitivity analysis

To explore the property of the proposed warranty (PW), Figure 1 is plotted where $\gamma = 0.8$, $c_m = 0.1$, $c_f = 0.1$, $\lambda = 2$, $\alpha = a = 0.2$ and $C_R = 8$. As shown in Figure 1, when m increases for $w = 1$, 1.05, 1.1, the warranty cost of 2D-RRPRW with R first increases from a value less than the warranty cost (which can be obtained by computing $\lim_{m\to\infty} WC$) of RPRW to a value greater than the warranty cost of RPRW, then decreases to the warranty cost of RPRW. However, for $w = 0.95$, the warranty cost of 2D-RRPRW with R first increases from a value greater than the warranty cost (which can be obtained by computing $\lim_{m\to\infty} WC$) of RPRW to a peak value, then decreases to the warranty cost of RPRW. These signal that when the warranty period w is longer, although the producer offers a refund to guarantee the user's fairness, the warranty cost of 2D-RRPRW with R still is less than the warranty cost of RPRW; but, under the case where the warranty period w is shorter, the refund to guarantee the user's fairness makes the warranty cost of 2D-RRPRW with R to increase to a value which is always greater than or equal to the warranty cost of RPRW. The above contrast indicates that by limiting the warranty period w, the warranty cost of 2D-RRPRW with R can be less than the warranty cost of classic RPRW. In addition, Figure 1 indicates that the warranty cost of 2D-RRPRW with R increases when w is increasing and m is fixed.

As shown in Figure 2 where $\lambda = 2$, $\gamma = 0.8$, $c_m = 0.1$, $c_f = 0.1$, $\alpha = a = 0.2$ and $C_R = 8$, when m increases for the same w, the warranty cost of 2D-RRPRW with R (i.e., PW_1) is equal to or greater than the warranty cost of PW_2 (i.e., 2D-RRPRW); the warranty cost of RPRW is still equal to or greater than the warranty cost of PW_2 (i.e., 2D-RRPRW). Note that 2D-RRPRW can be obtained by removing the refund from 2D-RRPRW with R. The above changes indicate that although the refund to guarantee the user's fairness can make the warranty cost of 2D-RRPRW with R to be less than the warranty cost of classic RPRW, it is still greater than the warranty cost of 2D-RRPRW, which is because the refund is shouldered by the producer.

4.2. Sensitivity analysis of users' RPRL with PM

To display whether the optimum RPRL with PM exists uniquely, we make Figure 3 where $\gamma = 1.2$, $c_f = 0.1$, $\alpha = a = 0.5$, $\lambda = 3$, $C_R = 10$, $m = 2$ and $w = 3$. Figure 3 shows that the optimum cost rate $CR(T^*)$ and planned time T^* exist uniquely. This means that the optimum RPRL with PM exists uniquely. From **Subplot (A)** where $C_r = 15$, it is found that when c_m increases, T^* is decreasing while

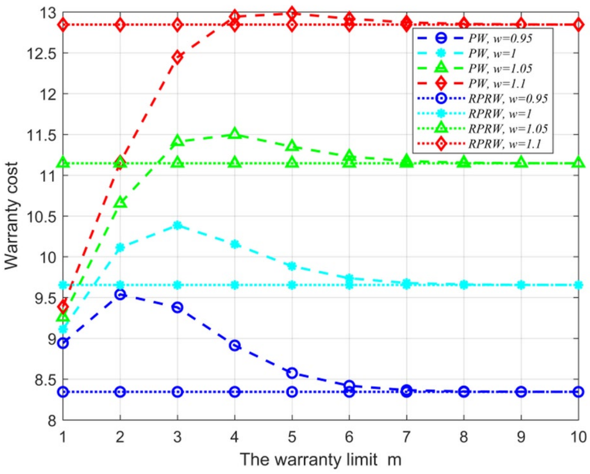

Fig. 1. Sensitivity analysis of 2D-RRPRW with R

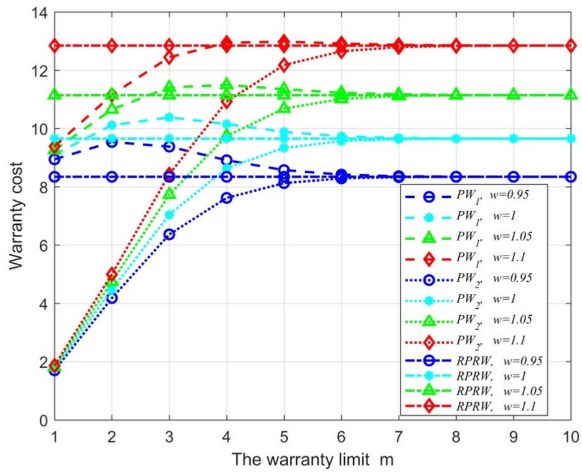

Fig. 2. Sensitivity analysis of 2D-RRPRW with R

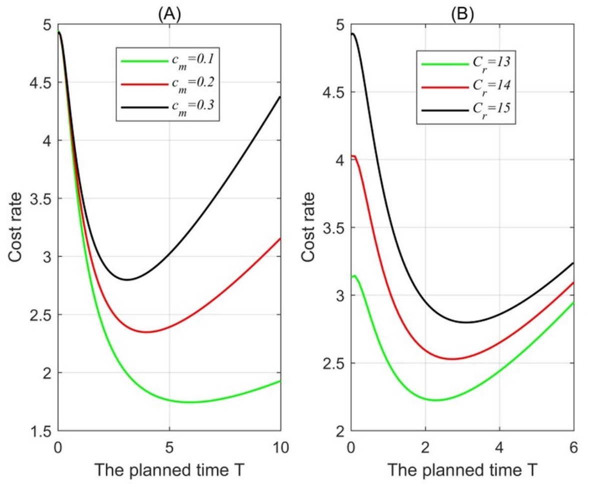

Fig. 3. The effect of c_m *and* C_r *on RPRL with PM*

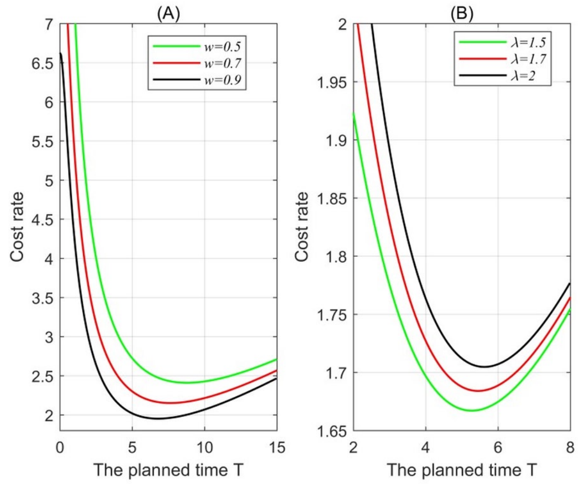

Fig. 4. The effect of w *and* λ *on RPRL with PM*

$CR(T^*)$ is increasing. This signals that the smaller minimal repair cost c_m is able to reduce the cost rate and extend the service span after the expiry of the warranty. From Subplot (B) where $c_m = 0.3$, it is discovered that when C_r increases, T^* increases and $CR(T^*)$ also increases. This illustrates that the greater replacement cost C_r is able to extend the service span after the expiry of the warranty while is not able to reduce the cost rate.

To indicate the effect of w and λ on RPRL with PM, we plot Figure 4 where $\gamma = 1.2$, $c_m = 0.1$, $c_f = 0.1$, $\alpha = a = 0.5$, $C_R = 10$, $C_r = 15$ and $m = 2$. As shown in Subplot (A) where $\lambda = 3$, the optimum cost rate $CR(T^*)$ and planned time T^* decreases when w increases and λ is fixed.. This indicates that a longer warranty period w is able to reduce the cost rate but is not able to extend the service span after the expiry of the warranty. From Subplot (B) where $w = 2$, it is found that T^* and $CR(T^*)$ increase when λ increases and w is fixed. From the overall viewpoint, the above phenomena signal that the shorter job cycles (i.e., the case that λ is greater) is able to extend the service span after the expiry of the warranty at the expense of the greater cost rate.

4.3. Performance illustration

In this paper, we have planned that RPRL with PM ensured the product's reliability after the expiry of the warranty. If RPRL with PM is removed from RPRL with PM, then RPRL with PM is transformed into RPRL in [21], which has been mentioned above. The expected cost rate related to RPRL has been offered in (38). These signal that RPRL with PM or RPRL can ensure the product's reliability after the expiry of the warranty. Form the user's perspective, a problem is how to select the best policy from RPRL with PM and RPRL to ensure the product's reliability after the expiry of the warranty. Obviously, the policy at which the service span after the expiry of the warranty is longer and the cost rate is lower is an ideal policy to ensure the product's reliability after the expiry of the warranty.

To explore whether the service span after the expiry of the warranty is longer and whether cost rate is lower, we make Figure 5 where $\gamma = 1.2$, $c_m = 0.5$, $c_f = 0.1$, $\alpha = a = 0.5$, $w = 3$, $\lambda = 3$, $C_R = 10$, $C_r = 20$ and $m = 2$. As indicated in Figure 5, compared RPRL, when RPRL with PM is used to ensure the product's reliability after the expiry of the warranty, the service span after the expiry of the warranty is longer and the cost rate is lower. These signal that the performance of RPRL with PM is superior to the performance of RPRL, thus RPRL with PM is an ideal policy to guarantee the product's reliability after the expiry of the warranty.

Note that if the information in Figure 5 is not obtained, then comparison method in [24] can be used to compare performances.

5. Conclusions

By taking the resultant force which is formed by the advanced technologies (such as in-situ sensors, industrial networks, control technologies) as a technological background, in this paper we firstly integrated the limited job cycles into classic RPRW and devised a producer's two-dimensional random RPRW with a refund (2D-RRPRW with R) to guarantee the product's reliability of the product that does succes-

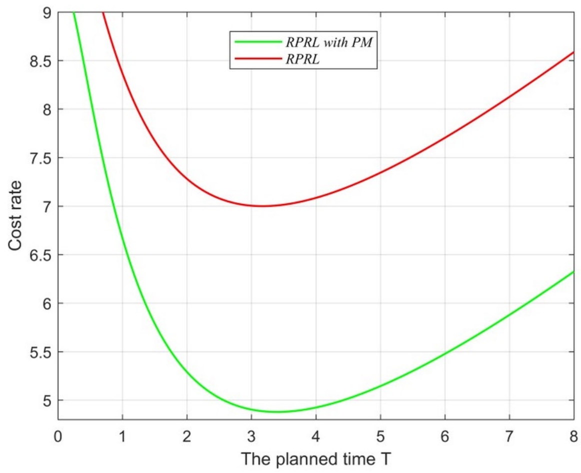

Fig. 5 Performance comparisons

sive missions at job cycles. A refund is used to guarantee the fairness of users whose warranty expiration occurs before the warranty period at the accomplishment of the limited number of job cycles. The warranty cost of2D-RRPRW with R was modeled and some special warranty models were obtained by altering warranty terms. By integrating the first job cycle into the service span after the expiry of the warranty, we investigated a users' random periodic replacement last with preventive maintenance (RPRL with PM) to further lengthen the service span after the expiry of the warranty and further reduce the failure frequency. Some classic cost rate models were provided by simplifying model. We used sensitivities analysis to explore the characteristics of the proposed approaches. It was shown that although 2D-RRPRW with R requires the producer provide a partial to the user, the producer can make the warranty cost of 2D-RRPRW with R to be less than the warranty cost of classic RPRW by limiting the warranty period; and RPRL with PM is superior to RPRL.

Acknowledgments

This article is supported by the National Social Science Fund of China (No. 2017BJY008), the Base and Basic Applied Study of Guangdong Province (No. 2020A1515011360), the National Natural Science Foundation of China (No. 72001026, 72161025, 71871181, 71961030), and Natural Science Basic Research Program of Shaanxi (2021JM-419).

References

1. Barlow RE, Proschan F. Mathematical theory of reliability. John Wiley & Sons, Hoboken 1965, http://10.1214/aoms/1177699826.
2. Cha JH, Finkelstein M, Levitin G. Optimal warranty policy with inspection for heterogeneous, stochastically degrading items. European Journal of Operational Research 2021; 289(3): 1142–1152, https://doi.org/10.1016/j.ejor.2020.07.045.
3. Hooti F, Ahmadi J, Longobardi M. Optimal extended warranty length with limited number of repairs in the warranty period. Reliability Engineering & System Safety 2020; 203: 107111, https://doi.org/10.1016/j.ress.2020.107111.
4. Hashemi, M, Asadi M, Tavangar M. Optimal maintenance strategies for coherent systems: A warranty dependent approach. Reliability Engineering & System Safety 2022; 217: 108027, https://doi.org/10.1016/j.ress.2021.108027.
5. He Z, Wang D, He S, Zhang Y, Dai A. Two-dimensional extended warranty strategy including maintenance level and purchase time: A win-win perspective. Computers & Industrial Engineering 2020; 141: 106294, https://doi.org/10.1016/j.cie.2020.106294.
6. Jung KM, Park M, Park DH. Cost optimization model following extended renewing two-phase warranty. Computers & Industrial Engineering 2015; 79: 188–194, https://doi.org/10.1016/j.cie.2014.10.016.
7. Luo M, Wu S. A comprehensive analysis of warranty claims and optimal policies. European Journal of Operational Research 2019; 276: 144–159, https://doi.org/10.1016/j.ejor.2018.12.034.
8. Liu B, Wu J, Xie M. Cost analysis for multi-component system with failure interaction under renewing free-replacement warranty. European Journal of Operational Research 2015; 243(3): 874–882, https://doi.org/10.1016/j.ejor.2015.01.030.
9. Li T, He S, Zhao X. Optimal warranty policy design for deteriorating products with random failure threshold. Reliability Engineering & System Safety, 2022; 218: 108142, https://doi.org/10.1016/j.ress.2021.108142.
10. Liu P, Wang G, Su P. Optimal replacement strategies for warranty products with multiple failure modes after warranty expiry. Computers & Industrial Engineering 2021; 153: 107040, https://doi.org/10.1016/j.cie.2020.107040.
11. Lin D, Jin B, Chang D. A PSO approach for the integrated maintenance model. Reliability Engineering & System Safety, 2020; 193: 106625, https://doi.org/10.1016/j.ress.2019.106625.
12. Liu P, Wang G, Su P. Optimal maintenance strategies for warranty products with limited repair time and limited repair number. Reliability Engineering & System Safety, 2021; 210: 107554, https://doi.org/10.1016/j.ress.2021.107554.
13. Nakagawa T. Random maintenance policies. Springer, London 2014, https://10.1007/978-1-4471-6575-0.
14. Park M, Jung KM, Park DH. Warranty cost analysis for second-hand products under a two-stage repair-or-full refund policy. Reliability Engineering & System Safety 2020; 193: 106596, https://doi.org/10.1016/j.ress.2019.106596.
15. Qiao P, Shen J, Zhang F, Ma Y. Optimal warranty policy for repairable products with a three-dimensional renewable combination warranty. Computers & Industrial Engineering 2022; 168: 108056, https://doi.org/10.1016/j.cie.2022.108056.
16. Qiu Q, Cui L. Gamma process based optimal mission abort policy. Reliability Engineering & System Safety, 2019; 190: 106496, https://doi.org/10.1016/j.ress.2019.106496.
17. Shang L, Si S, Sun S, Jin T. Optimal warranty design and post-warranty maintenance for products subject to stochastic degradation. IISE Transactions 2018; 50(10): 913–927, https://doi.org/10.1080/24725854.2018.1448490.
18. Shafiee M, Chukova S. Maintenance models in warranty: A literature review. European Journal of Operational Research, 2013; 229(3): 561–572, https://doi.org/10.1016/j.ejor.2013.01.017.
19. Shang L, Si S, Cai Z. Optimal maintenance–replacement policy of products with competing failures after expiry of the warranty. Computers & Industrial Engineering 2016; 98: 68–77, https://doi.org/10.1016/j.cie.2016.05.012.
20. Sheu SH, Tsai HN, Sheu UY, Zhang ZG. Optimal replacement policies for a system based on a one-cycle criterion. Reliability Engineering & System Safety 2019; 191: 106527, https://doi.org/10.1016/j.ress.2019.106527.

21. Shang L, Qiu Q, Wang X. Random periodic replacement models after the expiry of 2D-warranty. Computers & Industrial Engineering 2022; 164: 107885, https://doi.org/10.1016/j.cie.2021.107885.
22. Salem M B, Fouladirad M, Deloux E. Variance Gamma process as degradation model for prognosis and imperfect maintenance of centrifugal pumps. Reliability Engineering & System Safety, 2022; 223: 108417, https://doi.org/10.1016/j.ress.2022.108417.
23. Syamsundar A, Naikan V N A, Wu S. Extended arithmetic reduction of age models for the failure process of a repairable system. Reliability Engineering & System Safety, 2021; 215: 107875, https://doi.org/10.1016/j.ress.2021.107875.
24. Shang L, Wang H, Wu C, Cai Z. The post-warranty random maintenance policies for the product with random working cycles. Eksploatacja i Niezawodnosc – Maintenance and Reliability 2021; 23 (4): 726–735, http://doi.org/10.17531/ein.2021.4.15.
25. Wang L, Pei Z, Zhu H, Liu B. Optimising extended warranty policies following the two-dimensional warranty with repair time threshold. Eksploatacja i Niezawodnosc - Maintenance and Reliability 2018; 20(4): 523–530, http://dx.doi.org/10.17531/ein.2018.4.3.
26. Wang X, Xie W, Ye ZS, Tang L. Aggregate discounted warranty cost forecasting considering the failed-but-not-reported events. Reliability Engineering & System Safety, 2017; 168: 355-364, https://doi.org/10.1016/j.ress.2017.04.009.
27. Wang X, Liu B, Zhao X. A performance-based warranty for products subject to competing hard and soft failures. International Journal of Production Economics 2021; 233: 107974, https://doi.org/10.1016/j.ijpe.2020.107974.
28. Wang X, He K, He Z, Li L, Xie M. Cost analysis of a piece-wise renewing free replacement warranty policy. Computers & Industrial Engineering 2019; 135: 1047–1062, https://doi.org/10.1016/j.cie.2019.07.015.
29. Xie W, Shen L, Zhong Y. Two-dimensional aggregate warranty demand forecasting under sales uncertainty. IISE Transactions 2017; 49(5): 553–565, https://doi.org/10.1080/24725854.2016.1263769.
30. Ye Z S, Xie M. Stochastic modelling and analysis of degradation for highly reliable products. Applied Stochastic Models in Business and Industry, 2015; 31(1): 16–32, https://doi.org/10.1002/asmb.2063.
31. Ye Z S, Murthy D N P. Warranty menu design for a two-dimensional warranty. Reliability Engineering & System Safety 2016; 155: 21–29, https://doi.org/10.1016/j.ress.2016.05.013.
32. Yang L, Chen Y, Qiu Q, Wang J. Risk Control of Mission-Critical Systems: Abort Decision-Makings Integrating Health and Age Conditions. IEEE Transactions on Industrial Informatics, 2022; https://doi.org/ 10.1109/TII.2022.3141416.
33. Zhao X, Nakagawa T. Optimization problems of replacement first or last in reliability theory. European Journal of Operational Research 2012; 223(1): 141–149, https://doi.org/10.1016/j.ejor.2012.05.035.
34. Zhao X, Nakagawa T, Zuo M. Optimal replacement last with continuous and discrete policies. IEEE Transactions on Reliability 2014; 63: 868–880, http://dx.doi.org/10.1109/TR.2014.2337811.
35. Zhao X, Sun J, Qiu Q, Chen K. Optimal inspection and mission abort policies for systems subject to degradation. European Journal of Operational Research, 2021; 292(2): 610–621, https://doi.org/10.1016/j.ejor.2020.11.015.
36. Zhao X, Fan Y, Qiu Q, Chen K. Multi-criteria mission abort policy for systems subject to two-stage degradation process. European Journal of Operational Research, 2021; 295(1): 233–245, https://doi.org/10.1016/j.ejor.2021.02.043.