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SERIES OF ITERATED LOGARITHMS

Summary. In this paper certain properties of the series of iterated logarithms are discussed in more general context.

SZEREGI ITEROWANYCH LOGARYTMÓW

Streszczenie. W artykule pewne własności szeregów iterowanych logarytmów są rozważane w znacznie ogólniejszym kontekście.

1. Basic notions and properties

Let $\alpha \in (1, +\infty)$. Let us put

$$a_0(\alpha) := 1, \qquad a_{n+1}(\alpha) := \alpha^{a_n(\alpha)}$$

and

$$\log_\alpha^{(0)} x := x, \qquad \log_\alpha^{(n+1)} x := \log_\alpha(\log_\alpha^{(n)} x),$$

for every $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $x \in [a_n(\alpha), +\infty)$.

First the basic properties of sequence $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ will be presented.

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Theorem 1. *Sequence* $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ *is increasing for every* $\alpha \in (1, +\infty)$*. Limit* $\lim_{n\to+\infty} a_n(\alpha)$ *is finite if and only if* $\alpha \in (1, \sqrt[e]{e}]$*. Moreover, we have*

$$\lim_{n\to+\infty} a_n(\sqrt[e]{e}) = e.$$

The following corollary is an important consequence of Theorem 1.

Corollary 2. *For every* $\alpha \in (1, \sqrt[e]{e}]$ *and for every* $k \in \mathbb{N},\ k \geqslant 3$*, we have*

$$\sup\{n \in \mathbb{N}_0 \colon a_n(\alpha) \leqslant k\} = +\infty.$$

If $\alpha > \sqrt[e]{e}$ *then* $\lim_{n\to+\infty} a_n(\alpha) = +\infty$*. Therefore for every* $k \in \mathbb{N}$ *there exists* $n(k;\alpha) \in \mathbb{N}$ *such that* $a_n > k$ *for every* $n \in \mathbb{N}$ *satisfying* $n \geqslant n(k;\alpha)$*. In consequence, the integer number*

$$t_k(\alpha) := \max\{n \in \mathbb{N}_0 \colon a_n(\alpha) \leqslant k\}$$

is well defined.

Remark 3. In papers [1, 5, 8] there are discussed the properties of sequence $\{a_n(\alpha)\}_{n\in\mathbb{N}}$, among others, Theorem 1 is proven there.

Furthermore, it is shown there that if $\alpha \in (0,1)$ then sequence $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ is convergent if and only if $\alpha \in [e^{-e}, 1)$ (see [1, 8]). Cooper [5] proved that if $\sqrt[e]{e} < \alpha < \beta$ and for every $u \in \mathbb{N}$ the positive integer $v = v(u)$ is defined by inequalities

$$a_v(\alpha) < a_u(\beta) \leqslant a_{v+1}(\alpha),$$

then the difference $v - u$ is constant for all sufficiently large values $u \in \mathbb{N}$. Roughly speaking, every sequence $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ with $\alpha > \sqrt[e]{e}$ grows at the same rate.

2. Convergence of the series of iterated logarithms

It is a classical result (see [3, 4, 6, 9, 11]) that for every $\alpha \in (\sqrt[e]{e}, +\infty)$ and for every $l \in \mathbb{N}$ the series $\sum_{k\in\mathbb{N}} \left(\prod_{n=0}^{\min\{l, t_k(\alpha)\}} \log_\alpha^{(n)} k \right)^{-1}$ is divergent.

We are going to consider the convergence of the series of iterated logarithms having the most general form with respect to the upper index of multiplication, i.e. having the form $\sum_{k\in\mathbb{N}} \left(\prod_{n=0}^{t_k(\alpha)} \log_\alpha^{(n)} k \right)^{-1}$, where $\alpha \in (\sqrt[e]{e}, +\infty)$.

Theorem 4. *Let* $\alpha \in (\sqrt[e]{e}, +\infty)$. *Then the series* $\sum\limits_{k\in\mathbb{N}} \left(\prod\limits_{n=0}^{t_k(\alpha)} \log_\alpha^{(n)} k\right)^{-1}$ *is convergent if and only if* $\alpha \in (\sqrt[e]{e}, e)$.

Proof. Since the sequence $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ is increasing, we have $t_k = n$ for any positive integer $k \in J_n(\alpha) := \mathbb{N} \cap [a_n(\alpha), a_{n+1}(\alpha)),\ n \in \mathbb{N}$. It is easy to show that

$$\left| I_n(\alpha) - \sum_{k\in J_n(\alpha)} \left(\prod_{i=0}^{n} \log_\alpha^{(i)} k\right)^{-1} \right| \leqslant \left(\prod_{i=0}^{n} \log_\alpha^{(i)}(a_n(\alpha))\right)^{-1} = \\ = \left(\prod_{i=0}^{n} a_i(\alpha)\right)^{-1} \leqslant \alpha^{-n},$$

by the inequality $a_i(\alpha) \geqslant \alpha,\ i \in \mathbb{N}$, where

$$I_n(\alpha) := \int\limits_{a_n(\alpha)}^{a_{n+1}(\alpha)} \left(\prod_{i=0}^{n} \log_\alpha^{(i)} x\right)^{-1} dx = \\ = [(\ln\alpha)^{n+1} \log_\alpha^{(n+1)} x]_{a_n(\alpha)}^{a_{n+1}(\alpha)} = (\ln\alpha)^{n+1},\ n \in \mathbb{N}.$$

Hence we obtain at once that the series $\sum\limits_{k\in\mathbb{N}} \left(\prod\limits_{n=0}^{t_k(\alpha)} \log_\alpha^{(n)} k\right)^{-1}$ is convergent if and only if $\alpha \in (\sqrt[e]{e}, e)$, which completes the proof. □

Remark 5. Theorem 4 was discovered independently by many authors: Keung-Yan-Cheong and Cover [10], Beigel [2], Gurarie, Goldstern, Martin [7], and Wituła (his formulation, unpublished to date, is presented above).

3. Certain generalization of the problem of convergence of the series of iterated logarithms

We prove now the theorem which, independently of its internal beauty, will lead us to consider the problem of convergence of the series of iterated logarithms from the other point of view by comparing the convergence of series and the convergence of the proper element of these series.

Theorem 6. *Let* $a_n, b_n^{(i)} \in (0, +\infty),\ i,\ n \in \mathbb{N}$. *Assume that* $\lim_{n\to+\infty} a_n = 0$,

$$\sum_{n\in\mathbb{N}} b_n^{(i)} = +\infty \quad \textit{and} \quad \lim_{n\to+\infty} \frac{a_n}{b_n^{(i)}} = 0 \quad \textit{for every } i \in \mathbb{N}.$$

Then there exists an increasing sequence $\{r(n)\}_{n\in\mathbb{N}}$ *of positive integers such that*

$$\sum_{n\in\mathbb{N}} a_{r(n)} < +\infty \quad \textit{and} \quad \sum_{n\in\mathbb{N}} b_{r(n)}^{(i)} = +\infty, \quad \textit{for every } i \in \mathbb{N}.$$

Proof. Let us consider two increasing sequences $\{k(n)\}_{n\in\mathbb{N}}$ and $\{s(n)\}_{n\in\mathbb{N}}$ of positive integers such that

(i) $k(n) \leqslant s(n) < k(n+1),\ n \in \mathbb{N}$,

(ii) $2^t a_n \leqslant \min\{1, b_n^{(i)}\}$, for $n,\ t \in \mathbb{N},\ n \geqslant k(t)$ and $i = 1, \ldots, t$,

(iii) $\sum_{n=k(t)}^{s(t)} b_n^{(i)} \geqslant 1$, for every $i,\ t \in \mathbb{N},\ i \leqslant t$ and
$\sum_{n=k(t)}^{s(t)-1} b_n^{(i)} < 1$, for some $i \leqslant t$ if $s(t) > k(t)$.

Observe that if $s(t) = k(t)$ then

$$\sum_{n=k(t)}^{s(t)} a_n = a_{k(t)} \leqslant 2^{-t}.$$

On the other hand, if $s(t) > k(t)$ then by (iii) we can choose an index $i \leqslant t$ such that

$$\sum_{n=k(t)}^{s(t)-1} b_n^{(i)} < 1.$$

Hence we obtain the estimation

$$\sum_{n=k(t)}^{s(t)} a_n = \sum_{n=k(t)}^{s(t)-1} a_n + a_{s(t)} \leqslant 2^{-t} \sum_{n=k(t)}^{s(t)-1} b_n^{(i)} + 2^{-t} < 2^{-t+1}.$$

As a consequence of this fact we get

$$\sum_{t\in\mathbb{N}} \sum_{n=k(t)}^{s(t)} a_n \leqslant \sum_{t\in\mathbb{N}} 2^{-t+1} < +\infty.$$

Using (iii) we obtain

$$\sum_{t\in\mathbb{N}}\sum_{n=k(t)}^{s(t)} b_n^{(i)} = +\infty \quad \text{for every } i\in\mathbb{N}.$$

Finally, we deduce from the above that the increasing sequence $\{r(n)\}_{n\in\mathbb{N}}$ of all elements of set $\{n\in\mathbb{N}\colon (\exists t\in\mathbb{N})(k(t)\leqslant n\leqslant s(t))\}$ possesses the desired properties. □

Corollary 7. *Let us fix* $\alpha\in(\sqrt[e]{e},+\infty)$ *and* $l\in\mathbb{N}$*. Let us also set*

$$a_n = \left(\prod_{k=0}^{\min\{l,t_n(\alpha)\}} \log_\alpha^{(k)} n\right)^{-1}$$

and

$$b_n^{(i)} = \left(\prod_{k=0}^{\min\{l+1,t_n(\alpha)\}} \log_\alpha^{(k)} n\right)^{-1}$$

for every $i,\ n\in\mathbb{N}$*. Then there exists an increasing sequence* $\{r(n)\}_{n\in\mathbb{N}}$ *of positive integers such that*

$$\sum_{n\in\mathbb{N}}\left(\prod_{k=0}^{\min\{l,t_{r(n)}(\alpha)\}} \log_\alpha^{(k)}(r(n))\right)^{-1} = +\infty$$

and

$$\sum_{n\in\mathbb{N}}\left(\prod_{k=0}^{\min\{l+1,t_{r(n)}(\alpha)\}} \log_\alpha^{(k)}(r(n))\right)^{-1} < +\infty.$$

Corollary 8. *Let us put*

$$a_n = \left(\prod_{k=0}^{t_n(\alpha)} \log_\alpha^{(k)} n\right)^{-1}$$

and

$$b_n^{(i)} = \left(\prod_{k=0}^{\min\{i,t_n(\alpha)\}} \log_\alpha^{(k)} n\right)^{-1}$$

for every $i, n \in \mathbb{N}$ *and* $\alpha \in (\sqrt[e]{e}, +\infty)$*. Then there exists an increasing sequence* $\{r(n)\}_{n\in\mathbb{N}}$ *of positive integers such that*

$$\sum_{n\in\mathbb{N}} \left(\prod_{k=0}^{t_{r(n)}(\alpha)} \log_\alpha^{(k)}(r(n)) \right)^{-1} < +\infty$$

and

$$\sum_{n\in\mathbb{N}} \left(\prod_{k=0}^{\min\{i, t_{r(n)}(\alpha)\}} \log_\alpha^{(k)}(r(n)) \right)^{-1} = +\infty \qquad \text{for every } i \in \mathbb{N}.$$

We note that, if $\alpha \in (\sqrt[e]{e}, e)$ *then we can define* $r(n) = n,\ n \in \mathbb{N}$.

4. Generalisations to the bigger class of functions – obeying logarithms

The next theorems are the generalizations of two previous corollaries. Before their formulation we need some auxiliary definitions.

We assume that $f\colon (0, +\infty) \to (0, +\infty)$ is a function satisfying conditions

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} \frac{f(x)}{x} = 0$$

and then we use the notations

$$f^0(x) := x,\ f^k(x) := f(f^{k-1}(x)) \quad \text{for } k \in \mathbb{N},\ x \in (0, +\infty),$$

and

$$t(x) := \sup\{s \in \mathbb{N}_0\colon\ f^w(x) \geqslant 1 \text{ for every } w = 0, 1, \ldots, s\} \qquad \text{for } x \in [1, +\infty).$$

Theorem 9. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence of positive real numbers such that*

$$x_n \geqslant 1,\ n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} x_n = +\infty$$

and let $u_n^{(i)},\ n \in \mathbb{N},\ i \in \mathbb{N}$ *and* $v_n,\ n \in \mathbb{N}$ *be the sequences of positive integers such that the following conditions are satisfied*

(i) $u_n^{(i)} < v_n \leqslant t(x_n), \quad i,\ n \in \mathbb{N}$,

(ii) $\lim_{n\to+\infty} \left(\prod_{w=u_n^{(i)}+1}^{v_n} f^w(x_n) \right)^{-1} = 0$ *for every* $i \in \mathbb{N}$,

(iii) $\sum_{n=1}^{\infty} \left(\prod_{w=0}^{u_n^{(i)}} f^w(x_n) \right)^{-1} = +\infty$ *for every* $i \in \mathbb{N}$.

Then there exists a subsequence $\{y_n\}_{n\in\mathbb{N}}$ *of sequence* $\{x_n\}_{n\in\mathbb{N}}$ *such that*

$$\sum_{n\in\mathbb{N}} \left(\prod_{w=0}^{v_n} f^w(y_n) \right)^{-1} < +\infty$$

and

$$\sum_{n\in\mathbb{N}} \left(\prod_{w=0}^{u_n^{(i)}} f^w(y_n) \right)^{-1} = +\infty$$

for every $i \in \mathbb{N}$.

Proof. Let us define

$$a_n = \left(\prod_{w=0}^{v_n} f^w(x_n) \right)^{-1}$$

and

$$b_n^{(i)} = \left(\prod_{w=0}^{u_n^{(i)}} f^w(x_n) \right)^{-1}$$

for every $i, n \in \mathbb{N}$. Observe that a_n and $b_n^{(i)}$ satisfy the assumptions of Theorem 6. Therefore Theorem 9 follows from Theorem 6. □

The following result is an existential version of Theorem 9.

Theorem 10.

1. *Let* $k \in \mathbb{N}_0$*. Then there exists an increasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of positive reals such that*

$$\limsup(x_{n+1} - x_n) = +\infty,$$

$$\sum_{n\in\mathbb{N}} \left(\prod_{w=0}^{k} f^w(x_n) \right)^{-1} = +\infty$$

and

$$\sum_{n\in\mathbb{N}}\left(\prod_{w=0}^{k+1} f^w(x_n)\right)^{-1} < +\infty.$$

2. *Suppose that* $r\colon \mathbb{N}\to\mathbb{R}$ *satisfies condition* $\lim\limits_{n\to+\infty} r(n) = +\infty$. *Then there exists an increasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of positive reals such that*

$$\limsup(x_{n+1}-x_n) = +\infty,$$

$$\sum_{n\in\mathbb{N}}\left(\prod_{w=0}^{k} f^w(x_n)\right)^{-1} = +\infty$$

for every positive integer $k\in\mathbb{N}$, *and*

$$\sum_{n\in\mathbb{N}}\left(\prod_{w=0}^{r(n)} f^w(x_n)\right)^{-1} < +\infty.$$

Proof.
1. Let $k\in\mathbb{N}_0$. Then we can find an increasing sequence $\{y_n\}_{n\in\mathbb{N}}$ of positive reals such that

$$y_1\geqslant 1,\ y_{2n+1}-y_{2n}\geqslant n,\quad n\in\mathbb{N},$$

$$t(x)\geqslant k+1 \text{ for every } x\geqslant y_1,$$

$$f^k(x)\geqslant f^{k+1}(x)\geqslant 2^{n+1},\ x\geqslant y_n,\quad n\in\mathbb{N}.$$

For every $n\in\mathbb{N}$ we shall choose a finite and increasing sequence $z_u^{(n)}$, $u = 1,\ldots,v(n)$, $v(n)\in\mathbb{N}$, of elements belonging to interval (y_{2n-1},y_{2n}) and satisfying the inequality

$$1\leqslant\sum_{u=1}^{v(n)}\left(\prod_{w=0}^{k} f^w(z_u^{(n)})\right)^{-1} < 2.$$

The increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ of all elements $z_u^{(n)}$, $n,u\in\mathbb{N}$, $1\leqslant u\leqslant v(n)$, possesses the desired properties.

2. Let $\{r(n)\}_{n\in\mathbb{N}}$ denote a sequence of positive integers such that $\lim\limits_{n\to+\infty} r(n) = +\infty$. Then there exists an increasing sequence $\{y_n\}_{n\in\mathbb{N}}$ of positive reals such that

$$y_1\geqslant 1,\ y_{2n+1}-y_{2n}\geqslant n,\quad n\in\mathbb{N},$$

$$t(x)\geqslant r(n) \text{ and } f^k(x)\geqslant 2^n,\ x\geqslant y_n,\ n\in N,\ k=0,1,\ldots,n+1.$$

Let $z_u^{(n)}$, $u = 1, \dots, v(n)$, be chosen from interval (y_{2n-1}, y_{2n}), $n \in \mathbb{N}$, in such a way that

$$1 \leqslant \sum_{u=1}^{v(n)} \left(\prod_{w=0}^{r(n)-1} f^w(z_u^{(n)}) \right)^{-1} < 2.$$

It is clear that the increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ of all elements $z_u^{(n)}$, where $n, u \in \mathbb{N}, 1 \leqslant u \leqslant v(n)$, possesses the desired properties and therefore the proof is completed. □

Final remark. It is worth to note that Ukrainian mathematician Sljusarczuk has presented in works [12,13] the new logarithmic type (and many others) criteria for convergence of real series. Definitely, they generalize the classical criteria and are associated with the refinement of a logarithmic scale. For example the following one holds.

Theorem 11. *Let $a_n > 0$, $n \in \mathbb{N}$ and fix $p \in \mathbb{N}$. Let us put*

$$L_k(n) := \log_e^{(n)} n, \quad \textit{for every } k = 0, 1, 2, \dots$$

If $\limsup_{n\to\infty} L_{p+1}^{-1}(n) \log \left[a_n \prod_{k=0}^{p-1} L_k(n) \right]^{-1} > 1$ then the series $\sum a_n$ is convergent. On the other hand, if there exists $n_0 \in \mathbb{N}$ such that

$$L_{p+1}^{-1}(n) \log \left[a_n \prod_{k=0}^{p-1} L_k(n) \right]^{-1} \leqslant -1$$

for every $n \geqslant n_0$ then the series $\sum a_n$ is divergent.

For example, from this theorem we can deduce that the series

$$\sum_{n=100}^{\infty} \frac{L_3(n)^{-1-L_5^{-1}(n)}}{nL_1(n)L_2(n)}$$

is convergent but the series

$$\sum_{n=100}^{\infty} \frac{L_3(n)^{-1-L_4^{-1}(n)}}{nL_1(n)L_2(n)}$$

is divergent. For these series Theorem 6 could be also applied. Moreover we can formulate the theorems – substitutes of Theorems 9 and 10.

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Omówienie

Pewne własności szeregów iterowanych logarytmów podsunęły nam na myśl dyskusję podobnych relacji w ogólniejszym kontekście dowolnych ciągów liczb dodatnich. Otrzymano kilka interesujących twierdzeń. W artykule wspomniano też o wynikach ukraińskiego matematyka W.E. Sljusarczuka dotyczących nowych kryteriów zbieżności, zwłaszcza szeregów iterowanych logarytmów. Na tej podstawie zauważono, ze wyniki te pozwalają stosować otrzymane przez nas twierdzenia dla znacznie obszerniejszej klasy szeregów iterowanych logarytmów.