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NON-GENERATORS IN EXTENSIONS OF INFINITARY ALGEBRAS

A b s t r a c t. Contrary to the finitary case, the set $\Gamma(\mathbf{A})$ of all the non-generators of an infinitary algebra $\mathbf{A}$ is not necessarily a subalgebra of $\mathbf{A}$. We show that the phenomenon is ubiquitous: every algebra with at least one infinitary operation can be embedded into some algebra $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ is not a subalgebra of $\mathbf{B}$. As far as expansions are concerned, there are examples of infinite algebras $\mathbf{A}$ such that in every expansion $\mathbf{B}$ of $\mathbf{A}$ the set $\Gamma(\mathbf{B})$ is a subalgebra of $\mathbf{B}$. However, under relatively weak assumptions on $\mathbf{A}$, it is possible to get some expansion $\mathbf{B}$ of $\mathbf{A}$ such that $\Gamma(\mathbf{B})$ fails to be a subalgebra of $\mathbf{B}$.

1. Introduction

It is well-known that in a finitary algebra $\mathbf{A}$ the set $\Gamma(\mathbf{A})$ of all the non-generators is the intersection of all the maximal proper subalgebras of $\mathbf{A}$, hence $\Gamma(\mathbf{A})$ is a subalgebra of $\mathbf{A}$. Hansoul [3] proved that in the infinitary case $\Gamma(\mathbf{A})$ is not necessarily a subalgebra of $\mathbf{A}$; however Hansoul's proof is indirect and does not provide an explicit counterexample.

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Besides providing an explicit counterexample, we show that every algebra $\mathbf{A}$ with at least one infinitary operation can be embedded into some algebra $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ is not a subalgebra of $\mathbf{B}$. Moreover, assuming the Axiom of Choice (AC), $\mathbf{B}$ can be obtained in such a way that $B \setminus A$ is countable.

Let us now consider expansions instead of extensions, namely, we add operations rather than elements. We find conditions ensuring that some algebra $\mathbf{A}$ has (or has not) an expansion $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ fails to be a subalgebra of $\mathbf{B}$.

2. Preliminaries

We now recall the basic notions. See [2] for more details and unexplained notions. An *algebraic structure*, *algebra*, for short, is a set endowed with a family of operations and, possibly, constants. We allow *infinitary* operations, that is, operations depending on an infinite number of arguments. In detail, to every operation symbol f there is associated a possibly infinite set I. In each algebra whose type contains f, the symbol f is interpreted by some function $f_{\mathbf{A}} : A^I \to A$. As usual, when no risk of ambiguity is possible, we shall drop subscripts. A *finitary* algebra is an algebra having only finitary (= not infinitary) operations, that is, the I's above are always finite. The case of finitary algebras is the most studied in the literature [2].

Countable means either finite or denumerable. An algebra $\mathbf{A}$ is *countable* (*finite*) if its domain A is countable (finite). For convenience, we allow algebras with empty domain (when the type contains no constant). If $\mathbf{A}$ is an algebra and $X \subseteq A$, then $\langle X \rangle$ denotes the *subalgebra of* $\mathbf{A}$ *generated by* X, that is, the intersection of all the subalgebras of $\mathbf{A}$ which contain X. It is easily seen that $\langle X \rangle$ is indeed a subalgebra of $\mathbf{A}$ (by abuse of notation, here we do not distinguish between a subalgebra and its domain!) As usual, $\langle X, a \rangle$ is an abbreviation for $\langle X \cup \{a\} \rangle$.

If $\mathbf{A}$ is an algebra and $a \in A$, the element a is a *non-generator (of* $\mathbf{A}$*)* if, for every $X \subseteq A$, $\langle X, a \rangle = A$ implies $\langle X \rangle = A$. Otherwise, a is called a *relative generator*. Thus a is a relative generator if there is $X \subseteq A$ such that $\langle X, a \rangle = A$ but $\langle X \rangle \neq A$. As a stronger case of the latter notion, a is *indispensable* if a belongs to every generating set, equivalently, if $A \setminus \{a\}$ is a subalgebra. Originally considered in groups, non-generators have been subsequently studied in various special algebraic structures, as well as in the general universal setting. See, e. g., [1, 4, 5] for more details and references.

In the finitary case the set of all the non-generators of some algebra $\mathbf{A}$ is the domain for a subalgebra. This can be proved by observing that in the finitary case the set of non-generators is the intersection of all the maximal (proper) subalgebras of $\mathbf{A}$. The argument uses some algebraic properties of the lattice of subalgebras of a finitary algebra [4], does not work for infinitary algebras [3] and needs the axiom of choice [7, Form AL 9]. In Proposition 2.1 and Corollary 2.2 below we first present a more direct argument with a somewhat broader range of applicability. The argument is easy, but it has been probably overlooked by some authors.

For the sake of readability, if g is an infinitary operation with arguments indexed by some set I, we shall write $g(\dots, a_i, \dots)$ in place of $g((a_i)_{i \in I})$. Sometimes we need to fix some special element of I, call it 0, and then we shall write $g(a_0, \dots, a_i, \dots)$ when we need to know the argument of g at place 0. In any case, unless otherwise explicitly stated, we do not make any special assumption on the set I, in particular, we do not necessarily assume that I is countable, well-ordered, etc.

Informally, *terms* are defined by the usual recursive conditions: (a) every variable is a term; (b) if g is an operation with arguments indexed by I and each t_i ($i \in I$) is a term, then $g(\dots, t_i, \dots)$ is a term; (c) every term can be obtained by repeated iterations of (a) and (b). More formally, terms can be introduced as labeled trees without infinite branches. As in the finite case, on every algebra $\mathbf{A}$ of appropriate type, each term induces a *term-operation* obtained by evaluation, that is, assigning an element of A to each variable occurring in the term.

Proposition 2.1. *Suppose that* $\mathbf{A}$ *is a possibly infinitary algebra and* $a_1, \dots, a_n \in A$ *is a finite set of non-generators.*

If f *is a finitary* n*-ary term-operation of* $\mathbf{A}$*, then* $f(a_1, \dots, a_n)$ *is a non-generator.*

In particular, if g *is a possibly infinitary operation of* $\mathbf{A}$ *and* $(b_i)_{i \in I}$ *is a sequence of elements chosen from the finite set* $\{a_1, \dots, a_n\}$ *(hence repetitions in the sequence* $(b_i)_{i \in I}$ *are allowed), then* $g(\dots, b_i, \dots)$ *is a non-generator.*

Proof. Assume $\langle X, f(a_1, \dots, a_n) \rangle = A$, thus $\langle X, a_1, \dots, a_n \rangle = A$, since $f(a_1, \dots, a_n) \in \langle a_1, \dots, a_n \rangle$. Since a_n is a non-generator, then $\langle X, a_1, \dots, a_{n-1} \rangle = A$ and so on, in a finite number of steps we get $\langle X \rangle = A$. $\square$

If $\mathbf{A}$ and $\mathbf{B}$ are algebras with the same domain, $\mathbf{B}$ is an *expansion* of $\mathbf{A}$ if $\mathbf{B}$ has (possibly) more operations than $\mathbf{A}$, that is, the case $\mathbf{A} = \mathbf{B}$ is included. If $\mathbf{B}$ is an expansion of $\mathbf{A}$, then $\mathbf{A}$ is said to be a *reduct* of $\mathbf{B}$. Notice that the notion of expansion is distinct from *extension*. An algebra $\mathbf{B}$ is an extension of $\mathbf{A}$ just in case $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. For short, in an extension we add elements, in an expansion we add operations.

If $\mathbf{A}$ is an infinitary algebra, the *finitary reduct* $\mathbf{A}^{fr}$ of $\mathbf{A}$ is the reduct of $\mathbf{A}$ in which only the finitary operations are considered. Following an observation by the referee, we can work in a more general setting, since the notion of a non-generator does not really depend on the basic operations chosen for the algebra $\mathbf{A}$. Indeed, if $\mathbf{A}^c$ denotes the expansion of $\mathbf{A}$ obtained by adding all the term-operations definable in the type of $\mathbf{A}$, then an element $a \in A$ is a non-generator in $\mathbf{A}$ if and only if a is a non-generator in $\mathbf{A}^c$. The *complete finitary reduct* $\mathbf{A}^{cfr}$ of $\mathbf{A}$ is the finitary reduct of $\mathbf{A}^c$. Thus $\mathbf{A}^{cfr}$ has an operation for every finitary term-operation of $\mathbf{A}$.

Corollary 2.2. *If* $\mathbf{A}$ *is a (possibly infinitary) algebra, then the set* Γ *of the non-generators of* $\mathbf{A}$ *is a subalgebra of the finitary reduct* $\mathbf{A}^{fr}$ *of* $\mathbf{A}$*. More generally,* Γ *is* *closed under all finitary term-operations of* $\mathbf{A}$*, that is,* Γ *is a subalgebra of the complete finitary reduct* $\mathbf{A}^{cfr}$*.*

In particular, if $\mathbf{A}$ *is a finitary algebra, then the set of non-generators of* $\mathbf{A}$ *is a subalgebra of* $\mathbf{A}$*.*

The first statement in Corollary 2.2 does not follow from the last statement. It might happen that some element a is a non-generator in $\mathbf{A}$, but a is a relative generator in the finitary reduct of $\mathbf{A}$. Just consider an algebra $\mathbf{A}$ with only infinitary operations. The finitary reduct $\mathbf{A}^{fr}$ is then a set without operations, thus every element of $\mathbf{A}^{fr}$ is a relative generator, actually, an indispensable element. However, some element of the original infinitary algebra $\mathbf{A}$ might be a non-generator.

In the infinitary case the set of non-generators is not necessarily a subalgebra. However, there is a significant case in which this happens, see clause (1) in the next theorem. The theorem is taken from [6].

Theorem 2.3. *(1) In every complete semilattice the set of non-generators is a complete subsemilattice and is the intersection of all the maximal proper complete subsemilattices.*

(2) There is a complete lattice such that the set of non-generators is a complete sublattice, but it is not the intersection of all the maximal proper complete sublattices.

(3) There is a complete lattice such that the set of non-generators is not a complete sublattice.

In passing, we exploit the special feature of (complete) semilattices which provides the reason why 2.3(1) holds. Indeed, the proof of 2.3(1) from [6] shows that the first statement in the next proposition holds for complete semilattices.

Proposition 2.4. *Suppose that* $\mathbf{A}$ *is a possibly infinitary algebra such that each element of* $\mathbf{A}$ *is either indispensable or a non-generator. Then in* $\mathbf{A}$ *the set of non-generators is the intersection of all the maximal proper subalgebras.*

Proof. It is elementary to see that a non-generator belongs to every maximal proper subalgebra. On the other hand, an element a is indispensable if and only if $A \setminus \{a\}$ is a subalgebra (necessarily, proper maximal) of $\mathbf{A}$. The assumptions then imply that there is no other maximal proper subalgebra, thus the set of non-generators is the intersection of all the maximal proper subalgebras. □

3. Non-generators in extensions

Proposition 3.1. *There is an algebra* $\mathbf{B}$ *with a single operation depending on countably many arguments and such that the set of all the non-generators in* $\mathbf{B}$ *fails to be a subalgebra of* $\mathbf{B}$*.*

Proof. Let $B = \mathbb{N} \cup \{\infty\}$, where $\infty \notin \mathbb{N}$. On B consider the infinitary operation f depending on countably many arguments defined by

$$f(b_0, b_1, b_2, \dots) = \begin{cases} b_0 & \text{if the set } S = \{\, b_n \mid n \in \mathbb{N} \,\} \text{ is finite and } \infty \notin S, \\ 0 & \text{if } b_0 = b_1 = \infty, \\ n+1 & \text{if } b_0 = \infty \text{ and } b_1 = n, \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Because of the second and third clauses $\langle \infty \rangle = B$. Because of the first clause, if $S \subseteq \mathbb{N}$ and S is finite, then $\langle S \rangle = S$. On the other hand, if S is infinite, then, because of the fourth clause, $\langle S \rangle \supseteq \langle \infty \rangle = B$.

Hence, for $X \subseteq B$, we have $\langle X \rangle = B$ if and only if either $\infty \in X$, or X is infinite. This implies that every element of $\mathbb{N}$ is a non-generator and that ∞ is a relative generator. However, $\mathbb{N}$ is not a subalgebra of $\mathbf{B}$, due to the fourth clause. □

Another proof of Proposition 3.1 (with a quite different counterexample) can be found in [6].

We now show that every algebra with at least one infinitary operation can be extended to an algebra in which the set of all the non-generators fails to be a subalgebra.

For simplicity, we shall assume that, for every type and every operation f in the type, we have chosen one argument which shall always indicated as the first argument in expressions like $f(b_0, \dots, b_i, \dots)$. Technically, the following definition is dependent on the above choices; however, we shall later show that, with a bit more effort, a proof for Theorem 3.4 below can be given without performing any special choice of this kind.

Definition 3.2. If $\mathbf{A}$ and $\mathbf{B}$ are two possibly infinitary algebras of the same type without constants and $A \cap B = \emptyset$, let the *union* $\mathbf{C} = \mathbf{A} \cup \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ be defined as follows. The domain of $\mathbf{C}$ is $C = A \cup B$ and, for every operation f in the type, the interpretation of f on $\mathbf{C}$ is defined by

$$f_{\mathbf{C}}(c_0, \dots, c_i, \dots) = \begin{cases} f_{\mathbf{A}}(c_0, \dots, c_i, \dots) & \text{if } c_0, \dots, c_i, \dots \in A, \\ f_{\mathbf{B}}(c_0, \dots, c_i, \dots) & \text{if } c_0, \dots, c_i, \dots \in B, \\ c_0 & \text{otherwise.} \end{cases} \tag{2}$$

In a bit more general situation, we may allow exactly one between $\mathbf{A}$ and $\mathbf{B}$, say, $\mathbf{A}$, to have the type expanded with further constants. If this is case, constants are interpreted in $\mathbf{C}$ by the same elements as in $\mathbf{A}$. Thus in this case $\mathbf{C}$ and $\mathbf{A}$ have the same type, an expansion of the type of $\mathbf{B}$.

Lemma 3.3. *Under the above assumptions and definitions,* $\mathbf{A}$ *is a subalgebra of* $\mathbf{C}$*, and* $\mathbf{B}$ *is a subalgebra of the appropriate reduct of* $\mathbf{C}$*. Moreover, for every* $X \subseteq C$

(i) *X is a subalgebra of* $\mathbf{C}$ *if and only if both $X \cap A$ is a subalgebra of* $\mathbf{A}$ *and $X \cap B$ is a subalgebra of* $\mathbf{B}$.

(ii) $\langle X \rangle_{\mathbf{C}} = \langle X \cap A \rangle_{\mathbf{A}} \cup \langle X \cap B \rangle_{\mathbf{B}}$.

(iii) *An element $a \in A$ is a non-generator (indispensable) in* $\mathbf{A}$ *if and only if a is a non-generator (indispensable) in* $\mathbf{C}$. *The same holds for elements of* $\mathbf{B}$.

Theorem 3.4. *If* $\mathbf{A}$ *is an algebra with at least one infinitary operation, then* $\mathbf{A}$ *can be extended to some algebra* $\mathbf{C}$ *such that the set of non-generators of* $\mathbf{C}$ *fails to be a subalgebra of* $\mathbf{C}$.

Proof. Let $\mathbf{A}$ be an algebra with the infinitary operation f. If f depends on countably many arguments, consider the algebra $\mathbf{B}$ from Proposition 3.1. If f in $\mathbf{A}$ depends on uncountably many arguments, choose a countably infinite subset J of the arguments and define f in $\mathbf{B}$ in a way similar to (1), in such a way that $f_{\mathbf{B}}$ depends only on the arguments in J. In each case, expand the algebra $\mathbf{B}$ from Proposition 3.1 by adding a trivial operation $g_{\mathbf{B}}$ for every operation of $\mathbf{A}$ distinct from f, setting $g_{\mathbf{B}}(b_0, \dots, b_i, \dots) = b_0$.

The proof of Proposition 3.1 carries over even in this situation, thus the set of non-generators of $\mathbf{B}$, as defined here, is not a subalgebra of $\mathbf{B}$.

Let $\mathbf{C} = \mathbf{A} \cup \mathbf{B}$, as in Definition 3.2. By Lemma 3.3(i) and (iii) the set of the non-generators of $\mathbf{C}$ is not a subalgebra of $\mathbf{C}$. $\square$

3.1. A choiceless proof of Theorem 3.4

In the proof of Theorem 3.4 we have used the Axiom of Choice (AC) twice. We now show that the use of AC can be avoided.

We have used a consequence of AC in the proof of 3.4 by assuming that if I is some infinite set, then we can find a countably infinite $J \subseteq I$. We now modify the example provided in Proposition 3.1 in such a way that B can be endowed with an operation whose arguments depend on any infinite possibly not well-orderable set. As usual in a choiceless setting, a set S is *finite* if S can be put in a bijective correspondence with a natural number, *infinite* otherwise.

Lemma 3.5. *Suppose that I is an infinite set.*

There is an algebra $\mathbf{B}$ *such that B is in a bijective correspondence with the set $I \cup \{\infty\}$ ($\infty \notin I$),* $\mathbf{B}$ *has only one operation f, f depends on I-many arguments and the set of all the non-generators in* $\mathbf{B}$ *fails to be a subalgebra of* $\mathbf{B}$.

Proof. Let $B = \{ b_i \mid i \in I \} \cup \{\infty\}$, where the b_i's are chosen arbitrarily in such a way that $i \neq j \in I$ implies $b_i \neq b_j$ and ∞ is distinct from every b_i. In fact, we could have already taken $B = I \cup \{\infty\}$, but this might lead to notational confusion.

Pick some special element $\bar{b} \in B \setminus \{\infty\}$ (we do not need AC in order to perform this). Define an operation f on B depending on the set I of arguments by

$$
\begin{array}{ll}
f(\infty, \dots, \infty, \dots, \infty) = \bar{b}, & \\
f(\bar{b}, \dots, \bar{b}, \infty, \bar{b}, \dots, \bar{b}) = b_i & \text{with a single occurrence of } \infty \text{ at place } i, \\
f(\dots, d_i, \dots) = \bar{b} & \text{if } S = \{\, d_i \mid i \in I \,\} \text{ is finite and } \infty \notin S, \\
f(\dots, d_i, \dots) = \infty & \text{in all the remaining cases.}
\end{array}
$$

By the first two clauses, $\langle \infty \rangle = B$. We claim that if X is infinite, then $\langle X \rangle = B$. Indeed, pick some $\bar{x} \in X$. Then apply f to the sequence $(\dots, y_i, \dots)$, where $y_i = b_i$ if $b_i \in X$ and $y_i = \bar{x}$, otherwise (AC is not needed to construct this sequence). Then the outcome of f is ∞, by the fourth clause, thus $\langle X \rangle \supseteq \langle \infty \rangle = B$.

Hence if $X \subseteq B$, then $\langle X \rangle = B$ if and only if either X is infinite or $\infty \in X$. This implies that the set Γ of the non-generators of $\mathbf{B}$ is $B \setminus \{\infty\}$. However, Γ is not a subalgebra of $\mathbf{B}$, since $f(\dots, b_i, \dots) = \infty$. □

In the proof of Theorem 3.4 we have also used AC when $\mathbf{A}$ has infinitely many operations, since we have chosen some specific argument for every operation. We can do without AC by adding just one element to the union of $\mathbf{A}$ and $\mathbf{B}$.

Choiceless proof of Theorem 3.4.. Pick some infinitary operation $f_{\mathbf{A}}$ in $\mathbf{A}$ and construct an operation $f_{\mathbf{B}}$ on B as in the proof of Lemma 3.5. We shall construct an algebra $\mathbf{C}$ on $A \cup B \cup \{c\}$, where we can suppose that $A \cap B = \emptyset$ and $c \notin A \cup B$. Expand $\mathbf{B}$ in an appropriate way and join all the operations on $\mathbf{A}$ and $\mathbf{B}$ by setting to c all the otherwise undefined values. By the proof of Lemma 3.5, all the elements of $B \setminus \{\infty\}$ are non generators in $\mathbf{C}$, too, however $B \setminus \{\infty\}$ is not a subalgebra (we are essentially using a result analogue to Lemma 3.3, but notice that, as it stands, Lemma 3.3 does not hold in the present situation). □

3.2. Examples in specific classes

Given some algebra $\mathbf{A}$, the algebra $\mathbf{C}$ constructed in the proof of Theorem 3.4 has the same type of $\mathbf{A}$, but might be very different from $\mathbf{A}$. If we have some special class $\mathcal{K}$ of algebras in mind, possibly a variety, it is not necessarily the case that $\mathbf{C}$ belongs to $\mathcal{K}$. It is an open problem whether, in general, there is some construction which produces an algebra in $\mathcal{K}$ satisfying the conclusion of Theorem 3.4, of course, assuming suitable closure properties for $\mathcal{K}$.

In the present subsection we see that the analogue of Theorem 3.4 holds for a few special classes $\mathcal{K}$. The heart of the matter is that in several cases the construction in Definition 3.2 can be replaced by a more suitable "union" which preserves the condition that the algebras are in $\mathcal{K}$.

For the sake of brevity, let us say that an algebra $\mathbf{A}$ satisfies the *ngs-property* if the set of *n*on-*g*enerators of $\mathbf{A}$ is a *s*ubalgebra of $\mathbf{A}$. If $\mathcal{K}$ is a class of structures of the same type, we say that *the ngs-property fails extensively in* $\mathcal{K}$ if every algebra $\mathbf{A}$ in $\mathcal{K}$ can be extended to some algebra $\mathbf{B} \in \mathcal{K}$ such that the ngs-property fails in $\mathbf{B}$. In this terminology, Theorem 3.4 asserts that if $\mathcal{K}$ is the class of all structures in some type with at least one infinitary operation, then the ngs-property fails extensively in $\mathcal{K}$.

In the following proposition we notice that the ngs-property is not always preserved by taking products.

Proposition 3.6. *There are two complete distributive lattices with the ngs-property and whose product has not the ngs-property.*

Proof. In [6, Theorem 4] we have shown that in the complete lattice $2 \times (\omega^2 + 1)$ the set of non-generators fails to be a complete sublattice. Here 2 is the two-element lattice, and the ordinal $\omega^2 + 1$ is considered as a lattice with the structure induced by the linear order.

On the other hand, it is easy to see that both 2 and $\omega^2 + 1$ satisfy the ngs-property (actually, every complete linear order, thought of as a complete lattice, has the ngs-property). $\square$

We now need to fix some terminology about complete lattices. Recall that a *complete lattice* is a lattice in which every subset has a join and a meet. If we think of a complete lattice as an infinitary algebra with a meet and a join operation for each infinite cardinality, we turn out with a structure having a proper class of operations, which is inconvenient due to foundational issues. Of course, the problem arises only when dealing with classes of structures: when dealing with a single lattice $\mathbf{L}$ it is enough to work with only operations having at most $|L|$ arguments.

Henceforth, in order to avoid foundational issues, we shall deal with classes of *$<\kappa$-complete lattices*, those lattices in which every nonempty subset of cardinality $< \kappa$ has a join and a meet. Our constructions here are affected by the technical condition about the possible meet and join of the empty set. If also the empty set has a join and a meet, that is, the lattice has both a minimum and a maximum, we shall speak of a *bounded $<\kappa$-complete lattice*. In the bounded case a sublattice shares the maximum and the minimum with the parent lattice, while we do not require this condition when dealing with the class of (not necessarily bounded) $<\kappa$-complete lattices.

Proposition 3.7. *For every uncountable cardinal κ, the ngs-property fails extensively in each of the following classes.*

1. *The class of $<\kappa$-complete lattices.*
2. *The class of bounded $<\kappa$-complete lattices.*

3. The class of distributive $<\kappa$-complete lattices.

Proof. Given two lattices $\mathbf{L}$ and $\mathbf{M}$, without loss of generality with disjoint domains, their *ordinal sum* $\mathbf{L}+\mathbf{M}$ is the lattice induced by the ordering in which every element of $\mathbf{L}$ is taken to be less than each element of $\mathbf{M}$.

If $\mathbf{L}$ and $\mathbf{M}$ are $<\kappa$-complete (distributive), then $\mathbf{L}+\mathbf{M}$ is $<\kappa$-complete (distributive). Moreover, for every $N \subseteq L \cup M$, N is (the domain of) a substructure of $\mathbf{L}+\mathbf{M}$ if and only if both $N \cap L$ is a substructure of $\mathbf{L}$ and $N \cap M$ is a substructure of $\mathbf{M}$; in particular, both $\mathbf{L}$ and $\mathbf{M}$ are substructures of $\mathbf{L}+\mathbf{M}$. Furthermore, $\Gamma(\mathbf{L}+\mathbf{M}) = \Gamma(\mathbf{L}) \cup \Gamma(\mathbf{M})$.

Since, as recalled in the proof of Proposition 3.6, in [6, Theorem 4] we have constructed a complete distributive lattice $\mathbf{M}$ such that $\Gamma(\mathbf{M})$ is not a substructure of $\mathbf{M}$, then, for every uncountable κ and every (distributive) $<\kappa$-complete lattice $\mathbf{L}$, the lattice $\mathbf{L}+\mathbf{M}$ is a (distributive) $<\kappa$-complete extension of $\mathbf{L}$ in which the ngs-property fails.

The above arguments take care of cases (1) and (3).

To deal with case (2), suppose that $\mathbf{L}$ and $\mathbf{M}$ are bounded and, without loss of generality, that $L \cap M = \{0, 1\}$. Construct the *parallel sum* $\mathbf{L} \parallel \mathbf{M}$ on $L \cup M$ by declaring incomparable all pairs of elements $\ell \in L$ and $m \in M$, when both ℓ and m are distinct from 0 and 1. Distributivity is not preserved, in general, but all the other arguments above work, hence we get (2). $\square$

We do not claim that the following problems are difficult.

Problems 3.8. (a) Does the ngs-property fail extensively in the class of bounded $<\kappa$-complete distributive lattices?

(b) Is there a complete Boolean algebra for which the ngs-property fails?

(c) If $\mathbf{A}$ and $\mathbf{B}$ are algebras of the same type and the ngs-property fails in $\mathbf{A}$, does the ngs-property necessarily fail in $\mathbf{A} \times \mathbf{B}$?

(d) Study the ngs-property in infinitary algebras endowed with some kinds of, possibly partial, limit operations, or with some infinitary sum operations, as borrowed, for example, from topology or analysis.

4. Non-generators in expansions

Clearly, the assumption that the algebra $\mathbf{A}$ in Theorem 3.4 has an infinitary operation is necessary. Indeed, in every finitary algebra the set of non-generators is a subalgebra, by Corollary 2.2.

However, it follows trivially from Theorem 3.4 that, for every algebra $\mathbf{A}$, there is some extension $\mathbf{B}$ of some expansion $\mathbf{A}^+$ of $\mathbf{A}$ such that in $\mathbf{B}$ the set of non-generators fails to be a subalgebra. Can we do by expansions alone, that is, without extending the domain of the algebra? The answer is obviously no, in general, since every proper subalgebra of a finite algebra can be extended to a maximal proper subalgebra, and then the classical argument shows that in this case the set Γ of non-generators is the intersection of the maximal proper subalgebras, hence Γ is a subalgebra itself. As another counterexample, if some algebra $\mathbf{A}$ is generated by the set of its constants, equivalently, by the empty set, then every element of $\mathbf{A}$ is a non-generator, and this fact still holds in any expansion of $\mathbf{A}$.

In the following remark we shall see that there are similar examples even for infinite algebras without constants. However, we shall see in Proposition 4.3 that, under quite weak hypotheses, it is possible to expand some algebra (without extending it) in such a way that the non-generators fail to constitute a subalgebra.

Remark 4.1. (a) We first notice that

(*) If some algebra $\mathbf{A}$ has an element b such that $\langle b \rangle = A$ and b is indispensable, then the set Γ of non-generators of $\mathbf{A}$ is $A \setminus \{b\}$, hence Γ is a subalgebra of $\mathbf{A}$.

Indeed, b fails to be a non-generator, since it is indispensable. On the other hand, if $a \in A \setminus \{b\}$ and $\langle X, a \rangle = A$, then $b \in X$, since b is indispensable, and hence, by assumption, $\langle X \rangle \supseteq \langle b \rangle = A$.

(b) Now consider the algebra $\mathbf{A}$ over $\mathbb{Z} \cup \{b\}$, where b is a new element not in $\mathbb{Z}$. The algebra $\mathbf{A}$ has two unary operations s and p which are interpreted, respectively, as the successor and predecessor functions on $\mathbb{Z}$, and are such that $s(b) = p(b) = 0$. We claim that

(**) In any expansion $\mathbf{A}^+$ of $\mathbf{A}$ the set of non-generators of $\mathbf{A}^+$ is a subalgebra of $\mathbf{A}^+$.

Indeed, $\langle b \rangle_{\mathbf{A}} = A$, a fortiori, $\langle b \rangle_{\mathbf{A}^+} = A$ in any expansion $\mathbf{A}^+$ of $\mathbf{A}$. If b remains indispensable in $\mathbf{A}^+$, then, by (*), the set of non-generators of $\mathbf{A}^+$ is $A \setminus \{b\}$, hence it is a subalgebra. Otherwise, b is not indispensable in $\mathbf{A}^+$, hence $f(a_0, \dots, a_i, \dots) = b$, for some operation f and $a_0, \dots, a_i, \dots \in A \setminus \{b\}$.

Let $a \in A \setminus \{b\}$. Since $\langle a \rangle_{\mathbf{A}} = A \setminus \{b\}$, then $\langle a \rangle_{\mathbf{A}^+} \supseteq A \setminus \{b\}$, since $\mathbf{A}^+$ is an expansion of $\mathbf{A}$. Since $f(a_0, \dots, a_i, \dots) = b$ and $a_0, \dots, a_i, \dots \in A \setminus \{b\} \subseteq \langle a \rangle_{\mathbf{A}^+}$, then $\langle a \rangle_{\mathbf{A}^+} = A$. We have shown that $\langle a \rangle_{\mathbf{A}^+} = A$, for every $a \in A$.

If $\mathbf{A}^+$ has no constant in its type, then all the elements of $\mathbf{A}^+$ are relative generators, since $\langle \emptyset \rangle = \emptyset$ but $\langle \emptyset, a \rangle = \langle a \rangle = A$. Thus the set of non-generators is a subalgebra, recalling the convention that we consider an empty set as a subalgebra. If $\mathbf{A}^+$ has some constant c, then $\langle c \rangle = A$, but $c \in \langle X \rangle$, for every $X \subseteq A$, including the case $X = \emptyset$, hence $\langle X \rangle = A$, for every $X \subseteq A$. This trivially implies that every element of $\mathbf{A}^+$ is a non-generator, hence non-generators form a subalgebra in this case, as well.

The arguments in the above remark give a proof of the following proposition.

Proposition 4.2. *Suppose that $\mathbf{A}$ is an algebra with an indispensable element b such that $\langle b \rangle = A$. Suppose further that $\langle a \rangle = A \setminus \{b\}$, for every $a \in A \setminus \{b\}$.*

Then in every expansion $\mathbf{A}^+$ of $\mathbf{A}$ the set of non-generators of $\mathbf{A}^+$ is a subalgebra of $\mathbf{A}^+$.

We now give conditions under which some algebra can be actually expanded in such a way that non-generators fail to form a subalgebra.

Proposition 4.3. *Suppose that* $\mathbf{A}$ *is an algebra without constants and* Δ *is a proper nonempty subalgebra of* $\mathbf{A}$ *such that*

(*) whenever $a \in \Delta$, $X \subseteq \Delta$ *and* $\langle X, a \rangle = \Delta$, *then* $\langle X \rangle = \Delta$.

Then $\mathbf{A}$ *can be expanded to some algebra* $\mathbf{A}^+$ *such that in* $\mathbf{A}^+$ *the set of the non-generators fails to be a subalgebra.*

Notice that, under the assumptions in Proposition 4.3, Δ must be infinite. Indeed, $\langle \Delta \rangle = \Delta$, since Δ is a subalgebra of $\mathbf{A}$. Were Δ finite, we could obtain $\langle \emptyset \rangle = \Delta$ by applying (*) a finite number of times. This contradicts the assumption that $\mathbf{A}$ has no constant.

Proof. Add to $\mathbf{A}$ a unary operation f_c, for each $c \in A$, defined by

$$f_c(b) = \begin{cases} c & \text{if } b \notin \Delta; \\ b & \text{if } b \in \Delta. \end{cases}$$

Pick some $r \in A \setminus \Delta$; this is possible since Δ is assumed to be a proper subalgebra of $\mathbf{A}$. On A define an infinitary operation g depending on Δ-many arguments by

$$g(b_0, \dots, b_i, \dots) = \begin{cases} r & \text{if } \{\, b_i \mid b_i \text{ among the arguments of } g \,\} \supseteq \Delta; \\ b_0 & \text{otherwise.} \end{cases}$$

In words, g acts as the "first projection", unless every element of Δ appears among the arguments of g, in which case the outcome of g is r. Let $\mathbf{A}^+$ be the expansion of $\mathbf{A}$ obtained by adding the operation g, as well as all the unary operations f_c, for $c \in A$.

Because of the f_c's, if $b \notin \Delta$, then $\langle b \rangle_{\mathbf{A}^+} = A$, hence b is a relative generator, since $\mathbf{A}$, and hence $\mathbf{A}^+$, have no constant, thus $\langle \emptyset \rangle = \emptyset$, while $\langle \emptyset, b \rangle = A$.

We now claim that if $a \in \Delta$, then a is a non-generator in $\mathbf{A}^+$. Indeed, suppose that $\langle X, a \rangle_{\mathbf{A}^+} = A$. If $X \nsubseteq \Delta$, then $b \in X$, for some $b \notin \Delta$, hence $\langle X \rangle_{\mathbf{A}^+} \supseteq \langle b \rangle_{\mathbf{A}^+} = A$, by a comment above. Hence we can suppose that $X \subseteq \Delta$. Since Δ is a subalgebra of $\mathbf{A}$, then $\langle X, a \rangle_{\mathbf{A}} \subseteq \Delta$. Since $\langle X, a \rangle_{\mathbf{A}^+} = A$, instead, then it is necessary to apply the new operations in $\mathbf{A}^+$. To this aim, the operations f_c have no use, since $X \cup \{a\} \subseteq \Delta$ and Δ is closed under each f_c; actually, each f_c is the identity, when restricted to Δ. Hence we need to resort to g. We can apply g in a nontrivial way only if $\langle X, a \rangle_{\mathbf{A}} = \Delta$, but then $\langle X \rangle_{\mathbf{A}} = \Delta$, by assumption, in particular, $\langle X \rangle_{\mathbf{A}^+} \supseteq \Delta$. Then by applying g we get $r \in \langle X \rangle_{\mathbf{A}^+}$, and then $\langle X \rangle_{\mathbf{A}^+} = A$, since $r \in A \setminus \Delta$, hence $\langle r \rangle_{\mathbf{A}^+} = A$.

We have shown that Δ is the set of the non-generators of $\mathbf{A}^+$; however, Δ is not a subalgebra of $\mathbf{A}^+$, because of g and since Δ is nonempty. □

For example, Proposition 4.3 can be applied to every algebra with domain $\mathbb{Z} \cup X$, with $X \nsubseteq \mathbb{Z}$ and (only) a unary operation f which is defined as the successor function on $\mathbb{Z}$ and arbitrarily otherwise. Notice the contrast with the example in Remark 4.1(b).

5. Further remarks

In the next proposition we state some results showing that an algebra can be extended in order to obtain a lot of non-generators in the extension. We shall present proofs elsewhere[1].

Proposition 5.1. *(1) If* $\mathbf{A}$ *is a countable algebra with at least one operation of arity* ≥ 2*, then* $\mathbf{A}$ *can be extended to some algebra* $\mathbf{B}$ *over the set* $A \cup \{\infty\}$ *(*$\infty \notin A$*), in such a way that* A *is the set of all the non-generators of* $\mathbf{B}$ *and* ∞ *is indispensable in* $\mathbf{B}$.

(2) Every algebra $\mathbf{A}$ *with at least one operation of arity* ≥ 1 *can be extended to some algebra* $\mathbf{B}$ *in such a way that* $B \setminus A \neq \emptyset$ *and in* $\mathbf{B}$*: (a) every element of* A *is a non-generator and (b) every element of* $B \setminus A$ *is indispensable. Henceforth, by Proposition 2.4, in* $\mathbf{B}$ *the set of non-generators is the intersection of all the maximal proper subalgebras.*

(3) Every algebra $\mathbf{A}$ *with at least one operation of arity* ≥ 1 *can be extended to some algebra* $\mathbf{C}$ *such that every* $c \in C$ *is a non-generator in* $\mathbf{C}$.

Remark 5.2. (a) Contrary to Proposition 5.1, in particular, contrary to Clause (3), it is not always the case that an algebra can be extended to an algebra in which all the elements are relative generators. Indeed, if the type of some algebra $\mathbf{A}$ has a constant c, then (the interpretation) of c is a non-generator, hence c remains a non-generator in any extension (and in any expansion) of $\mathbf{A}$.

(b) However, if $\mathbf{A}$ has no constant, then $\mathbf{A}$ can be expanded to an algebra in which all the elements are relative generators. Just add to $\mathbf{A}$ all possible unary operations. Then $\langle a \rangle = \langle a, \emptyset \rangle = A$ in the expansion, but $\langle \emptyset \rangle = \emptyset$, since there is no constant.

(c) The situation with indispensable elements is different. If $\mathbf{B}$ is either an extension or an expansion of $\mathbf{A}$, $a \in A$ and a is indispensable in $\mathbf{B}$, then $B \setminus \{a\}$ is a subalgebra of $\mathbf{B}$, hence $A \setminus \{a\}$ is a subalgebra of $\mathbf{A}$, thus a is indispensable in $\mathbf{A}$, as well.

In conclusion, if $\mathbf{A}$ has some element a which is not indispensable, then we cannot extend or expand $\mathbf{A}$ in such a way that a becomes indispensable.

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