CALCULATION OF STRESS-STRAIN STATE ELASTIC PLATE WITH NOTCH OF AN ARBITRARY SMOOTH CONTOUR

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Abstract

The paper contains comparing calculations of the stress fields in an elastic plate with notch along the arc of a circle, ellipse or parabola obtained by analytic-numerical method based on complex Kolosov-Mushelishvili potentials and by numerical variation-difference method. These fields differ by no more than 2%, which, in particular, indicates the reliability of such numerical implementation.

Key words: semi-plane, plate, notch, variation-difference method, stress field

1 Introduction

Investigation of the stress-strain state of the plate structural elements weakened by notch, [2] is a necessary step in the calculation of their strength and reliability. Since these structural elements have finite dimensions or curvilinear boundary, the possibility of the application of analytical methods for solving the corresponding boundary value problems [4] is significantly limited, and in most cases impossible.

In this paper, we provide comparison of the obtained solutions of plane elasticity problems on uniaxial loading of a plate structural element with notch of an arbitrary smooth contour by analytic-numerical method using the complex Kolosov-Mushelishvili potentials [7] and numerical method using the variation-difference method [6].

2 Analytic-numerical method for solving the problem

Let us find the stress state of a plate of the thickness h, which is simulated by the half-plane, on the surface of which a notch is made of an arbitrary smooth contour [7]. We assume that the half-plane extends to infinity by normal stress of value P (Figure 1), and the boundary of the half-plane with notch is free from stresses.

Choose a Cartesian coordinate system Oxy, directing the axis Ox along the straight edge, and the vertical axis upward. The curve traced by the notch is denoted by L, the straight line portion of the boundary of the half-plane by L'. The lower half-plane of the plane xOy is denoted by S^-, the upper one by S^+.

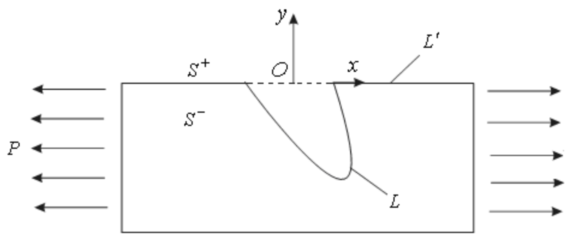

Figure 1. Plate element with notch under uniaxial loading

According to the formulation of the problem we have the following boundary conditions:

$$\sigma_{yy} = 0, \quad \sigma_{xy} = 0, \quad x \in L'; \; N = T = 0, \quad t \in L,$$

where N and T are the normal and tangential components of the vector of stresses on L respectively.

To solve the problem, we introduce the complex Kolosov-Mushelishvili potentials $\Phi(z)$ and $\Psi(z)$ [2] and present them in the form

$$\Phi(z) = \Phi_1(z) + \Phi_2(z) + \frac{p}{4}, \quad \Psi(z) = \Psi_1(z) + \Psi_2(z) - \frac{p}{2}.$$

Here $\Phi_2(z)$ and $\Psi_2(z)$ are complex potentials which are holomorphic in the lower half-plane and must ensure that the zero boundary conditions on the axis $y = 0$, are fulfilled, and corrective complex potentials $\Phi_1(z)$ and $\Psi_1(z)$ are responsible for the implementation of the boundary conditions on the surface of the notch.

Analytically extending the function $\Phi_2(z)$ from the region S^- over the region S^+ and solving the corresponding problem of linear conjugation, we obtain a singular integral equation [7], which we solve numerically using the method of mechanical quadratures [5].

3 Variation-difference method for solving the problem

We consider the plane problem of elasticity theory in a finite region V with curved boundary Σ (see Figure 1), which simulates the stress-strain state in a plate with notch of an arbitrary smooth contour. From the mathematical point of view it consists in solving equations of equilibrium in a plate [1]

$$\left(C_{ijkl}u_{k,l}\right)_{,j} = 0, \tag{1}$$

using mixed boundary conditions on the surface Σ

$$C_{ijkl}u_{k,l}n_j \,\big|_{\Sigma} = P_i \ . \tag{2}$$

Here C_{ijkl} are the components of the elastic modulus tensor; u_i, P_i , n_j are the components of the displacement vector, surface forces, and the external normal to the surface Σ respectively; $u_{i,j} \equiv \partial u_i / \partial x_j$. We assume the summation from one to two by the same indices that occur twice in one expression.

For numerical solution of problem (1) - (2) it is convenient to use its variation formulation [3], which is to minimize the Lagrangian

$$L = \int_V W dV - \int_\Sigma P_i \ u_i d\Sigma, \tag{3}$$

where $W = \frac{1}{2}C_{ijkl}u_{i,j}u_{k,l}$ is the energy density of elastic deformation.

We write the Lagrangian (3) in the canonical field V_0 , which can be a rectangle or a region composed of rectangles. For this purpose we use a discrete bijection mapping of the grid in a curvilinear region V to a uniform rectangular grid $N_1 \times N_2$ of the region V_0 (Figure 2)

$$x_i = x_i(\beta^1, \beta^2) \quad (i = 1,2), \tag{4}$$

Then $J = \det(A_i^j)$, $g_{ij} = A_i^m A_j^m$, where $A_i^j = \partial x_i / \partial \beta^j$ is the Jacobi matrix of this mapping. Using (4) we write the energy density of the deformation W in the coordinates $\vec{\beta}$

$$W = \frac{1}{2} C^{ijkl} u_{i,j} u_{k,l} = \frac{1}{2} C^{ijkl}\left(\vec{\beta}\right) B_j^m B_l^n u_{i|m} u_{k|n} = \frac{1}{2} D^{imkn}\left(\vec{\beta}\right) u_{i|m} u_{k|n}$$

where $u_{i|m} \equiv \partial u_i / \partial \beta^m$, $B_j^m = \partial \beta^m / \partial x_j$, $D^{imkn} = C^{ijkl} B_j^m B_l^n$.

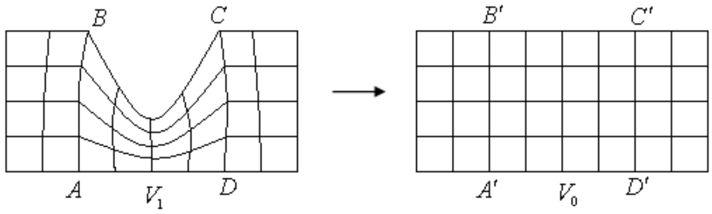

Figure 2. Mapping of a grid in the curvilinear region V onto the uniform rectangular grid in the region V_0

Thus, the Lagrangian in the rectangle V_0 will look like:

$$L_0 = \frac{1}{2} \int_{V_0} J D^{imkn} u_{i|m} u_{k|n} dv - \int_{\Sigma_0} q\left(\vec{\beta}\right) P_i u_i d\Sigma, \tag{5}$$

where $q(\vec{\beta}) = \begin{cases} \sqrt{g_{11}}, & \beta^2 = \{0, l_2\}, \\ \sqrt{g_{22}}, & \beta^1 = \{0, l_1\}. \end{cases}$

Replacing in (5) all continual function by grid ones, integrals by finite sums, and derivatives by difference derivatives, we obtain the difference analogue of the Lagrangian L_0^h using the discrete analogue of mapping (4), which should not be given analytically, in particular, to be conformal. It is sufficient to have one correspondence between nodes in the curvilinear V_1 and model V_0 regions. To determine the stationary point L_0^h we obtain a system of linear algebraic equations

$$\partial L^h / \partial v_\beta^h (i_1, i_2) = 0, \quad i_\alpha = 1,2,\ldots,N_\alpha, \quad \alpha, \beta = 1,2 . \tag{6}$$

This approach leads to the impossibility of the use of direct methods for solving the system (6) due to the accumulation of errors of rounding. However, it was done with a combined iterative process that implements the scheme of the gradient method and the method with Chebyshev set of iterative parameters [8]. The complexity of its practical implementation is selection of iterative parameters.

The described variation-difference method in domains with curved boundary is implemented as a software on FORTRAN.

4 Results

For example, the calculations of the components of the stress tensor on the notch and near it done, if its boundary is an arc of the circle, ellipse or parabola.

In Figure 3 and Figure 4 there are shown the graphs of dimensionless stresses $\sigma^0_{\theta\theta} \equiv \sigma_{\theta\theta}/P$, $\sigma^0_{xx} \equiv \sigma_{xx}/P$ and $\sigma^0_{yy} \equiv \sigma_{yy}/P$ for the notches along the arc of the circle. Here in after, $2l$ is the width of the notch (along the axis Ox); δ is the depth of the notch (along the axis Oy); $a = \delta/l$ is a dimensionless parameter relative absorption; $\sigma^0_{\theta\theta}$ are dimensionless circumferential stresses on the notch, $\sigma^0_{xx}, \sigma^0_{yy}$ are dimensionless normal stresses on a segment $x^0 \equiv x/l = 0$, $y^0 \equiv y/\delta \in [-5,-1]$ (along the axis Oy below the groove). The hatched lines represent stress obtained by the analytic-numerical method, and the solid lines by variation-difference method.

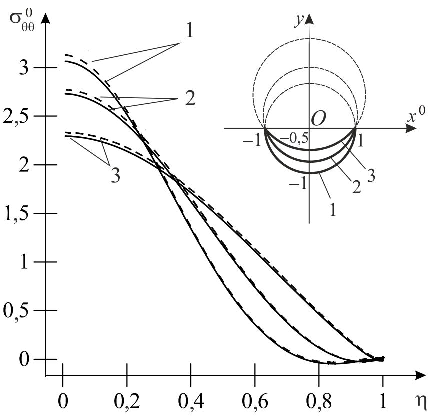

Figure 3. Stress $\sigma^0_{\theta\theta}$ on the notch in an arc of the circle for $l = 1$at different values of δ

In Figure 3, the curves *1* are constructed for $\delta = 1$, curves *2* for $\delta = 0{,}75$ curves *3* for $\delta = 0{,}5$. As seen from this figure, the stress $\sigma^0_{\theta\theta}$ (actually the coefficient of stress concentration) in the top of the notch ($\eta = 0$) achieves its greatest value in the case of notch along the semicircle (curve 1). And it is only slightly higher than typical for the Kirsch problem value 3.

Here in after, the accuracy of the results is of four significant digits (the error of about 0,1%). Monitoring convergence and accuracy of analytic-numerical and numerical solution is conducted by comparing the studied variables on the grids with single and double number of nodes.

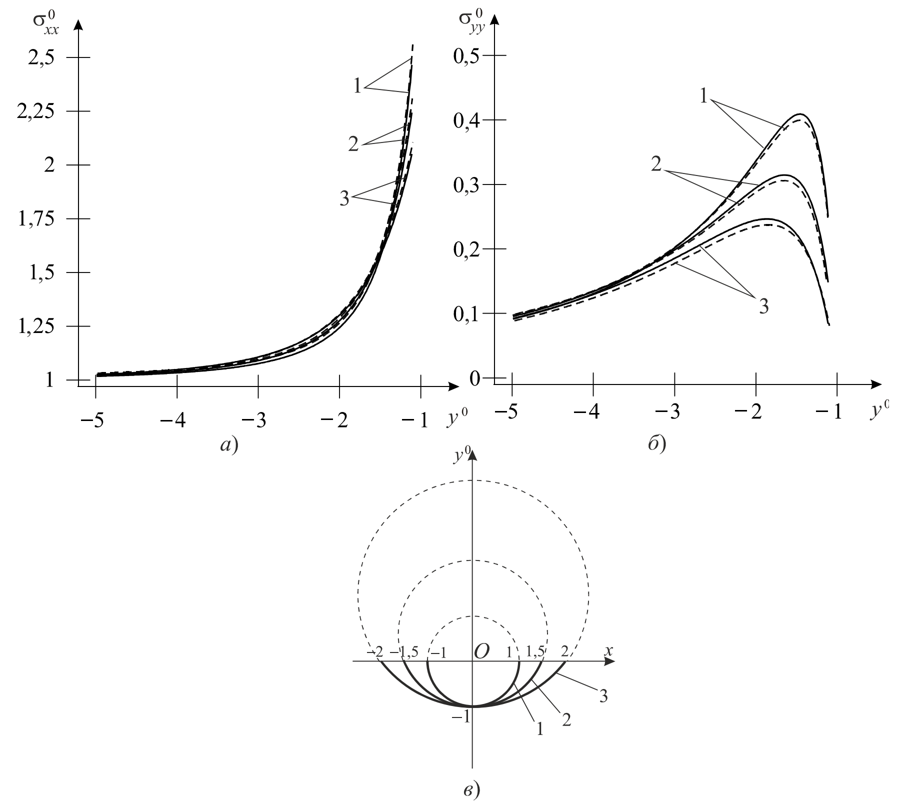

Figure 4. Stresses σ^0_{xx} and σ^0_{yy} on extension of the axis of symmetry of the notch along the arc of a circle δ=1 for various values of l.

In Figure 4, the curves *1* are constructed for $l = 1$, curves *2* for $l = 1{,}5$, curves *3* for $l = 2$. As shown in Figure 4, the normal stress σ^0_{xx} much lower of the notch ($y = -5$) is almost equal P, and at the top of the notch it is obvious that $\sigma^0_{xx} = \sigma^0_{\theta\theta}$.

Figure 5 and Figure 6 show the relevant graphs of stresses $\sigma^0_{\theta\theta}$, σ^0_{xx} and σ^0_{yy} for the notches along the arc of the ellipse.

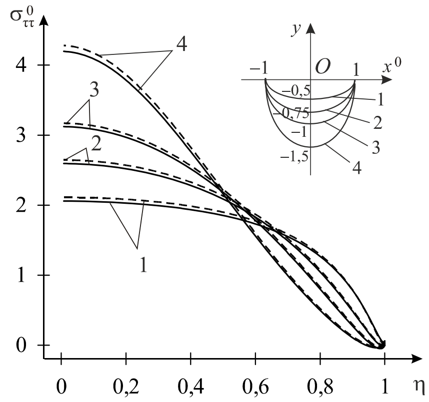

Figure 5. Stresses $\sigma_{\tau\tau}^{0}$on the notch along the arc of the ellipse for $l = 1$ and various values of δ.

In Figure 5, the curves *1* are constructed for $\delta = 0{,}5$, curves *2* for $\delta = 0{,}75$, curves *3* for $\delta = 1$, curves *4* for $\delta = 1{,}5$.

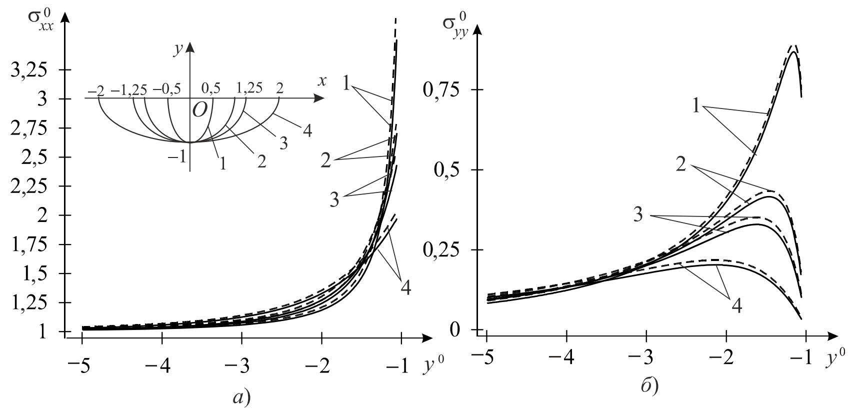

Figure 6. Stresses σ_{xx}^{0} and σ_{yy}^{0} on the extension of the symmetry axis of the notch along the arc of the ellipse for $\delta = 1$ and various values of l

In Figure 6, the curves *1* are constructed for $l = 0{,}5$, curves *2* for $l = 1$, curves *3* for $l = 1{,}25$, curves *4* for $l = 2$.

Figure 7 and Figure 8 show the relevant graphs of stresses $\sigma_{\tau\tau}^{0}$, σ_{xx}^{0} and σ_{yy}^{0} for the notches along the arc of the parabola.

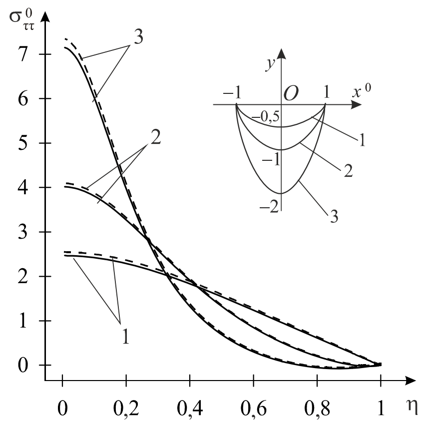

Figure 7. Stresses $\sigma^0_{\tau\tau}$ on the notch along the arc of the parabola for $l = 1$ and various values of δ

In Figure 7, the curves *1* are constructed for $\delta = 0{,}5$, curves *2* for $\delta = 1$, curves *3* for $\delta = 2$.

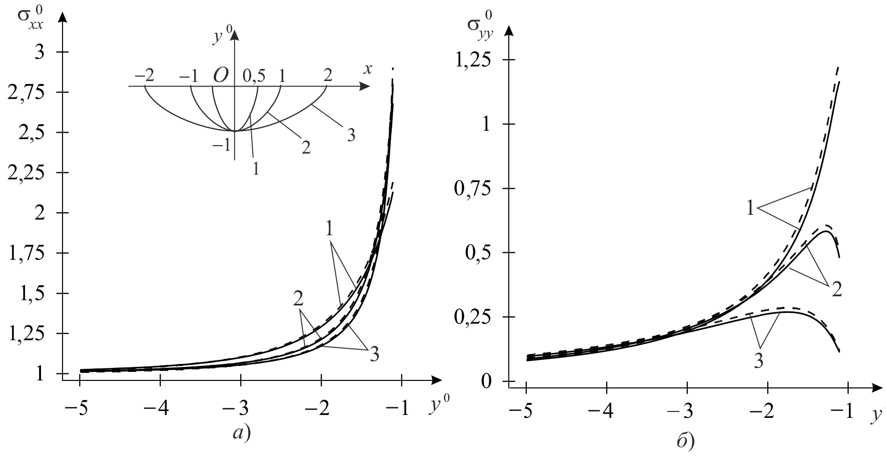

Figure 8. Stresses σ^0_{xx} and σ^0_{yy} on the extension of the symmetry axis of the notch along the arc of the parabola for $\delta = 1$ and various values of l

In Figure 8, the curves *1* are constructed for $l = 0{,}5$, curves *2* for $l = 1$, curves *3* for $l = 2$.

Figure 9 shows the change of the stresses $\sigma^0_{\tau\tau}$ for the three types of notches of the same depth (arcs of circles, parabolas and ellipses that pass through three fixed points), which enables us to identify the influence of notch shape on the stress state of the plate.

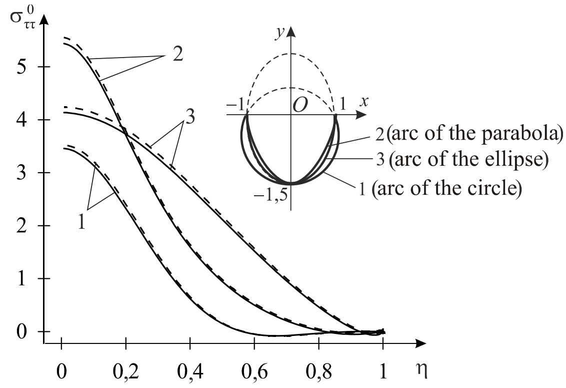

Figure 9. Stresses $\sigma^0_{\tau\tau}$ on the surface of the notches of the same depth along the arc of the circle, parabola, and ellipse for r $l = 1, \delta = 1{,}5$.

In Figure 9, curve *1* concerns the circle, curve *2* the parabola, and curve *3* the ellipse.

5 Conclusions

As is shown in Figures 5-8, the given stresses at the top of the notch along the arc of an ellipse or a parabola significantly increase with increasing of the relative depth of the notch (while increasing its depth or decreasing width). As is shown in Figure 9, sharpness of the obviously also enlarges the level of stress.

As is shown in Figures 3-9, the stress fields obtained by analytic-numerical and variation-difference methods differ by no more than 2%. This discrepancy can be explained by the fact that the analytical solution domain is unbounded, while the numerical calculation was carried out, obviously, for a finite field.

Thus, the developed method of numerical determination of stress and their concentrations agrees at solving plane elasticity problems in plates with notch.

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