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# NUMERICAL CRITERION FOR THE DURATION OF NON-CHAOTIC TRANSIENTS IN ODEs

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**Abstract:** The paper proposes an original numerical criterion for the duration analysis of non-chaotic transients based on the Euclidean norm of a properly defined vector. For this purpose, transient trajectories, prior to their entering a small neighbourhood of the limit cycle, are used. The vector has been defined with its components constituting the lengths of the sections, which connect the origin of the coordinate system with appropriately determined transient trajectory points. The norm of the vector for the analysis of non-chaotic transients has also been applied. As an assessment criterion of transients, the convergence of the norm to small neighbourhood of the limit cycle with the assumed accuracy is used. The paper also provides examples of the application of this criterion to the Van der Pol oscillators in the case of periodic oscillations.

**Key words:** numerical criterion, non-chaotic transients, limit cycle, ODEs systems, Van der Pol oscillator

## 1. INTRODUCTION

An exact assessment of transient behaviour in physical systems is very important for many practical reasons. For example, in real-time digital simulators, which are applied in the power industry, the exact assessment of transients is also essential for practical implementations in short-circuit protection systems. The transient behaviour analysis is also essential when dealing with short circuits in the power grid. Control systems, especially the ones used in nonlinear systems, must account not only for the transient phenomena that occur in them but also for their duration. In the analyses of nonlinear systems, neither the superposition principle can be used nor can the transient component be separated. It can be said that the transient phenomenon is fully integrated into the mathematical model of the system and constitutes an inseparable problem to be solved. This is the source of additional difficulties of transient analysis. In addition, the transient dynamics in nonlinear systems depends not only on the parameters of the system but also on the choice of initial conditions. Additionally, it should be emphasised that chaotic systems are particularly sensitive to the latter, where small changes in the initial conditions can cause great differences in the nature of the final solution. An example of this is the system of Lorenz equations [1, 2].

After the introduction of state variables, nonlinear differential equations are transformed into a system of the first-order ordinary differential equation (ODEs), called the state equations. The purpose of these transformations is to simplify the equations so as to be able to use numerical methods to solve them [3, 4]. A great number of physical systems are modelled by second-order ODEs, for example, the Duffing or Van der Pol equations [3, 5, 6, 7, 8, 9]. These equations, after their transformation into a system of equations, are analysed on the phase plane. As a result, for stable systems, it is possible to obtain specific phase portraits whose trajectories are convergent to the limit cycle. In the transients, these trajectories are irregular lines that turn into periodic cycles after the transient time ttr [10].

In contrast to the paper [10], in which the non-chaotic transients in the Duffing equations were examined on the basis of the analyses of cycle fields, in this paper, the Euclidean norm of a properly defined vector was applied to the analyses of the Van der Pol's non-chaotic transients.

The literature available on the subject is very extensive. We will now present a brief description of the selected studies regarding the analysis of the transient behaviour occurring in nonlinear systems [11-17].

Paper [11] presents a general overview of works on transient behaviour in chaotic systems, while monograph [12] and dissertation [18] discuss in detail the physical phenomena taking place in the Duffing and Van der Pol systems including the methods of their analyses.

Specific transient behaviour is discussed in Zumdieck et al. [13]. Long chaotic transients occur in complex networks of pulse-coupled oscillators. It has been shown that small changes in the structure of the system have a decisive influence on its dynamics. The paper [2] presents an interesting analysis of the impact of initial conditions on the course of transient trajectories. To this end, a system of three Lorenz equations has been used. Chaotic transients and super transients in spatially extended systems have been described in Tél and Lai [14]. These specific states occur in the systems described by PDEs (partial differential equations), e.g., the Navier-Stokes equations. Also, the noteworthy results on transient analysis are presented in Cooper et al. [15] and Sabarathinam et al. [16]. In the former one, several methods of controlling an autonomous Van der Pol oscillator have been analysed. It presented transient trajectories using different control methods and non-zero initial conditions, entering the limit cycle of the system. The latter work discusses an original Duffing system with smooth cubic nonlinearity in the form of a memristor. It shows a phase portrait and time evolutions of state variables for non-zero initial conditions and selected system parameters. In the paper by Vahedi et al. [17], the Duffing oscillator has been used for the analyses of distributed-generation (DG) units. The paper analyses an oscillator operating in both chaotic and periodic vibrations under an appropriate control mode. It presents a characteristic phase portrait of the system with transient trajectories of the system and characteristic phase portraits. In the case of periodic vibrations, transient trajectories enter the limit cycle. Papers [19, 20] make use of the transients to estimate encoded parameters of the Duffing and Van der Pol nonlinear systems.

The paper proposes an original criterion for determining the duration of the non-chaotic transient in ODEs, based on the Euclidean norm of a properly defined vector.

The criterion is illustrated by example analyses for the duration of transient processes in physical systems described by the Van der Pol's nonlinear equations.

In our paper, we consider nonlinear systems with a stable small neighbourhood of the limit cycle and the temporal length of the non-chaotic transient time denoted by ttr. The proposed method of estimating the transient time ttr is illustrated by analysing the Van der Pol equations [8, 21]. The above equations were developed in the first half of the 20th century. They have been a subject of many scientific studies, and the phenomena described in them have been frequently discussed. The computations have been made using a proprietary program written in C++. The fourth-order Runge-Kutta method was applied in the numerical calculations. Some selected results were compared with the computations obtained by applying the Wolfram Mathematica software.

## 2. NUMERICAL CRITERION FOR THE DURATION OF NON-CHAOTIC TRANSIENTS IN ODEs

An accurate time definition for the duration of the non-chaotic transients in nonlinear systems, considering its practical applications, is very important. It can be applied in the analysis of the operation of automation systems, the evaluation of the power systems security and the analysis of the operation of electronic systems.

The time $t_{tr}$, in which the transient trajectory enters a small neighbourhood of the limit cycle, is assumed as the end of the duration of the non-chaotic transients. For further considerations, a constant number of $N$ points was assumed for each cycle loop of the cycle for determining Euclidean norms (Fig. 1). In this case, the assumed $N$ coincides with the step of numerical calculations.

Each point in $N$ corresponds to the section $OP_i$ $(i = 1,2,\dots,N)$. Thus, we define vector, $Y$:

$$Y = [y_1, y_2, \dots, y_N] \tag{1}$$

wherein:

$$y_i = OP_i = ({x_{1,i}}^2 + {x_{2,i}}^2)^{1/2}, \quad (i = 1,2,\dots,N) \tag{2}$$

where $x_1$ and $x_2$ are the values of state variables in the resulting time intervals

The Euclidean norm is assigned to vector $Y$:

$$\|Y\| = \left(\Sigma_{i=1}^{N} y_i^2\right)^{\frac{1}{2}} \tag{3}$$

where $N$ is the assumed constant number points for each cycle loop.

In interval time, we have a series of $\|Y\|_k$ norms:

$$\|Y\|_k = \left(\left(\Sigma_{i=1}^{N} y_i^2\right)^{\frac{1}{2}}\right) k, \ (k = 1,2,\dots,L) \tag{4}$$

where $L$ is the total number of cycles investigated.

For stable systems, the series of norms $\|Y\|_k$ $(k = 1,2,\dots,L)$ converges to a stable norm, which represents a small neighbourhood of the limit cycle.

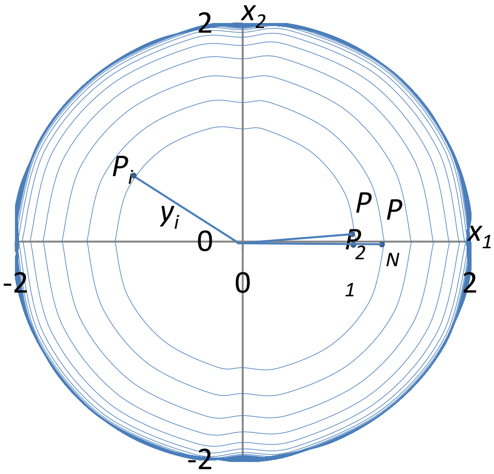


**Fig. 1.** Phase portrait with non-chaotic transients of the Van der Pol system Eq. (6), $\mu = 0.1$, $x_1(0) = 1$, $x_2(0) = 0$

Considering the above-mentioned facts, a definition of the duration of non-chaotic transients in nonlinear systems can be presented as follows.

In a stable system, described by the system of equations $\dot{x} = f(x(t), u(t), t)$, where $x(t) \in R^n$ and $u(t) \in R^n$, the duration of non-chaotic transients is determined by time $t_{tr}$, in which the transient trajectory tends to a small neighbourhood of the limit cycle with the condition:

$$\frac{|\|Y\|_k - (\|Y\|_m)_k|}{(\|Y\|_m)_k} \geq \varepsilon, k = L-1, L-2, \dots \tag{5}$$

where for each $k$, $(\|Y\|_m)_k$ represents the mean value of the norms calculated for cycles $L, \dots, k+1$ and $\varepsilon$ is a sufficiently small number.

It should be noted that the non-chaotic transients last as long as the standard deviations $\|Y\|_k$ in criterion (5) are greater by $\varepsilon$ than the mean value of $(\|Y\|_m)_k$. The condition is checked cyclically from the end of the integration interval, i.e., from small deviations of the norm $\|Y\|_k$ for a more precise determination of the mean value of $(\|Y\|_m)_k$. The first cycle satisfying the condition (5) determines the last cycle of the non-chaotic transients and therefore its duration $t_{tr}$.

## 3. APPLICATION OF THE PROPOSED CRITERION FOR THE VAN DER POL EQUATION

The Van der Pol equation without the driving force is very often presented after the application of the Liénard transformation:

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 + \mu\left(x_2 - \frac{{x_2}^3}{3}\right). \tag{6}$$

The Van der Pol equation applies to systems in which there is a nonlinear damping $\mu(1 - x^2)$, where $\mu$ is a small parameter.

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Using the $\mu = 0.1$ to solve the system (6) with initial conditions $x_1(0) = 1, x_2(0) = 0$, it is possible to obtain a typical phase portrait shown in Fig. 1. The convergence of $\|Y\|_k$ to $(\|Y\|_m)_k$ is illustrated in Fig. 2. Assuming $\varepsilon = 0.005$, the transient time $t_{tr}$ is found to be $t_{tr} = 52.06$.

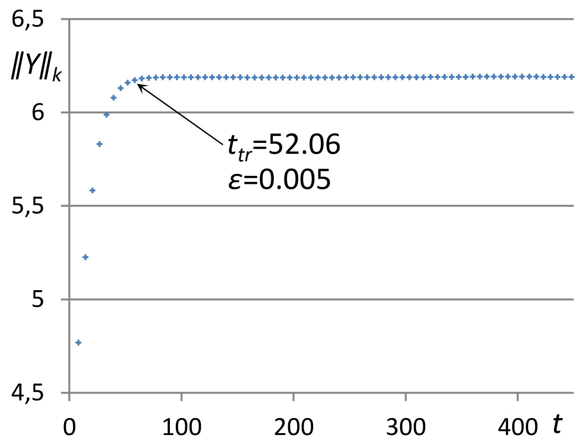


**Fig. 2.** Chart of $\|Y\|_k = f(t)$, $\mu = 0.1, x_1(0) = 1, x_2(0) = 0$

An example application of the proposed criterion for the dependence of the transient time on the value of the damping coefficient $\mu$ is given for the driven Van der Pol equation.

This equation is most often used to describe electronic vibration generators, although it can be found in other fields of knowledge [21, 22].

The Van der Pol equation with a driving function b1cos(b2t) has the following form:

$$\frac{d^2z}{dt^2} + \frac{\mu(1-z^2)dz}{dt} + z = b_1\cos(b_2t), \tag{7}$$

where $\mu$ is the parameter of the electronic circuit, while $b_1$ and $b_2$ are the driving parameters.

Eq. (7) can be represented as a system of equations, taking $z(t) = x_1(t),\ dz/dt = x_2(t)$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\mu(1 - {x_1}^2)x_2 - x_1 + b_1\cos(b_2t). \tag{8}$$

In further considerations, we adopt damping parameter values $\mu$ changing in the range 0.01÷0.2 and take:

$$b_1 = 1.0, b_2 = 250. \tag{9}$$

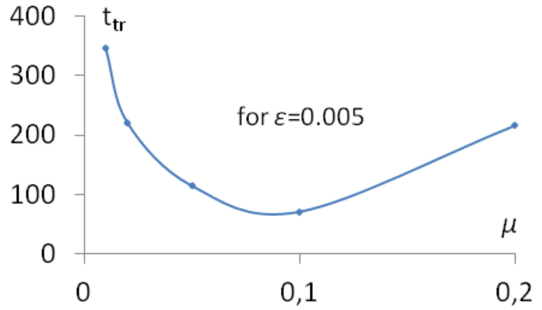


**Fig. 3.** Dependence of the non-chaotic transient time $t_{tr}$ on the value of $\mu$

As a result of solving the system (8), we obtain the phase portrait characterising the periodic vibrations. The individual steps of the determination of transient time $t_{tr}$ are presented earlier in Section 2.

Fig. 3 presents the chart of the time duration $t_{tr}$ of non-chaotic transient dependence on the value of the dumping coefficient $\mu$ for the Van der Pol Eq. (8), based on the proposed numerical criterion.

An example of calculating the transients time $t_{tr}$ based on the proposed method for the point $\mu = 0.05$ in Fig. 3 is shown in Figs. 4 and 5.

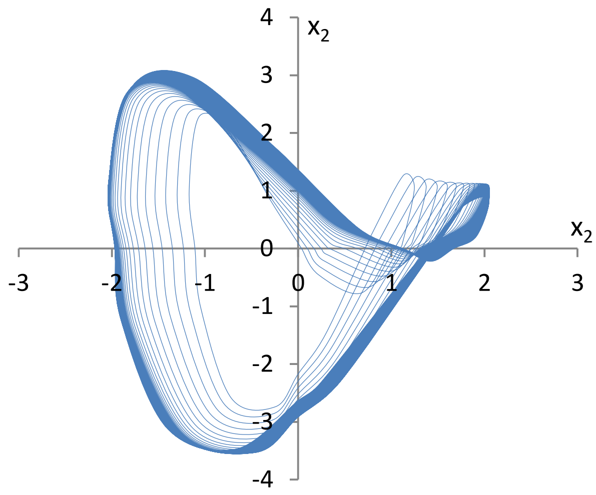


**Fig. 4.** Phase portrait of Van der Pol system, $\mu = 0.05$, $b_1 = 1.0$, $b_2 = 250$ $x_1(0) = 1, x_2(0) = 0$

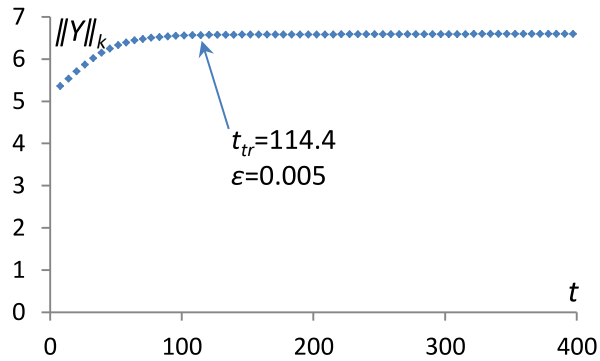


**Fig. 5.** Chart for $\|Y\|_k = f(t)$, $\mu = 0.05$, $b_1 = 1.0$, $b_2 = 250$ $x_1(0) = 1, x_2(0) = 0$

For higher values of the damping factor $\mu >$ 0.2 Eq. (8) in the considered time interval in the system, there are vibrations that characterise the exemplary attractor shown in Fig. 6. In this case, the $\|Y\|_k$ norm does not converge. The chart of the $\|Y\|_k$ norm for the phase portrait from Fig. 6 is shown in Fig. 7.

In further work, the generalisation of the duration of non-chaotic transients into higher-order ODEs equations will be continued. Also, the study of the influence of $\varepsilon$ and the other system parameters of the duration on non-chaotic transients will be continued.

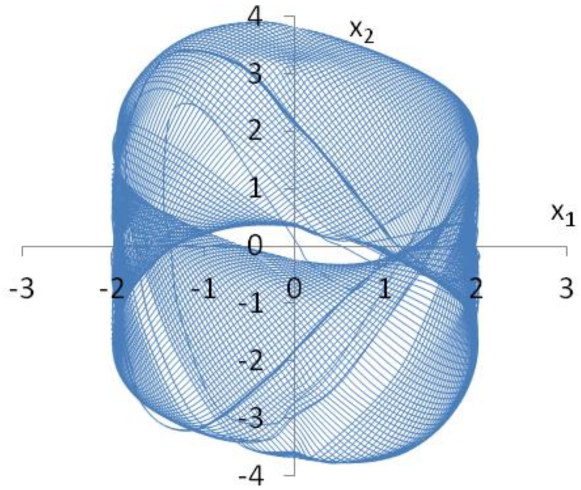


**Fig. 6.** Phase portrait of Van der Pol system, $\mu = 0.3, b_1 = 1.0, b_2 = 250, x_1(0) = 1, x_2(0) = 0$

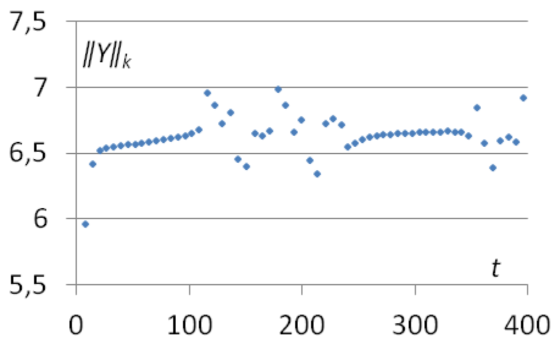


**Fig. 7.** Chart for $\|Y\|_k = f(t), \mu = 0.3, b_1 = 1.0, b_2 = 250, x_1(0) = 1, x_2(0) = 0$

## 4. FINAL REMARKS AND CONCLUSIONS

The paper proposes an original numerical criterion for the duration analysis of non-chaotic transients based on the Euclidean norm of a properly defined vector for second-order ODEs systems. The analysed systems are assumed to be stable, and their transient trajectories converge to a small neighbourhood of the limit cycle. In order to assess time $t_{tr}$, a series of Euclidean norms $\|Y\|_k$ of vector $Y$ are used. The vector's components include the length values of $OP_i$ sections in successive trajectory cycles, where $OP_i = ({x_{1,i}}^2 + {x_{2,i}}^2)^{1/2}$. In a nonlinear stable system with the limit cycle, the $\|Y\|_k$ series is convergent to the mean value of $(\|Y\|_m)_k$ calculated from the end of the time interval, characteristic of the phase portrait reflecting the periodic oscillation. The proposed criterion is illustrated by examples of the analyses for the duration of transient processes in physical systems described by Van der Pol nonlinear equations. The proposed criterion is relatively simple and easy to apply to many practical engineering issues.

The method can be also generalised to higher-order ODEs. For example, when analysing the non-chaotic transients modelled by Lorenz equations, the tests should be carried out in space $R^3$, with $OP_i = ({x_{1,i}}^2 + {x_{2,i}}^2 + {x_{3,i}}^2)^{1/2}$. In this case, the series of norms $\|Y\|_k$ converges to the mean norm $(\|Y\|_m)_k$ computed in three-dimensional space.

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