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THE EXISTENCE OF CONSENSUS OF A LEADER-FOLLOWING PROBLEM WITH CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, consensus of a leader-following problem is investigated. The leader-following problem describes a dynamics of the leader and a number of agents. The trajectory of the leader is given. The dynamics of each agent depends on the leader trajectory and others agents' inputs. Here, the leader-following problem is modeled by a system of nonlinear equations with Caputo fractional derivative, which can be rewritten as a system of Volterra equations. The main tools in the investigation are the properties of the resolvent kernel corresponding to the Volterra equations, and Schauder fixed point theorem. By study of the existence of an asymptotically stable solution of a suitable Volterra system, the sufficient conditions for consensus of the leader-following problem are obtained.

Keywords: leader-following problem, Caputo fractional differential equation, consensus, nonlinear system, Schauder fixed point theorem.

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1. INTRODUCTION

We investigate the leader-following problem given as a system of equations with Caputo fractional derivative of the following form

$$^{C}D^{\alpha}_{0+}x_i(t) = f(t, x_i(t)) + \gamma \sum_{j=1}^{N} a_{ij}\big(x_j(t) - x_i(t)\big) + \gamma d_i\big(x_0(t) - x_i(t)\big), \tag{1.1}$$

$i = 1, 2, \ldots, N$, with initial conditions

$$x_0(0) = x_0^\star \in \mathbb{R}, \;\; x_i(0) = x_i^\star \in \mathbb{R},$$

where $x_0 : \mathbb{R}_+ \to \mathbb{R}$ fulfills condition

$$^{C}D^{\alpha}_{0+}x_0(t) = f(t, x_0(t)). \tag{1.2}$$

Here $x_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 0, 1, \ldots, N$ are unknown functions, γ, a_{ij}, d_i are some real constants, $i, j = 1, 2, \ldots, N$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. Throughout this paper we assume that $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By ${}^C D^\alpha_{0+} g(t)$ we denote Caputo fractional differential of fractional order $\alpha \in (0, 1)$ of a function $g : \mathbb{R}_+ \to \mathbb{R}$.

Definition 1.1 ([12]). Let $\alpha \in (0, 1)$. The function

$$ {}^C D^\alpha_{0+} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(\tau)}{(t-\tau)^\alpha} d\tau $$

is called Caputo left-side derivative (shortly Caputo derivative) of fractional order α of a function g.

The organization of this paper is the following. In Section 2, the brief history of fractional calculus is described. Next, in Section 3, the motivation of introduction of the system (1.1) as a model of flocking is presented, and the definition of consensus is given. In Section 4, we recall some useful lemmas and we also introduce some useful notation. Finally, in Section 5, the existence of asymptotically stable solution is investigated, and the sufficient conditions for consensus of the leader-following problem are obtained.

2. FRACTIONAL CALCULUS

Fractional calculus is a natural extension of ordinary calculus, where integrals and derivatives are defined for arbitrary real orders. The origins of fractional calculus go back to seventeenth century, when in 1695 the derivative of order 0.5 was described by Leibniz. Since that time, several different fractional derivatives have been introduced, for example: Riemann–Liouville, Hadamard, Grunwald–Letnikov, Caputo. The choice of an appropriate fractional derivative (or integral) depends on the studied problem, and for this reason a large number of works devoted to different fractional operators appeared in literature. Also nowadays, fractional calculus theory attracts attention of many authors, see for example [14, 15]. The Caputo and Riemann–Liouville fractional operators are two broadly used in various fields. It is known (see [17]) that Caputo is one of the most appropriate operators to discuss problems involving a fractional differential equation with initial condition.

3. MODEL OF FLOCKING

The motivation for investigation of system (1.1) is the following. Flocking is an emergent behavior that can be observed in many multi-agent systems and represents the situation that autonomous agents, using only limited environmental information, organize themselves in an ordered motion. For example, Cucker and Smale in [6, 7] proposed the model to describe the evolution of a flock with a finite quantity of agents. Girejko *et al.* investigated consensus models on isolated time scales: Krause's model in [9], and Cucker-Smale model (see [10]).

Recently, an interesting topic of investigation is the consensus of a group of agents with the leader, where the leader is a special agent whose motion is independent of all the other agents and is followed by all the other ones. In this paper, we investigate the system proposed by Yu, Jiang and Hu in [19], where the authors adopted the Caputo fractional operators to model the multi-agent system dynamics and analyze the consensus by investigation of algebraic graph theory. We described the dynamics of each agent labeled i, $i = 1, 2, \dots, N$, by the following nonlinear Caputo fractional differential equation

$$ {}^{C}D^{\alpha}_{0+}x_i(t) = f(t, x_i(t)) + u_i(t), \quad x_i(0) = x_i^{\star} \in \mathbb{R}, \tag{3.1} $$

where unknown function $x_i(t)$ represents the state of i-th agent at time $t \in \mathbb{R}_+$, u_i is i-th agent input, which can only use local information from its neighbor agents. The trajectory of the leader, labeled as $i = 0$, is given by the following condition

$$ {}^{C}D^{\alpha}_{0+}x_0(t) = f(t, x_0(t)), \quad x_0(0) = x_0^{\star} \in \mathbb{R}. \tag{3.2} $$

Function u_i is introduced in the following way. Let $a_{ij} \in \mathbb{R}$, $d_i \in \mathbb{R}$, for $i = 1, 2, \dots, N$, and let $A = [a_{ij}]$ and

$$ D = diag(d_1, d_2, \dots, d_N) \text{ be a given } N \times N \text{ matrices.} \tag{3.3} $$

The Laplacian matrix $L = [l_{ij}]$ of A (see [7]) is defined as

$$ l_{ii}(t) = \sum_{j \neq i} a_{ij} \quad \text{and} \quad l_{ij}(t) = -a_{ij}. \tag{3.4} $$

We investigate the problem given by (3.1), where

$$ u_i(t) = \gamma \sum_{j=1}^{N} a_{ij}\big(x_j(t) - x_i(t)\big) + \gamma d_i\big(x_0(t) - x_i(t)\big), \quad i = 1, 2, \dots, N, \tag{3.5} $$

and $\gamma \in \mathbb{R}$. Obviously, by putting (3.5) into (3.1), we get equation (1.1).

Definition 3.1. The multi-agent system (1.1) is said to be achieved the leader-following consensus if any solutions $\tilde{x}(t)$ and $\bar{x}(t)$ of system (1.1) satisfy

$$ \lim_{t\to\infty} |\tilde{x}_i(t) - \bar{x}_i(t)| = 0 \quad \text{for} \quad i = 1, 2, \dots, N, \tag{3.6} $$

and for any initial conditions $\tilde{x}_i^{*}, \bar{x}_i^{*} \in \mathbb{R}$.

If a solution of system (1.1) satisfies condition (3.6) for $\tilde{x}^{*}, \bar{x}^{*} \in U \subset \mathbb{R}^N$, the leader-following consensus is achieved locally.

4. USEFUL LEMMAS AND NOTATIONS

The following lemmas are used in the proof of the main result of this paper.

Lemma 4.1 (Schauder fixed point theorem, [1]). *Let Ω be a nonempty, closed and convex subset of a Banach space and let $T : \Omega \to \Omega$ be a continuous mapping, such that $T(\Omega)$ is a relatively compact subset of Ω. Then T has a fixed point in Ω.*

Next, we introduce the following notations $z := [z_1, z_2, \dots, z_N]^T$, $z_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, N$. Set

$$\mathcal{B} := \{z : \mathbb{R}_+ \to \mathbb{R}^N,\ z_i \text{ is bounded and continuous for any } i = 1, 2, \dots, N\},$$

where

$$\|z\| := \sup_{t\geq 0}\{|z(t)|\}. \tag{4.1}$$

Obviously, $\mathcal{B}$ is a Banach space with the norm defined by (4.1). Space $\mathcal{C}_l$ is defined as follows:

$$\mathcal{C}_l := \{z : \text{there exists } \lim_{t\to\infty} z(t) \in \mathbb{R}^N\}, \tag{4.2}$$

$$\lim_{t\to\infty} z(t) = [\lim_{t\to\infty} z_1(t), \lim_{t\to\infty} z_2(t), \dots, \lim_{t\to\infty} z_N(t)]^T.$$

Let $\rho > 0$ be a constant, and let

$$\mathcal{B}_\rho := \{z : \|z\| \leq \rho\}. \tag{4.3}$$

Set

$$\mathcal{C}_{l,\rho} := \mathcal{C}_l \cap \mathcal{B}_\rho, \tag{4.4}$$

where $\mathcal{C}_l$ is defined by (4.2).

The following definition will be used in the sequel.

Definition 4.2 ([3,5]). Let $S_T := \{(t,s) : 0 \leq s \leq t \leq T\}$. Function $E(t,s)$ is weakly singular on the set S_T if it is discontinuous in S_T, but for each $t \in [0,T]$, $E(t,s)$ has at most finitely many discrete discontinuities in the interval $0 \leq s \leq t$ and for every continuous function $\phi : [0,T] \to \mathbb{R}$

$$\int_0^t E(t,s)\phi(s)ds \quad \text{and} \quad \int_0^t |E(t,s)|ds$$

both exist and are continuous on $[0,T]$. If $E(t,s)$ is weakly singular on S_T for every $T > 0$, then it is weakly singular on the set $S := \{(t,s) : 0 \leq s \leq t < \infty\}$.

Exactly as in [2], we can derive the following auxiliary lemma. In [2] the functions considered are assumed to be continuous but the careful analysis of the proof allows us to formulate it for weakly singular functions. Since the only change arises in the notion of the integral, i.e. we replace the Riemann Integral with the Lebesgue one and omit the sets of measure zero.

Assume that $K : S \to \mathbb{R}$ is weakly singular on the set S.

Lemma 4.3. *Suppose that for a function K, the following hypotheses are fulfilled:*

(i) *there exists $M > 0$ such that*

$$\int_0^t |K(t,s)|ds \leq M \ \textit{for} \ t \in \mathbb{R}_+,$$

(ii) *for all $\beta > 0$, one has*

$$\lim_{t\to\infty} \int_0^\beta K(t,s)ds = 0,$$

(iii)

$$\lim_{t\to\infty} \int_0^t K(t,s)ds = 1.$$

Then for every $z \in C_l$,

$$\lim_{t\to\infty} \int_0^t K(t,s)z(s)ds = \lim_{t\to\infty} z(t).$$

On the space C_l the following compactness criterion holds.

Lemma 4.4 ([15])**.** *A family $\mathcal{A} \subset C_l$ is relatively compact if and only if*

(a) *$\mathcal{A}$ is uniformly bounded,*
(b) *$\mathcal{A}$ is equicontinuous on compact subsets of $\mathbb{R}_+$,*
(c) *$\mathcal{A}$ is equiconvergent.*

Definition 4.5. A family $\mathcal{A} \subset C_l$ is called equiconvergent if for every $\varepsilon > 0$, there exists a $M = M(\varepsilon) > 0$, such that for all $z \in \mathcal{A}$, and for all $t_1, t_2 \geq M$, $|z(t_1) - z(t_2)| \leq \varepsilon$.

5. EXISTENCE OF A BOUNDED AND ASYMPTOTICALLY STABLE SOLUTION

Let us consider system (1.1)–(1.2). Setting

$$\begin{gathered} x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T, \ \ x_0^v(t) = [x_0(t), x_0(t), \ldots, x_0(t)]^T, \\ F(t, x(t)) = \big[f(t, x_1(t)), f(t, x_2(t)), \ldots, f(t, x_N(t))\big]^T, \\ F(t, x_0^v(t)) = \big[f(t, x_0(t)), f(t, x_0(t)), \ldots, f(t, x_0(t))\big]^T, \end{gathered} \tag{5.1}$$

we can rewrite system (1.1)–(1.2) as follows:

$${}^C D_{0+}^{\alpha}\big(x(t) - x_0^v(t)\big) = F(t, x(t)) - F(t, x_0^v(t)) - \gamma B\big(x(t) - x_0^v(t)\big), \tag{5.2}$$

where $B = D + L$, D and L are defined by (3.3) and (3.4), respectively.

In the following Remark, the equivalency of system (1.1)–(1.2) and equation (5.2) is proved in the case $N = 3$. For $N \neq 3$, the proof is analogous and hence omitted.

Remark 5.1. Assume that $N = 3$. To show that system (1.1)–(1.2) and equation (5.2) are equivalent it is enough to prove that

$$[u_1(t), u_2(t), u_3(t)]^T = -\gamma B\big(x(t) - x_0^v(t)\big), \tag{5.3}$$

where $u_i(t)$ is defined by (3.5). Set

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

The Laplacian matrix of matrix A is

$$L = \begin{bmatrix} a_{12} + a_{13} & -a_{12} & -a_{13} \\ -a_{21} & a_{21} + a_{23} & -a_{23} \\ -a_{31} & -a_{32} & a_{31} + a_{31} \end{bmatrix}.$$

We can rewrite (3.5) in the following form:

$$\begin{cases} u_1(t) = \gamma\Big(a_{12}\big(x_2(t) - x_1(t)\big) + a_{13}\big(x_3(t) - x_1(t)\big) + d_1\big(x_0(t) - x_1(t)\big)\Big), \\ u_2(t) = \gamma\Big(a_{21}\big(x_1(t) - x_2(t)\big) + a_{23}\big(x_3(t) - x_2(t)\big) + d_2\big(x_0(t) - x_2(t)\big)\Big), \\ u_3(t) = \gamma\Big(a_{31}\big(x_1(t) - x_3(t)\big) + a_{32}\big(x_2(t) - x_3(t)\big) + d_3\big(x_0(t) - x_3(t)\big)\Big). \end{cases}$$

From the other hand

$$\begin{aligned}
&-\gamma B\big(x(t) - x_0^v(t)\big) \\
&= -\gamma(L + D)\big(x(t) - x_0^v(t)\big) \\
&= -\gamma\left(\begin{bmatrix} a_{12} + a_{13} & -a_{12} & -a_{13} \\ -a_{21} & a_{21} + a_{23} & -a_{23} \\ -a_{31} & -a_{32} & a_{31} + a_{31} \end{bmatrix} + \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\right) \begin{bmatrix} x_1(t) - x_0(t) \\ x_2(t) - x_0(t) \\ x_3(t) - x_0(t) \end{bmatrix} \\
&= -\gamma \begin{bmatrix} a_{12} + a_{13} & -a_{12} & -a_{13} \\ -a_{21} & a_{21} + a_{23} & -a_{23} \\ -a_{31} & -a_{32} & a_{31} + a_{31} \end{bmatrix} \begin{bmatrix} x_1(t) - x_0(t) \\ x_2(t) - x_0(t) \\ x_3(t) - x_0(t) \end{bmatrix} - \gamma \begin{bmatrix} d_1\big(x_1(t) - x_0(t)\big) \\ d_2\big(x_2(t) - x_0(t)\big) \\ d_3\big(x_3(t) - x_0(t)\big) \end{bmatrix} \\
&= \gamma \begin{bmatrix} a_{12}\big(x_2(t) - x_1(t)\big) + a_{13}\big(x_3(t) - x_1(t)\big) \\ a_{21}\big(x_1(t) - x_2(t)\big) + a_{23}\big(x_3(t) - x_2(t) \\ a_{31}\big(x_1(t) - x_3(t)\big) + a_{32}\big(x_2(t) - x_3(t) \end{bmatrix} - \gamma \begin{bmatrix} d_1\big(x_1(t) - x_0(t)\big) \\ d_2\big(x_2(t) - x_0(t)\big) \\ d_3\big(x_3(t) - x_0(t)\big) \end{bmatrix}.
\end{aligned}$$

Hence, (5.3) holds.

Let us denote

$$e_i(t) = x_i(t) - x_0(t) \text{ for } i = 1, 2, \ldots, N, \tag{5.4}$$

and

$$h(t, e(t)) := F(t, e(t) - x_0^v(t)) - F(t, x_0^v(t)) - (\gamma B + I)e(t), \tag{5.5}$$

where

$$h(t, e(t)) = \big[h_1(t, e_1(t)), h_2(t, e_2(t)), \ldots, h_N(t, e_N(t))\big]^T,$$

and I denotes the identity $N \times N$ matrix. Hence, system (5.2) takes the following form

$${}^C D_{0+}^{\alpha} e(t) = e(t) + h(t, e(t)), \tag{5.6}$$

with initial condition

$$e(0) = [x_1^\star, \ldots, x_N^\star]^T - [x_0^\star, \ldots, x_0^\star]^T \in \mathbb{R}^N.$$

Equation (5.6) can be transformed into an equivalent Volterra integral equation

$$e(t) = e(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(e(s) + h(s, e(s))\big) ds, \tag{5.7}$$

where Γ is the gamma function (see [8,11,13]). By $C : \mathbb{R}_+ \to \mathbb{R}_+$, we denote the kernel of Volterra equation (5.7), that is,

$$C(t-s) := \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1}, \quad t \geq s \geq 0. \tag{5.8}$$

We observe that the kernels of Volterra integral equations obtained from Caputo fractional differential equations with $0 < \alpha < 1$ are weakly singular. Equation (5.7) can be written as a Volterra equation

$$e(t) = e(0) - \int_0^t C(t-s)\big(e(s) + h(s, e(s))\big) ds. \tag{5.9}$$

It is known (see, for example [16, p. 191] or [11]), that function e is a solution of equation (5.7) if and only if e satisfies the variation of constants equation in the form

$$e(t) = y(t) - \int_0^t R(t-s) h(s, e(s)) ds, \tag{5.10}$$

where the function $y : \mathbb{R}_+ \to \mathbb{R}^N$, $y(t) = [y_1(t), y_2(t), \ldots, y_N(t)]^T$, is given by

$$y(t) = e(0) - \int_0^t R(t-s) e(0) ds. \tag{5.11}$$

The function R is known as the resolvent kernel of C and in its domain satisfied equation (see [4, p. 55])

$$R(t-s) = C(t-s) - \int_0^t C(t-v)R(v-s)ds,$$

or equivalently

$$R(t-s) = C(t-s) - \int_0^t R(t-v)C(v-s)ds.$$

The resolvent kernel R corresponding to the given kernel C is unique.

For simplicity, putting $t := t-s$, we can rewrite C and R as a one variable functions. The resolvent kernel R satisfies the following conditions (see [16, 18]) for $t \geq 0$

$$0 \leq R(t) \leq C(t), \;\; R(t) \to 0 \;\text{ as }\; t \to \infty, \tag{5.12}$$

and

$$\text{if } C \notin L^1[0,\infty), \text{ then } \int_0^\infty R(t)dt = 1. \tag{5.13}$$

Next, we prove the existence result of asymptotically stable solution of equation (5.10). Since equations (5.10) and (5.9) are equivalent, in this way we prove an existence result of asymptotically stable solution of equation (5.9). Using this result, we present the sufficient conditions for the leader-following consensus of system (3.1)–(3.2), which is in fact equivalent to system (1.1)–(1.2).

Definition 5.2. A function $\tilde{e}$ is said to be an asymptotically stable solution of equation (5.10) if for every $\varepsilon > 0$, there exists a $M = M(\varepsilon)$, such that for every $t \geq M$, and for every other solution $\bar{e}$ of equation (5.10), the following inequality holds

$$\|\tilde{e}(t) - \bar{e}(t)\| \leq \varepsilon.$$

Notice that, since f is a continuous function, any coordinate of vector of F has this property. Condition (5.5) and linearity of term $\gamma(B+I)e$ imply that any coordinate of function h is continuous, too. Suppose that there exists $l = \left[l_1, l_2, \dots, l_N\right]^T \in \mathbb{R}^N$, such that

$$\lim_{t\to\infty} h(t,z) = l \text{ uniformly with respect to } z \in \mathcal{C}_{l,\rho}, \tag{5.14}$$

where h is defined by (5.5). Set

$$m_\rho := \sup_{t\geq 0}\{|h(t,z)| : z \in \mathcal{C}_{l,\rho}\} < \infty. \tag{5.15}$$

Since $\alpha \in (0,1)$, by (5.8), we have

$$C(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \notin L^1[0,\infty), \tag{5.16}$$

because of

$$\int_0^\infty |C(t)|dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} dt = \infty.$$

From the above and (5.13), we get $\int_0^\infty R(t)dt = 1$. By (5.11), we have

$$\lim_{t\to\infty} y(t) = e(0)\left(1 - \lim_{t\to\infty} \int_0^t R(t-s)ds\right) = e(0)\left(1 - \lim_{u\to\infty} \int_0^{u+s} R(u)du\right) = e(0).$$

Therefore $\sup_{t\geq 0}\{|y(t)|\} < \infty$. Hence, we set

$$\|y\| = \sup_{t\geq 0}\{|y(t)|\} =: m_y. \tag{5.17}$$

Theorem 5.3. *Let condition* (5.14) *be satisfied. If there exists* $\rho > 0$*, such that*

$$m_y + m_\rho < \rho, \tag{5.18}$$

then equation (5.10) *has at least one solution* $e : \mathbb{R}_+ \to \mathbb{R}^N$ *in* $\mathcal{C}_{l,\rho}$*, where* $\mathcal{C}_{l,\rho}$ *is defined by* (4.4)*. Moreover, every solution of equation* (5.10) *in* $\mathcal{C}_{l,\rho}$ *is asymptotically stable.*

Proof. Notice that $\mathcal{C}_{l,\rho}$ is a nonempty closed and convex subset of the Banach space $\mathcal{B}$.

For $e \in \mathcal{C}_{l,\rho}$, we define map T

$$(Te)(t) = y(t) - \int_0^t R(t-s)h(s,e(s))ds, \tag{5.19}$$

where y satisfies (5.11).

From (5.16) and (5.13), we see that conditions (i) and (iii) follow immediately, and by (5.12), the resolvent kernel R satisfies (ii). Hence, the function R satisfies assumptions of Lemma 4.3. On virtue of Lemma 4.3, by (5.14), we get

$$\lim_{t\to\infty} \int_0^t R(t-s)h(s,e(s))ds = l \quad \text{for} \quad e \in \mathcal{C}_{l,\rho}. \tag{5.20}$$

By (5.11) and (5.13), we obtain

$$\lim_{t\to\infty} y(t) = [0,0,\ldots,0]^T. \tag{5.21}$$

From (5.20) and (5.21), we get

$$\lim_{t\to\infty}(Te_i)(t) = l.$$

Moreover, using (5.19), by (5.13), (5.15), (5.17) and (5.18), we obtain

$$|(Te_i)(t)| \le |y_i(t)| + \int_0^t |R(t-s)||h(s,e_i(s))|ds \le m_y + m_\rho \le \rho$$

for $i = 1, 2, \ldots, N$. This implies that $\|(Te_i)(t)\| \le \rho$. Hence, $(Te)(t) \in \mathcal{C}_{l,\rho}$ for any $e \in \mathcal{C}_{l,\rho}$. It means that $T(\mathcal{C}_{l,\rho}) \subset \mathcal{C}_{l,\rho}$.

By (5.13), we have

$$R \in L^1[0,\infty). \tag{5.22}$$

Since the convolution of a continuous function and an L^1 function is continuous, the function y defined by (5.11) is continuous for $t \ge 0$. Function h is a continuous function, then we get that T is a continuous mapping.

Since $T(\mathcal{C}_{l,\rho}) \subset \mathcal{C}_{l,\rho}$ we have that $T(\mathcal{C}_{l,\rho})$ is uniformly bounded.

Next, we prove that $T(\mathcal{C}_{l,\rho})$ is equicontinuous on compact subsets of $\mathbb{R}_+$. To show it, it is sufficient to show that $T(\mathcal{C}_{l,\rho})$ is equicontinuous on interval $[0,\mu]$, for any $\mu > 0$. Since y is continuous for $t \ge 0$, hence y is uniformly continuous on $[0,\mu]$.

Condition (5.22) implies that $\int_0^t R(t-s)ds$ is continuous in t for all $t \ge 0$. So more, for $t_2 \ge t_1$ the resolvent kernel has the following property

$$\int_0^{t_1} \big[R(t_1-s) - R(t_2-s)\big]ds \to 0 \text{ as } |t_1-t_2| \to 0.$$

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $t_1, t_2 \in [0,\mu]$ with $|t_1-t_2| < \delta$ implies $|y(t_1) - y(t_2)| < \frac{\varepsilon}{3}$ and $\int_0^{t_1} |R(t_1-s) - R(t_2-s)|ds < \frac{\varepsilon}{3m_\rho}$. By (5.22), we have $\int_{t_1}^{t_2} |R(t_2-s)|ds < \frac{\varepsilon}{3m_\rho}$. Hence, for $e \in \mathcal{C}_{l,\rho}$ and $t_1, t_2 \in [0,\mu]$, $t_1 \le t_2$, we get

$$\begin{aligned}
\|(Te)(t_1) - (Te)(t_2)\| \le{}& \|y(t_1) - y(t_2)\| \\
&+ \Big\| \int_0^{t_1} \big(R(t_1-s) - R(t_2-s)\big)h(s,e(s))ds \Big\| \\
&+ \Big\| \int_{t_1}^{t_2} R(t_2-s)h(s,e(s))ds \Big\| \\
\le{}& \frac{\varepsilon}{3} + m_\rho \frac{\varepsilon}{3m_\rho} + m_\rho \frac{\varepsilon}{3m_\rho} = \varepsilon.
\end{aligned}$$

This shows that $T(\mathcal{C}_{l,\rho})$ is equicontinuous on compact subsets of $\mathbb{R}_+$. Therefore, by Lemma 4.4, the set $T(\mathcal{C}_{l,\rho})$ is relatively compact.

Hence, all assumptions of Lemma 4.1 are satisfied. It ensures that there exists at least one solution of equation (5.10) in $\mathcal{C}_{l,\rho}$.

Finally, we prove that all solutions of (5.10) in $\mathcal{C}_{l,\rho}$ are asymptotically stable. Let us denote

$$\phi(t) := \sup\{|h(t,e) - l|, e \in \mathcal{C}_{l,\rho}\}$$

for all $t \in \mathbb{R}_+$, where l is defined by (5.14). Obviously, $\lim\limits_{t\to\infty} \phi(t) = 0$. By virtue of Lemma 4.3 (iii), we have

$$\lim_{t\to\infty} \int_0^t R(t,s)\phi(s)ds = 0. \tag{5.23}$$

Let $\tilde{e}, \bar{e} \in \mathcal{C}_{l,\rho}$ be solutions of (5.10). It means $\tilde{e}(t) = (T\tilde{e})(t)$ and $\bar{e}(t) = (T\bar{e})(t)$. Thus for all $t \geq 0$, we have

$$\begin{aligned}
&\Big\|\tilde{e}(t) - \bar{e}(t)\Big\| \\
&= \Big\| \int_0^t R(t-s)\big(h(s,\tilde{e}(s)) - h(s,\bar{e}(s))\big)ds\Big\| \\
&= \Big\| \int_0^t R(t-s)\Big(\big(h(s,\tilde{e}(s)) - l\big) - \big(h(s,\bar{e}(s)) - l\big)\Big)ds\Big\| \\
&\leq \Big\| \int_0^t R(t-s)\big(h(s,\tilde{e}(s)) - l\big)ds\Big\| + \Big\| \int_0^t R(t-s)\big(h(s,\bar{e}(s)) - l\big)ds\Big\| \\
&= \sup_{t\geq 0}\left\{\Big\| \int_0^t R(t-s)\big(h(s,\tilde{e}(s)) - l\big)ds\Big\|\right\} + \sup_{t\geq 0}\left\{\Big\| \int_0^t R(t-s)\big(h(s,\bar{e}(s)) - l\big)ds\Big\|\right\} \\
&\leq \sup_{t\geq 0}\left\{\int_0^t \Big\|R(t-s)\Big\|\Big\|h(s,\tilde{e}(s)) - l\Big\|ds\right\} + \sup_{t\geq 0}\left\{\int_0^t \Big\|R(t-s)\Big\|\Big\|h(s,\bar{e}(s)) - l\Big\|ds\right\} \\
&= 2\int_0^t \Big\|R(t-s)\Big\|\phi(t)ds.
\end{aligned}$$

From the above and (5.23), we get $|\tilde{e}(t) - \bar{e}(t)| \to 0$ as $t \to \infty$. It means that every solution of (5.10) in $\mathcal{C}_{l,\rho}$ is asymptotically stable. This ended the proof. $\square$

Corollary 5.4. *If there exists $\rho > 0$, such that*

$$m_\rho < \rho, \tag{5.24}$$

then the leader-following consensus of system (1.1) *with the leader trajectory given by* (1.2) *and initial conditions in* $[-\rho,\rho]^N$ *is achieved locally.*

Proof. Since $\lim_{t\to\infty} y(t) = 0$, then there exists $M > 0$, such that

$$\sup_{t\geq M}\{|y(t)|\} \leq \frac{\rho - m_\rho}{2}. \tag{5.25}$$

Condition (5.25) and (5.24) implies that condition (5.18) is satisfied for all $t \geq M$. Since that, all assumptions of Theorem 5.3 are satisfied for $t \geq M$ and then the thesis holds. It means that $|\tilde{e}(t) - \bar{e}(t)| \to 0$ as $t \to \infty$. By (5.4), we get $|\tilde{x}(t) - \bar{x}(t)| \to 0$ as $t \to \infty$. Taking into account equivalency of equation (5.10) and system (1.1)–(1.2), it means that the leader-following consensus of system (1.1)–(1.2) is achieved. □

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