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Analysis of Vibrations of Plate Strip with Concentrated Masses Using Tolerance Averaging Technique

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Abstract

In this note vibrations of thin periodic plate strips with periodically distributed system of two concentrated masses are analysed. Moreover, it is assumed that every concentrated mass is connected to a string, which cause the effect of damping in vibrations. Governing equation for such structure is defined as a differential equation with highly oscillating, periodic and non-continuous coefficients. In order to solve the equation, tolerance averaging technique is applied. As a result, governing equations with constant coefficients are obtained. In an example, derived model is used to calculate lower and higher frequencies of the travelling wave related to the internal periodic structure.

Keywords: periodic plate strip, vibrations, tolerance averaging technique

1. Introduction

In this paper thin plate strips with span L are considered. It is assumed, that these plate strips have certain internal periodic microstructure related to a system of two concentrated masses, distributed periodically along the x_1-axis. Additionally, there are strings attached to concentrated masses, which make it possible to observe the effect of damping on plate's strips vibrations. Given system of concentrated masses and strings makes it possible to distinguish a small, repeatable element, called the *periodicity cell*. The span of every *cell* is equal to l, which is called *the microstructure parameter* and is small compared to the plate span L.

Vibrations of these structures are described by the governing equation with highly oscillating, periodic, non-continuous coefficients, which is not a good tool to analyse special problems. Hence, investigations of such structures can be performed using different models. The most popular one is based on the homogenization method, which uses e.g. effective plate stiffness (cf. [3]). However, equations of these models neglect the effect of the microstructure size on the plate strip behaviour. Thus, to take into account this effect the tolerance averaging technique is used to average the differential equation of this plate strip. As a result, governing equations with constant coefficients are obtained.

The main aim of this paper is to derive exact formulas for frequencies of the travelling wave for the plate strip using the tolerance averaging technique, which was proposed and explained by Woźniak and Wierzbicki [5], Woźniak, Michalak and Jędrysiak (eds.) [4]. Afterwards, some numerical examples of the plate strips behaviour are presented.

2. Modelling foundations

Let $Ox_1x_2x_3$ be an orthogonal Cartesian coordinate system and define t as the time coordinate. It is also assumed, that our considerations are treated as independent of x_2-coordinate. Let us introduce the following denotations: $x \equiv x_1$, $z \equiv x_3$, $x \in [0, L]$, $z \in [-h/2, h/2]$, where h is the constant thickness of the plate. Hence, it can be assumed that the plate strip is described in the interval $\Lambda = (0, L)$, with the basic cell $\Omega \equiv [-l/2, l/2]$ in the interval $\overline{\Lambda}$, where l is the length of the basic cell, called *a microstructure parameter*. For further transformations, it is crucial, that the *microstructure parameter l* satisfies conditions: $l << L$ and $h << l$. Deflections of the plate strip are denoted as $w(x,t)$ ($x \in \overline{\Lambda}$, $t \in (t_0, t_1)$).

Let us assume, that the material properties of the plate strip $E(x), \rho(x), x \in \Lambda,$ are periodic functions in x. Hence, functions describing the mass density per unit area of midplane μ and the bending stiffness B can be stated as follows:

$$\mu(x) \equiv h\rho(x), \qquad B \equiv \frac{h^3}{12(1-\nu^2)} E(x). \tag{1}$$

Moreover, the plate strip is connected to a system of periodically distributed strings, which are described by the damping parameter $c(x)$. In order to apply the tolerance averaging technique, the parameter $c(x)$ must satisfy all conditions of a periodic function.

It is assumed, that the plate strip fulfils prerequisites of the Kirchhoff-type thin plate theory. Denoting the derivative of x by ∂, and the time derivative by dots, the partial differential equation of the fourth order for deflection $w(x,t)$ takes the following form:

$$\partial\partial[B(x)\partial\partial w(x,t)] + c(x)\dot{w}(x,t) + \mu(x)\ddot{w}(x,t) = 0, \tag{2}$$

with coefficients being highly oscillating, non-continuous, periodic functions in x. Equation (2) describes free vibrations of the plate strip with the effect of damping on its vibrations and stands a starting point for further investigations in the framework of the tolerance averaging technique.

In the tolerance modelling procedure some introductory concepts, like: an *averaging operator*, a *slowly varying function*, a *tolerance-periodic function* and a *highly oscillating function*, are used. These concepts were presented in a various literature, for example: by Woźniak and Wierzbicki [5].

3. Modelling assumptions

There are two main assumptions in the tolerance averaging technique. The first of them is *the micro-macro decomposition* of the plate strip deflection w, which can be formulated as follows:

$$w(x,t) = W(x,t) + g^A(x)Q^A(x,t), \quad A = 1, \ldots, N, \quad x \in \Lambda, \tag{3}$$

where $W(\cdot,t)$ is *the macrodeflection* of the plate strip, $Q^A(\cdot,t)$ are *the fluctuation amplitudes* and $g^A(\cdot)$ are the known *fluctuation shape functions*. Functions $W(\cdot,t)$ and $Q^A(\cdot,t)$ are the new basic kinematic unknowns, which are for every t slowly varying functions.

The tolerance averaging approximation is the second modelling assumption. Assuming that the terms $O(\delta)$ are negligibly small, the following relations can be proved in the course of modelling:

$$
\begin{aligned}
&<\Phi>(x) = <\overline{\Phi}>(x) + O(\delta), \qquad <\Phi F>(x) = <\Phi>(x)F(x) + O(\delta),\\
&<\Phi\partial_\alpha(g^A F)>(x) = <\Phi\partial_\alpha g^A>(x)F(x) + O(\delta),\\
&x \in \Lambda;\ \alpha = 1,2;\ A = 1,\ldots,N;\ 0 < \delta << 1;
\end{aligned}
\tag{4}
$$

where δ is a tolerance parameter, Φ is tolerance periodic function, $\overline{\Phi}$ is a periodic approximation of Φ, F is a slowly varying function and $g^A(\cdot)$ is a fluctuation shape function.

4. Modelling procedure

Basing on the introductory concepts presented in [4] nad [5] and the calculations developed by Jędrysiak in [1] and Jędrysiak and Michalak [2], the modelling procedure can be outlined as follows.

As mentioned before, the starting point is the Kirchhoff-type thin plate free vibrations differential equation (2). In order to obtain equations with constant coefficients, some transformations must be performed. These transformations are: substituting the micro-macro decomposition (3) to equation (2), applying the averaging operator and using the tolerance averaging approximations (4). As a result, we arrive at a system of equations for $W(\cdot,t)$ and $Q^A(\cdot,t)$ in the form:

$$
\begin{aligned}
&<B>\partial\partial\partial\partial W + <B\partial\partial g^A>\partial\partial Q^A + <c>\dot{W} + \underline{<cg^A>}\dot{Q}^A + <\mu>\ddot{W} = 0,\\
&<B\partial\partial g^A>\partial\partial W + <B\partial\partial g^A\partial\partial g^B>Q^A +\\
&+\underline{<cg^A>}\dot{W} + \underline{<cg^A g^B>}\dot{Q}^A + \underline{<\mu g^A g^B>}\ddot{Q}^A = 0.
\end{aligned}
\tag{5}
$$

In the system of equations (5) the first equation describes vibrations of the plate strip in the macro scale, while the second stands for the system of N equations, which refers to microvibrations. It can be observed, that only the underlined terms are dependent on the microstructure parameter l. Keeping in mind the fact, that coefficients in the system of equations above are constant, it is possible to obtain a convenient solution describing free vibrations of the plate strip, including both the effect of the microstructure and damping.

5. Frequencies of plate strip free vibrations with the influence of damping

In this section a homogenous weightless and unbounded plate strip along the *x*-axis is considered. Periodicity of the structure is related to a system of two periodically distributed concentrated masses M_1 and M_2 and strings attached to those masses. Strings are described by their damping parameters, c_1 and c_2 respectively, cf. Figure 1.

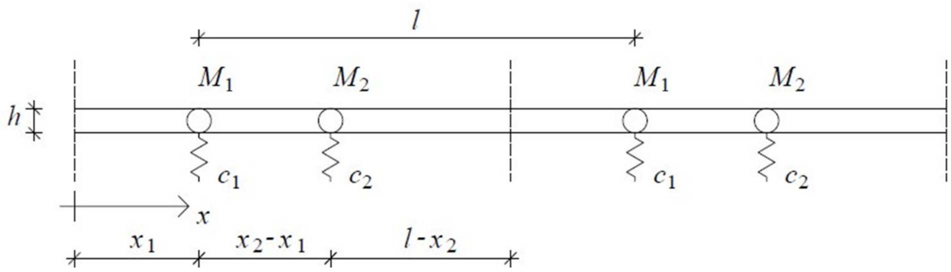

Figure 1. The plate strip with a system of two periodically distributed concentrated masses and strings

In the further investigations Young's modulus E, Poisson's ratio ν and thickness h of the plate are assumed to be constant. Moreover, the plate mass is negligibly small when compared to concentrated masses M_1 and M_2.

According to a structure of the periodicity cell of plate strips and bearing in mind the normalizing condition $<\mu g>=0$, only one fluctuation shape functions g^A, A=1, is assumed. Denoting as follows:

$$
\begin{aligned}
&g \equiv g^1 \\
&\tilde{D} \equiv < B >, && \hat{D} \equiv < B\partial\partial g \partial\partial g >, \\
&\tilde{m} \equiv < \mu >, && \hat{m} \equiv l^{-4} < \mu g g >, \\
&\tilde{c} \equiv < c >, && \bar{c} \equiv l^{-2} < c g >, && \hat{c} \equiv l^{-4} < c g g >,
\end{aligned}
\tag{6}
$$

equations (5) take the form:

$$
\begin{aligned}
&\tilde{D}\partial\partial\partial\partial W + \tilde{c}\dot{W} + \tilde{m}\ddot{W} + l^2\bar{c}\dot{Q} = 0, \\
&l^2\bar{c}\dot{W} + \hat{D}Q + l^4\hat{c}\dot{Q} + l^4\hat{m}\ddot{Q} = 0.
\end{aligned}
\tag{7}
$$

Equations (7) stand for a system of equations for the macrodeflection W and the fluctuation amplitude Q. The first equation describes fundamental vibrations of the plate strip (e.g. lower frequencies of the travelling wave), while the second refers to microstructural vibrations (related to higher frequencies of the travelling wave). Solutions to those equations can be assumed in the form:

$$W(x,t) = A_W \exp[i(kx - \omega t)],$$
$$Q(x,t) = A_Q \exp[i(kx - \omega t)], \tag{8}$$

where A_W, A_Q are amplitudes, k is a wave number, t is a time coordinate and ω is a frequency. After some transformations formulas for the lower (ω_-) and higher ($\omega_{+:1}$) frequencies can be obtained as roots of the characteristic equation in the form:

$$l^4 \tilde{m}\hat{m}\omega^4 + l^4(\tilde{m}\hat{c} + \tilde{c}\hat{m})i\,\omega^3 - (k^4 l^4 \tilde{D}\hat{m} + l^4 \tilde{c}\hat{c} - l^4 \overline{c}\overline{c} + \tilde{m}\hat{D})\omega^2 +$$
$$-(k^4 l^4 \tilde{D}\hat{c} + \tilde{c}\hat{D})i\,\omega + k^4 \tilde{D}\hat{D} = 0. \tag{9}$$

By solving the equation above, it is possible to obtain four different roots of the characteristic equation: a pair of numbers, which refers to lower frequencies of the plate strip's free vibrations (macrovibrations) and a pair of complex conjugate numbers, which describes higher frequencies of the structure (vibrations related to microstructure).

6. Eigenvalue problem

Coefficients in equations (7) are strongly dependent on the type of assumed fluctuation shape function g^A. In the following calculations, the exact fluctuation shape function is derived as a solution to an eigenvalue problem on the periodicity cell. In the case under consideration, eigenvalue problem takes the following form:

$$B\,\partial\partial\partial\partial g(x) - \mu(x)\lambda^2 g(x) = 0, \tag{10}$$

where B is the stiffness defined by $(1)_2$ and $g(x)$ is a periodic function related to eigenvalue $\lambda \equiv \alpha l$ (α is the wave number). Assuming that the plate mass is negligibly small when compared to the concentrated masses and applying proper periodic boundary conditions and the normalizing condition $\langle\mu g\rangle=0$, it is possible to obtain only one eigenfunction $g(x)$, which describe a shape of free vibrations of the cell.

In order to obtain the exact fluctuation shape function $g(x)$, methods known from the structural mechanics can be used. For each point, in which the concentrated mass is posed to the plate, equilibrium equations for transversal forces and moments can be written. By applying certain boundary conditions in the form:

$$g(0) = g(l), \qquad \partial g(0) = \partial g(l),$$
$$\partial\partial g(0) = \partial\partial g(l), \quad \partial\partial\partial g(0) = \partial\partial\partial g(l), \tag{11}$$

we arrive at the characteristic equation in the form of determinant equal to zero:

$$\det L_{pr} = 0, \qquad p, r = 1..4. \tag{12}$$

As a result, the second order equation for ω is obtained. Hence, it is possible to derive one eigenvalue ω^2. Basing on the obtained eigenvalue, the exact values of deflections along the periodicity cell can be calculated similarly to deflections of beams.

7. Results of calculations

Using the tolerance averaging technique is a convenient way of investigating plate strips behaviour in the micro-scale. In this section, several numerical examples are presented in order to verify obtained formulas.

Let us assume, that the concentrated mass M_2 is a mass of reference, to which the mass M_1 is compared. Similarly, let the damping coefficient c_2 be a reference value for the coefficient c_1. As a result, the following denotations can be made:

$$M_1 \equiv \zeta M_2, \qquad\qquad c_1 \equiv \xi c_2, \tag{13}$$

where ζ is a mass ratio and ξ is a damping coefficient ratio.

Calculation examples has been performed for several different calculation cases. In every case, it has been assumed that the plate strip thickness h is equal to $0.1l$. Additionally, in order to obtain the exact dimensionless parameters of free vibrations frequencies, the ratio between reference mass M_2 and stiffness coefficient B are defined as follows:

$$B \equiv \frac{h^3}{12} M_2 \tag{14}$$

The calculation cases differ from each other with mass distributions (coordinates x_1 and x_2), mass proportion and values (M_1 and M_2), and dispersion coefficients (c_1 and c_2). For details, cf. Table 1.

Results are shown in Table 1 and Table 2 and in the form of charts describing the microvibrations amplitude versus time coordinate, cf. Figure 2. Additionally, all the calculations are performed for different values of dimensionless wave number $q \equiv kl \in [-\pi;\pi]$, cf. Table 2.

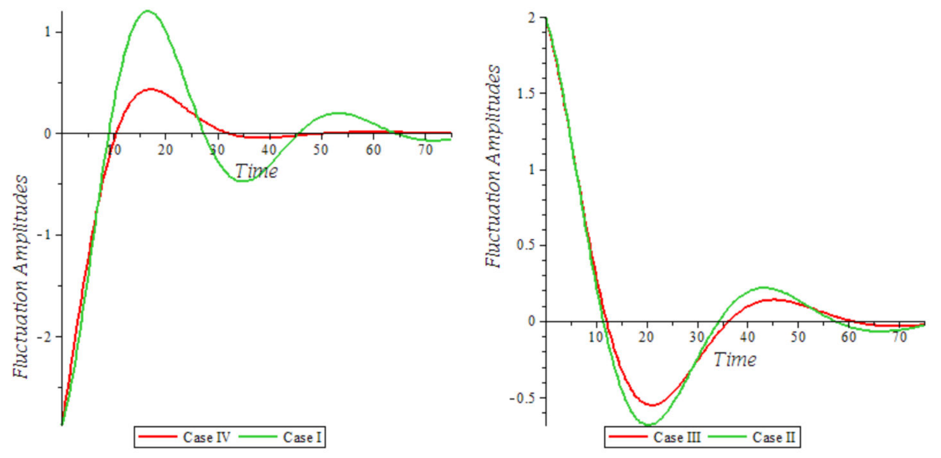

Figure 2. Damping of fluctuation amplitudes in chosen cases

Table 1. Lower and higher frequencies of travelling wave in calculation cases

and for dimensionless wave number q=1.0

Case	Mass coordinates		Mass proportions		Dispersion coefficients		Lower frequencies	Higher frequencies
	x_1	x_2	M_1	M_2	c_1	c_2		
I	0,25l	0,75l	1	1	10^{-1}	10^{-1}	-0,00042 -0,09958	-0,05±0,1718i
II	0,25l	0,75l	3	1	10^{-1}	10^{-1}	-0,00021 -0,09979	-0,05±0,1372i
III	0,25l	0,75l	3	1	$2\cdot10^{-1}$	10^{-1}	-0,00012 -0,18557	-0,057±0,1297i
IV	0,25l	0,75l	1	1	$2\cdot10^{-1}$	$2\cdot10^{-1}$	-0,00021 -0,19979	-0,10±0,1483i
V	0,4l	0,7l	1	1	10^{-1}	10^{-1}	-0,00042 -0,09958	-0,05±0,2070i
VI	0,4l	0,7l	3	1	10^{-1}	10^{-1}	-0,00021 -0,09979	-0,05±0,1665i
VII	0,4l	0,7l	3	1	$2\cdot10^{-1}$	10^{-1}	-0,00012 -0,18328	-0,058±0,1595i
VIII	0,4l	0,7l	1	1	$2\cdot10^{-1}$	$2\cdot10^{-1}$	-0,00021 -0,19979	-0,10±0,1880i

Table 2. Comparison of lower and higher frequencies of travelling wave in different calculation cases depending on different dimensionless wave number q

Case	Lower frequencies			Higher frequencies		
	q=0.1	q=1.0	q=2.0	q=0.1	q=1.0	q=2.0
I	$-4,2\cdot10^{-8}$ -0,10000	-0,00042 -0,09958	-0,00718 -0,09282	-0,05±0,1718i	-0,05±0,1718i	-0,05±0,1718i
II	$-2,1\cdot10^{-8}$ -0,10000	-0,00021 -0,09979	-0,00345 -0,09655	-0,05±0,1372i	-0,05±0,1372i	-0,05±0,1372i
III	$-1,2\cdot10^{-8}$ -0,18568	-0,00012 -0,18557	-0,00192 -0,18391	-0,057±0,1297i	-0,057±0,1297i	-0,057±0,1297i
IV	$-2,1\cdot10^{-8}$ -0,20000	-0,00021 -0,19979	-0,00339 -0,19661	-0,10±0,1483i	-0,10±0,1483i	-0,10±0,1483i
V	$-4,2\cdot10^{-8}$ -0,10000	-0,00042 -0,09958	-0,00718 -0,09282	-0,05±0,2070i	-0,05±0,2070i	-0,05±0,2070i
VI	$-2,1\cdot10^{-8}$ -0,10000	-0,00021 -0,09979	-0,00345 -0,09655	-0,05±0,1665i	-0,05±0,1665i	-0,05±0,1665i
VII	$-1,2\cdot10^{-8}$ -0,18340	-0,00012 -0,18328	-0,00192 -0,18157	-0,058±0,1595i	-0,058±0,1595i	-0,058±0,1595i
VIII	$-2,1\cdot10^{-8}$ -0,20000	-0,00021 -0,19979	-0,00339 -0,19661	-0,10±0,1880i	-0,10±0,1880i	-0,10±0,1880i

8. Final remarks

In this paper *the tolerance averaging technique* has been used to obtain the governing equations with constant coefficients for thin plate strips with internal periodic structure. By analyzing results shown in Tables 1 and 2 and Figure 2, it can be observed that:

- the tolerance model is a convenient tool for the analysis of micro- and macro vibrations in case, in which the effect of damping has to be taken into account;
- the lower frequencies of the travelling wave are dependent on the dimensionless wave number *q*;
- as long as the mass proportions ζ and the dimensionless wave number *q* are constant and $\xi = 1$, the lower frequencies do not depend on the mass distribution;
- the obtained values of the higher frequencies are complex numbers, which real part describes the damping of vibrations while the imaginary part describes the period of vibrations;
- the mass proportions and coordinates of the concentrated masses have an influence on the imaginary part of the higher frequency, but it seems that they have no effect on its real part;
- the damping coefficients affect both real and imaginary part of the higher frequencies;
- the higher frequencies are not dependent on the dimensionless wave number *q*;
- higher damping coefficients make vibration amplitudes decrease faster.

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