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GLOBAL SOLUTIONS FOR A NONLINEAR KIRCHHOFF TYPE EQUATION WITH VISCOSITY

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Abstract. In this paper we consider the existence and asymptotic behavior of solutions of the following nonlinear Kirchhoff type problem

$$u_{tt} - M\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle u - \delta\triangle u_t = \mu|u|^{\rho-2}u \quad \text{in } \Omega\times]0,\infty[,$$

where

$$M(s) = \begin{cases} a - bs & \text{for } s \in [0, \frac{a}{b}[, \\ 0, & \text{for } s \in [\frac{a}{b}, +\infty[. \end{cases}$$

If the initial energy is appropriately small, we derive the global existence theorem and its exponential decay.

Keywords: global solutions, nonlinear Kirchhoff type problem, exponential decay.

Mathematics Subject Classification: 35L80, 35L70, 35B33, 35J75.

1. INTRODUCTION

In this work we consider the following nonlocal problem

$$\begin{gathered} u_{tt} - M\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle u - \delta\triangle u_t = \mu|u|^{\rho-2}u \quad \text{in } \Omega\times]0,\infty[, \\ u = 0 \quad \text{on } \Gamma\times]0,\infty[, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, $\delta, \mu > 0$ and

$$M(s) = \begin{cases} a - bs & \text{for } s \in [0, \frac{a}{b}[, \\ 0 & \text{for } s \in [\frac{a}{b}, +\infty[, \end{cases} \tag{1.2}$$

$a, b > 0, \rho > 2$.

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When $M(s) = a + bs$, $s \geq 0$, $a > 0$, $b \geq 0$, $\delta = 0 = \mu$ and Ω is a finite open interval, equation (1.1) was introduced by [16] in the study of nonlinear vibrations of the elastic string and is called the wave equation of Kirchhoff type after his name. See also [17]. We should refer to [10,22] for a deduction of the model for a non-homogeneous material. Moreover, it is said a degenerate equation when $M(s)$ has zeros and a nondegenerate one when $M(s) \geq m_0 > 0$ for all $s \geq 0$. The global existence for real analytic initial data was proved in [5] and [33], while the global existence of small C^∞ and Sobolev solutions was established in [9] and [13]. The question of global solutions for arbitrary data from Sobolev spaces is still open. When equations have some dissipative terms u_t, $(-\triangle)u_t$, $\triangle^2 u_t$, etc., we can prove the existence of global solutions, and moreover some decay properties. There are many contributions on various mathematical subjects of the mixed problem (1.1) with $M(s) > 0, \delta > 0$. The authors, in [21, 23, 28, 38] studied existence results and decay rate of the solutions. Considering a polynomial nonlinearity, global existence and stability results was proved in [43]. In [19], was analyzed local existence and blow-up of the solution for nonlinear wave equations of Higher-order Kirchhoff type. For a power logarithmic source, in [8,42] was shown global existence and asymptotic behavior of solutions. For the degenerate case $M(s) \geq 0$, it was investigated global existence and asymptotic behavior in [25, 26, 30, 40]. Also in this case, but with a more general nonlinearity, in [2] was considered the existence and stability of the global solution. A large number of results on the solutions to problem (1.1) have been established by many authors through various approaches and assumptive conditions (see [1, 3, 4, 7, 11, 24, 31, 32, 41] and references therein). Some papers with various kinds of Kirchhoff operators are shown in [34].

In [44] investigated the existence and multiplicity of nontrivial solution for the new nonlocal problem

$$\begin{gathered} -\left(a - b\int_\Omega |\nabla u|^2\,dx\right)\triangle u = |u|^{\rho-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma. \end{gathered} \tag{1.3}$$

Also see [15, 35, 39] for generalizations of (1.3).

It is opportune to observe that when $a = 0 = \delta$ the equation (1.1) becomes the quasilinear non well posed problem

$$\begin{gathered} u_{tt} + \left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u = \mu|u|^{\rho-2}u \quad \text{in } \Omega\times]0,\infty[, \\ u = 0 \quad \text{on } \Gamma\times]0,\infty[, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in \Omega, \end{gathered}$$

which can be seen as a boundary value problem for the potential equation as in [20]. The example of Hadamard [14] is the case of the potential equation in $\mathbb{R}^2$. This question has some interest in the study of the optimal control for singular distributed system and it seems to be essentially open.

Motivated for their works, it is interesting to investigate the global solvability of (1.1) with the nonlocal operator given in (1.3). More precisely, under appropriate assumptions imposed on the initial data and the source term, we shall establish global existence of solutions by using of Tartar method [37] combined with suitable a priori estimates including $|\triangle u(t)|$ in addition to the usual energy estimate. To our best knowledge, it is the first attempt to study the properties of the solutions for such kind of equations.

The outline of this manuscript is the following. In Section 2, we prepare some lemmas needed for our arguments and state the local existence theorem. In Section 3, we prove the global existence of solutions and its exponential decay.

2. PRELIMINARIES

Throughout this paper the functions are all real valued and the notations are as usual, in particular we shall denote the usual L^p-norm by $\|\cdot\|_p$, $(p \geq 1)$ and the inner product $(u, v) = \int\limits_\Omega uv\,dx$. Moreover, C, C_i $(i = 1, 2, \ldots)$ denote various positive constants and they may be different at each appearance.

Lemma 2.1 ([12, Lemmas 7.12 and 7.16], Sobolev–Poincaré Inequality)**.** *Let ρ be a number with $2 \leq \rho < \infty$ $(n = 1, 2)$ or $2 \leq \rho \leq \frac{2N}{N-2}$ $(n \geq 3)$, then there is a positive constant $c_* = c(\Omega, \rho)$ such that*

$$\|u\|_\rho \leq c_* \|\nabla u\|_2 \quad \textit{for } u \in H_0^1(\Omega). \tag{2.1}$$

Thus, the norm $\|\nabla u\|_2$ is equivalent to the usual norm in $H_0^1(\Omega)$.

Lemma 2.2 ([6, Theorem 1], Gagliardo–Nirenberg Inequality)**.** *Let $1 \leq r < q \leq \infty$ and $q \leq p$. Then the inequality*

$$\|u\|_q \leq C\|u\|_{W^{m,q}}^\theta \|u\|_r^{1-\theta} \quad \textit{for } u \in W^{m,q}(\Omega) \cap L^r(\Omega)$$

holds with some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} + \frac{m}{N} - \frac{1}{q}\right)^{-1},$$

provided that $0 < \theta \leq 1$ $(0 < \theta < 1$ if $p = \infty)$.

Lemma 2.3 ([27, Lemma 3])**.** *Let $\Phi(t)$ be a bounded positive function on $[0, +\infty[$ satisfying, for some constant $k_0 > 0$,*

$$\Phi(t) \leq k_0(\Phi(t) - \Phi(t+1)) \quad \textit{on } [0, +\infty[.$$

Then, we have

$$\Phi(t) \leq \Phi(0)e^{-k_1 t} \quad \textit{on } [0, +\infty[,$$

where $k_1 = \log\left(\frac{k_0}{k_0 - 1}\right)$.

Definition 2.4. A weak solution of (1.1) is a function $u : [0,T[\to H_0^1(\Omega)\cap H^2(\Omega)$ satisfying

$$\frac{d}{dt}(u(t),w) + M\left(\int_\Omega |\nabla u|^2\,dx\right)(\nabla u,\nabla w) + \delta(\nabla u_t, w) = \mu(|u|^{\rho-2}u, w)$$

for all $w \in H_0^1(\Omega)\cap H^2(\Omega)$ (see [36]).

For the sake of completeness, we recall the following local existence result, which may be proved by the Banach contraction mapping principle (see [29]).

Theorem 2.5 (Local existence)**.** *Let $M(s)$ be a nonnegative locally Lipschitz function for $s \geq 0$. We assume that $f(u)$ is a nonlinear function such that $f(0) = 0$ and*

$$|f(u) - f(v)| \leq k_1(|u|^\alpha + |v|^\alpha)|u - v|$$

with some constant k_1, and

$$0 \leq \alpha \leq 4/(N-4) \text{ if } N \geq 5 \quad (0 \leq \alpha < +\infty \text{ if } N \leq 4).$$

If the initial data $\{u_0, u_1\}$ belong to $(H_0^1(\Omega)\cap H^2(\Omega)) \times L^2(\Omega)$ then there exists $T = T(\|\triangle u_0\|_2, \|\nabla u_1\|_2) > 0$ such the problem (1.1) admits a unique local weak solution u satisfying

$$u \in C^0\left([0,T); H_0^1(\Omega)\cap H^2(\Omega)\right)$$

and

$$u_t \in C^0\left([0,T); L^2(\Omega)\right) \cap L^2\left([0,T); H_0^1(\Omega)\right).$$

Moreover, at least one of the following statements is valid:

(i) $T = +\infty$,
(ii) $\|u_t(t)\|_2^2 + \|\triangle u(t)\|_2^2 \to +\infty$ *as* $t \to T^-$.

Now, we set

$$B_\rho = \sup_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\|u\|\rho}{\|\nabla u\|_2}, \quad \gamma_1 = \frac{b}{4a}, \quad \gamma_2 = \frac{B_\rho^\rho}{\rho a}.$$

Following Tartar's ideas, we define the function

$$h(\lambda) = \frac{1}{4}\lambda^2 - \gamma_1\lambda^4 - \frac{3}{2}\gamma_2\lambda^\rho.$$

Then

$$h'(\lambda) = \lambda\left(\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}\rho\gamma_2\lambda^{\rho-2}\right).$$

So, choosing $\lambda \in \mathbb{R}$, such that

$$0 \leq \lambda^2 \leq \frac{1}{16\gamma_1} \quad \text{and} \quad 0 \leq \lambda^{\rho-2} \leq \frac{1}{6\rho\gamma_2},$$

we get that these λ's satisfy the inequality

$$\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}\rho\gamma_2\lambda^{\rho-2} \geq 0$$

and $h'(\lambda) \geq 0$ for $0 \leq \lambda \leq \lambda_1$, where

$$\lambda_1 = \min\left\{(16\gamma_1)^{-1/2}, (6\rho\gamma_2)^{-1/(\rho-2)}\right\}.$$

Thus $h(0) = 0$ and $h(\lambda) \geq 0$ for all $\lambda \in [0, \lambda_1]$.

From this, we get

$$h_0(\lambda) = \frac{1}{2}\lambda^2 - \gamma_1\lambda^4 - \gamma_2\lambda^\rho \geq \frac{1}{4}\lambda^2 + \frac{1}{2}\gamma_2\lambda^\rho, \quad \forall\lambda \in [0, \lambda_1]. \tag{2.2}$$

The energy associated with the problem (1.1) is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) \quad \text{for } u \in H_0^1(\Omega),$$

where

$$J(u(t)) = \frac{a}{2}\|\nabla u(t)\|_2^2 - \frac{b}{4}\|\nabla u(t)\|_2^4 - \frac{1}{\rho}\|u(t)\|_\rho^\rho.$$

Multiplying equation (1.1) by $u_t(t)$ and integrating it over Ω, we obtain

$$\frac{d}{dt}E(t) + \delta\|\nabla u_t(t)\|_2^2 = 0. \tag{2.3}$$

Therefore, $E(t)$ is a nonincreasing function on t, and

$$E(t) + \delta\int_0^t \|\nabla u_t(s)\|_2^2\, ds = E(0). \tag{2.4}$$

From now on, for simplicity, we will take $\delta = 1 = \mu$.

3. GLOBAL EXISTENCE AND EXPONENTIAL DECAY

In this section we state the main results of this paper. Firstly, we give the following two propositions.

Proposition 3.1. *If the local solution* $u(t)$ *of* (1.1) *satisfies* $0 < \|\nabla u(t)\|_2 < \lambda_1$ *on* $[0, T_0]$*, then*

$$J(u(t)) \geq a\left(\frac{1}{4}\|\nabla u(t)\|_2^2 + \frac{\gamma_2}{2}\|\nabla u(t)\|_2^\rho\right) \tag{3.1}$$

and

$$\|\nabla u(t)\|_2 \leq \left[\frac{4}{a}E(t)\right]^{1/2}. \tag{3.2}$$

Proof. It is obvious from (2.2). In fact,

$$J(u(t)) \geq ah_0(\|\nabla u(t)\|_2) \geq a\left(\frac{1}{4}\|\nabla u(t)\|_2^2 + \frac{\gamma_2}{2}\|\nabla u(t)\|_2^\rho\right).$$

So (3.1) holds. Also

$$E(u(t)) \geq J(u(t)) \geq \frac{a}{4}\|\nabla u(t)\|_2^2,$$

which implies (3.2). □

Proposition 3.2. *Let u be a local solution of* (1.1). *Under the assumption of Proposition 3.1, the energy $E(t)$ satisfies*

$$E(t) \leq C_E E(0) e^{-kt}, \tag{3.3}$$

where $k = \ln\left(\frac{k_0}{k_0-1}\right)$, k_0 is defined in (3.8) *and $C_E = \max\{1, \sigma_0\}$, with σ_0 given in* (3.10).

Proof. First, we suppose that $T_0 > 1$. Integrating (2.3) from t to $t+1$, $0 < t < T_0 - 1$, we find

$$\int_t^{t+1} \|\nabla u_t(s)\|^2\, ds = E(t) - E(t+1) \equiv F^2(t).$$

Using the mean value theorem for integrals, there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2 \leq 2F(t), \quad i = 1, 2. \tag{3.4}$$

Next, multiplying (1.1) by u and integrating over Ω, we obtain

$$\begin{aligned} &a\|\nabla u(t)\|_2^2 - b\|\nabla u(t)\|_2^4 - \|u(t)\|_\rho^\rho \\ &= \|u_t(t)\|_2^2 - (\nabla u_t(t), \nabla u(t)) - \frac{d}{dt}(u_t(t), u(t)). \end{aligned} \tag{3.5}$$

On the other hand, it follows from the Sobolev–Poincaré inequality and (3.2) that

$$\begin{aligned} \|u(t)\|_\rho^\rho &\leq B_\rho^\rho \|\nabla u(t)\|_2^\rho \leq B_\rho^\rho \|\nabla u(t)\|_2^{\rho-2} \|\nabla u(t)\|_2^2 \\ &\leq B_\rho^\rho \left[\frac{4}{a}E(0)\right]^{(\rho-2)/2} \|\nabla u(t)\|_2^2 \end{aligned}$$

and

$$b\|\nabla u(t)\|^4 \leq b\left[\frac{4}{a}E(0)\right]\|\nabla u(t)\|_2^2.$$

Thus, we get

$$b\|\nabla u(t)\|_2^4 + \|u(t)\|_\rho^\rho \le \frac{1}{a}\left[B_\rho^\rho\left(\frac{4}{a}E(0)\right)^{(\rho-2)/2} + \frac{4b}{a}E(0)\right](a\|\nabla u(t)\|_2^2)$$
$$\equiv (1-\eta_0)(a\|\nabla u(t)\|_2^2), \quad 0<\eta_0<1. \tag{3.6}$$

See Remark 3.4 for the justification of condition (3.6). Then

$$\eta_0 a\|\nabla u(t)\|_2^2 \le a\|\nabla u(t)\|_2^2 - b\|\nabla u(t)\|_2^4 - \|u(t)\|_\rho^\rho \equiv I(t). \tag{3.7}$$

From (2.4) and (3.5), integrating the resultant inequality over $[t_1,t_2]$ we have

$$\eta_0 a\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\,ds$$
$$\le \int_{t_1}^{t_2} I(s)\,ds \le c_*^2\int_{t_1}^{t_2}\|\nabla u_t(s)\|_2^2\,ds + \int_{t_1}^{t_2}|(\nabla u_t(s),\nabla u(s))|\,ds - (u_t(t),u(t))|_{t_1}^{t_2}$$
$$\le c_*^2F^2(t) + \int_{t_1}^{t_2}\|\nabla u_t(s)\|_2\|\nabla u(s)\|_2\,ds + c_*^2\sum_{i=1}^{2}\|\nabla u_t(t_i)\|_2\|\nabla u(t_i)\|_2$$
$$\le c_*^2F^2(t) + \left[\left(\int_t^{t+1}\|\nabla u_t(s)\|_2^2\,ds\right)^{1/2} + c_*^2\sum_{i=1}^{2}\|\nabla u_t(t_i)\|\right]\sup_{s\in[t,t+1]}\|\nabla u(s)\|_2$$
$$\le c_*^2F^2(t) + (4c_*^2+1)F(t)\left(\frac{4}{a}E(t)\right)^{1/2},$$

where we have used (3.2) and (3.4) at the last inequality.

On the other hand, integrating (2.3) over $[t,t_2]$, noting that $E(t_2)\le 2\int_{t_1}^{t_2}E(s)\,ds$ due to $t_2-t_1\ge\frac{1}{2}$, using (3.8) and the Young inequality, we have

$$E(t) = E(t_2) + \int_t^{t_2}\|\nabla u_t(s)\|_2^2\,ds$$
$$\le 2\int_{t_1}^{t_2}E(s)\,ds + \int_t^{t+1}\|\nabla u_t(s)\|_2^2\,ds$$
$$\le (c_*^2+1)\int_t^{t+1}\|\nabla u_t(s)\|^2\,ds + a\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\,ds$$
$$\le \left(c_*^2+\frac{c_*^2}{\eta_0}+1\right)F^2(t) + \frac{1}{2}\left(\frac{2(4c_*^2+1)}{\eta_0\sqrt{a}}\right)^2F^2(t) + \frac{1}{2}E(t).$$

Thus

$$E(t) \leq k_0(E(t) - E(t+1)),$$

where

$$k_0 = 2\left[\left((c_*^2 + \frac{c_*^2}{\eta_0} + 1\right) + \frac{1}{2}\left(\frac{2(4c_*^2+1)}{\eta_0\sqrt{a}}\right)^2\right] + 1. \tag{3.8}$$

Then, noting (2.4) and applying Lemma 2.3 we have

$$E(t) \leq E(0)e^{-kt} \quad \text{for } 0 \leq t \leq T_0. \tag{3.9}$$

In the case when $0 \leq t \leq 1$, since $E(t)$ is bounded, we have

$$E(t) \leq \sigma_0 E(0)e^{-kt} \quad \text{for some } \sigma_0 > 0. \tag{3.10}$$

So, from (3.9) and (3.10) we obtain (3.3). □

Theorem 3.3. *Let $N = 3$ and $4 < \rho < 6$. Assume further that $\{u_0, u_1\}$ belong to $(H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$ with*

$$\|\nabla u_0\| < \min\left\{\left(\frac{a}{b}\right)^{1/2}, \lambda_1\right\}, \quad [4E(0)]^{1/2} < \lambda_1, \tag{3.11}$$

then problem (1.1) *admits a unique global solution satisfying*

$$u \in C([0,+\infty[; H_0^1(\Omega) \cap H^2(\Omega)),$$
$$u_t \in C([0,+\infty[; L^2(\Omega)) \cap L^2((0,+\infty[; H_0^1(\Omega)),$$

and the energy satisfies

$$E(t) \leq Ce^{-kt} \quad \textit{for } t \geq 0, \tag{3.12}$$

with some constant $k > 0$.

Remark 3.4. It is easy to see that, from (2.2), the condition $(4E(0))^{1/2} < \lambda_1$ implies

$$\beta = B_\rho^\rho \left(\frac{4}{a}E(0)\right)^{(\rho-2)/2} + \frac{4b}{a}E(0) < 1,$$

which will be used in the proof of Theorem 3.3.

Proof. Let $u(t)$ be a unique solution of the problem (1.1) in the sense of Theorem 2.5 on $[0, T_0[$, with T_0 the maximal time where the solution exists. First, we note that under the assumption (3.11) we get, from (3.7) that $I(t) > 0$ for $t \in [0, T_0[$.

Multiplying (1.1) by $-2\triangle u$ and integrating over Ω we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|\triangle u\|_2^2 - 2\int_\Omega u_t \triangle u\, dx\right) + 2a\|\triangle u\|_2^2 \\ &= -2\int_\Omega |u|^{\rho-2}u\triangle u\, dx - 2\|\nabla u_t\|_2^2 + 2b\|\nabla u\|_2^2\|\triangle u\|_2^2. \end{aligned} \tag{3.13}$$

Multiplying (3.13) by $\epsilon,\ 0<\epsilon\leq 1$ and multiplying (2.3) by 2 and adding them together, we get

$$\begin{aligned}&\frac{d}{dt}E^*(t)+2(1-\epsilon)\|\nabla u_t(t)\|_2^2+2a\epsilon\|\triangle u(t)\|_2^2\\&\leq -2\epsilon\int_\Omega |u|^{\rho-2}u\triangle u\,dx+2\epsilon b\|\nabla u(t)\|_2^2\|\triangle u(t)\|_2^2,\end{aligned}\tag{3.14}$$

where

$$E^*(t)=2E(t)-\int_\Omega u_t(t)\triangle u(t)\,dx+\epsilon\|\triangle u(t)\|_2^2.$$

As $I(u(t))>0$ and

$$\left|2\epsilon\int_\Omega u_t(t)\triangle u(t)\,dx\right|\leq 2\epsilon\|u_t(t)\|_2^2+\frac{\epsilon}{2}\|\triangle u(t)\|_2^2,$$

we have

$$E^*(t)\geq (1-2\epsilon)\|u_t(t)\|_2^2+\epsilon\|\triangle u(t)\|_2^2.$$

Now, choosing $\epsilon=\frac{2}{5}$ we have

$$E^*(t)\geq \frac{1}{5}(\|u_t(t)\|_2^2+\|\triangle u(t)\|_2^2).$$

On the other hand, we see from Lemma 2.2 and (3.2) that

$$\begin{aligned}\left|2\int_\Omega |u|^{\rho-2}u\triangle u\,dx\right|&\leq 2(\rho-2)\|u(t)\|^{\rho-2}_{\frac{3}{2}(\rho-2)}\|\nabla u(t)\|_6^2\\&\leq 2c_*^\rho(\rho-2)\|\nabla u(t)\|_2^{\rho-2}\|\triangle u(t)\|_2^2\leq C_{10}E^*(t)\end{aligned}$$

and

$$\|\nabla u(t)\|_2^2\|\triangle u(t)\|_2^2\leq \frac{4}{a}E(0)E^*(t),$$

where

$$C_{10}=10c_*^\rho(\rho-2)\left(\frac{4E(0)}{a}\right)^{(\rho-2)/2}.$$

Hence, integrating (3.14) over $]0,t[$ we get

$$E^*(t)\leq E^*(0)+\int_0^t C_{11}E^*(s)\,ds$$

where $C_{11}=\frac{2}{5}\left[C_{10}+10b\left(\frac{4E(0)}{a}\right)\right]$. Then, by the Gronwall inequality, it follows that

$$E^*(t)\leq E^*(0)\exp(C_{11}t).$$

Therefore, by Theorem 2.5, we have $T_0=\infty$. Moreover, from Proposition 3.2 we obtain the decay estimate (3.12). □

Remark 3.5. It seems to be interesting to study a global solution for Kirchhoff equation with nonlinear source and boundary damping term or with nonlinear boundary damping and source term, i.e.

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u - \triangle u_t = |u|^{\rho-2}u \quad \text{in } \Omega\times]0,\infty[,$$
$$u = 0 \quad \text{on } \Gamma_0\times]0,\infty[,$$
$$M\left(\int_\Omega |\nabla u|^2\,dx\right)\frac{\partial}{\partial\nu}u + \frac{\partial}{\partial\nu}u_t = g(u_t) \quad \text{on } \Gamma_1\times]0,\infty[,$$
$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x\in\Omega,$$

and

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u - \triangle u_t = 0 \quad \text{in } \Omega\times]0,\infty[,$$
$$u = 0 \quad \text{on } \Gamma_0\times]0,\infty[,$$
$$M\left(\int_\Omega |\nabla u|^2\,dx\right)\frac{\partial}{\partial\nu}u + \frac{\partial}{\partial\nu}u_t = |u|^{\rho-2}u \quad \text{on } \Gamma_1\times]0,\infty[,$$
$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x\in\Omega,$$

with $M(s)$ given in (1.2). In [18] the authors considered the global solvability with these boundary conditions, but with $M(s) \approx a + bs^r$, for all $s \geq 0$, $a, b > 0$, $r \geq 1$.

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