Tanguy Christian, 0000-0002-8155-5352
Orange Innovation, Châtillon, France, christian.tanguy{at}orange.com

Influence of transients on end-to-end availability for a meshed network

Keywords

availability, exponential distribution, gamma distribution, meshed networks, transient effects

Abstract

The disruption of services must be kept at a minimum in telecommunications networks so that the consequences are not too severe and their durations are as short as possible. Maintenance policies often rely on the steady-state availabilities of each element of the system, and focus on the system's weak links. The end-to-end (or two-terminal) availability – a standard performance index – of a meshed network has long been studied, but mainly for small systems, and assuming constant values for the availability of each element. When taken into account, the time-dependent contributions of links and nodes to the system unavailability were computed using exponential failure and repair distributions. In this work we revisit the meshed network first proposed by Walter, Esch, and Limbourg (ESREL 2008), and compute the end-to-end availability between two nodes, where the individual contributions of links and nodes are kept. This allows the ranking of links and nodes, using well-known performance indices (Birnbaum, Risk Reduction Worth, etc.). We can thus determine the elements that should receive due attention in maintenance and resilience studies. However, as the steady-state availability may not always be a lower bound to the transient availability in the case of non-exponential failure and repair distributions, we have studied the influence of such configurations on the time-dependent behaviours of all the aforementioned quantities. We then discuss the influence of uncertainty in the availability values, and compare the results obtained for the all-terminal reliability, another often-used performance criterion of networks.

1. Introduction

Telecommunications networks must recover quickly after failures, natural events, cyberattacks, and so on. Determining the weak links of the system allows the development of effective maintenance strategies. Standard approaches are mostly based on the knowledge of the steady-state availabilities of the various components of the whole system.

Recent publications have shown important transient variations of the availability in several telecommunications subfields: 5G systems and Network Virtualization Functions studies (Mauro et al., 2017, 2018), high availability of cluster configurations (Distefano et al., 2010), and communication channels in the European railway industry (Carnevali et al., 2015), to cite but a few. They demonstrate that the availability may oscillate for an extended period of time.

Resilience issues have also initiated a large body of work, and it is worth noticing that, especially in the last few years, time-dependent aspects of resilience have come to the fore in urban and commodities infrastructures (Li et al., 2020; Ouyang & Dueñas-Osorio, 2012; Lin & El-Tawil, 2020; Zeiler et al, 2017; Zeng et al., 2021). Such studies may require important computational effort, because the investigated systems may be very large. Another fruitful approach has been to consider medium-sized systems, for which the number of parameters remains tractable while allowing to get insights about the behaviour of larger systems. Such a configuration has recently been studied (Eid, 2021; Tanguy, 2022a), in which the time-dependent contributions of nodes and links to the

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global all-terminal availability (the all-terminal reliability or availability, usually written Rel_A in the literature, represents the probability that all nodes of a network are connected) have been assessed, when components undergo failures and repairs obeying exponential distributions.

This approach is very promising for the description of telecommunications networks, and we have decided to apply some of our former results in the case of non-exponential distributions (Tanguy et al., 2019; Tanguy, 2020) to investigate the assessment of potential weak links of the network, and determine when the assumption of exponential distributions may be questionable.

In the present work, we consider another performance measure of telecommunication networks, namely the end-to-end (or point-to-point) availability, also called two-terminal availability in the literature (usually written as $\mathrm{Rel}_2(S \to T)$, the probability of operation between source S and destination T). For such a measure, contributions of nodes and links cannot always be separated. We show that, similarly to what occurs for the all-terminal availability in the Eid configuration (Eid, 2021; Tanguy, 2022a), transient effects may not be neglected, especially when failure time distributions are not restricted to exponentials.

The chapter, in which we assume no correlation between elements of the system, is organized as follows. In Section 2, we present the medium-sized network architecture proposed in (Walter et al., 2008), along with the main assumptions on the availabilities of the nodes and links. Section 3 describes the method used to obtain the end-to-end availability between source and destination. This expression is much simpler when all nodes and links are supposed identical, with availabilities ρ and p, respectively. In Section 4, we compute a few performance measures (Birnbaum, Raw Achievement Worth (RAW), Risk Reduction Worth (RRW), etc.) in order to determine the weakest links of the system, so that maintenance efforts are focussed on the proper network elements, be they nodes or links. Section 5 is devoted to the study of the relative influence of links and nodes to the total end-to-end availability. We show that it may strongly depend on transients (during the mission time) even if in the long term, the steady-state behaviour is recovered, as expected. We discuss in Section 6 the influence of another factor, namely the uncertainty about the Mean Time To Failure (MTTF) and Mean Time To Repair (MTTR) on the ranking of the system's components. We also compare the results obtained with those derived for the all-terminal availability. We conclude with remarks about the assessment of the reliability of large systems (Kołowrocki, 2004).

2. Description of network

2.1. Graph representation of network

The network considered in this work has been proposed in (Walter et al., 2008), and is represented by the graph displayed in Figure 1. The performance index considered here is the two-terminal reliability or availability, depending on the context (their expressions are formally identical, namely the probability that the source and destination nodes are connected). It has long been known that the computation of the two-terminal reliability for the most general graphs is complex even when nodes are *perfect* (they do not fail), and when edges of the graph have the same reliability/availability p. It may become cumbersome even for a small number of nodes in the underlying graph (Beichelt & Tittmann, 2012).

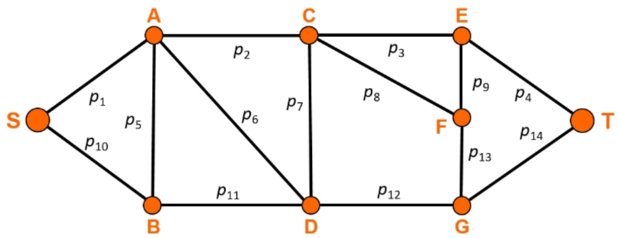

Figure 1. Network considered in this study (after Walter et al., 2008).

In Figure 1 the source node is S, the destination node is T. The graph is undirected. Intermediate nodes and links are labelled.

2.2. Reference numerical data

As mentioned in the Introduction, the availabilities of nodes and links are often assumed to take their steady-state values, and do not vary with time. When a time-dependence is introduced, it is mostly through the use of exponential distributions for lifetimes and repairs. For this reason, we shall consider the numerical values given in Table 1 for the failure and repair rates of all the nodes and links of the system, assuming identical elements for the two families of equipment.

Table 1. Failure and repair rate values used for reference in this study

	Failure rate λ (hour^{-1})	Repair rate μ (hour^{-1})
Node	0.0009	0.0091
Edge	0.0300	0.1700

From Table 1, it is easy to derive the steady-state values p_∞ and ρ_∞ of the links and nodes availabilities, respectively, obtained by the well-known expression $\mu/(\mu + \lambda)$ (Henley & Kumamoto, 1991; Kuo & Zuo, 2003; Rausand & Høyland, 2004):

$$p_\infty = 0.85, \tag{1}$$

$$\rho_\infty = 0.91. \tag{2}$$

These values will serve as reference when studying the influence of transients for the whole system's availability.

2.3. Exponential distributions

When the failure and repair distributions are exponentials, one can use the well-known formula (Kuo & Zuo, 2003; Rausand & Høyland, 2004) for the average availability $A(t)$

$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu}\, e^{-(\lambda + \mu)\, t}. \tag{3}$$

Application of (3) for the values of Table 1 gives the time-dependent average availabilities $p(t)$ and $\rho(t)$ of links and nodes, respectively:

$$p(t) = \frac{85}{100} + \frac{15}{100}\, e^{-t/5}, \tag{4}$$

$$\rho(t) = \frac{91}{100} + \frac{9}{100}\, e^{-t/100}. \tag{5}$$

2.4. Gamma distributions

Gamma distributions (Rausand & Høyland, 2004; Pham, 2006) are, after exponentials, among the most often used distributions in reliability theory. We consider in the following the gamma distribution defined by its density

$$f(t) = \frac{(\alpha\, \lambda)^\alpha\, t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\alpha\, \lambda\, t}, \tag{6}$$

where α is the so-called shape parameter, and Γ is the Euler gamma function. The definition (6) ensures that the Mean Time To Failure (MTTF) is still equal to $1/\lambda$. Several works have studied the time-dependent availability when the lifetime obeys a gamma distribution (Pham-Gia & Turkkan, 1999; Rao & Naikan, 2015; Sarkar & Chaudhuri, 1999; Tanguy et al., 2019; Tanguy, 2022b). It is possible to express $p(t)$ and $\rho(t)$ exactly and analytically in some cases, for instance when α is an integer, a possibility that we shall use in the following.

3. Calculation of two-terminal reliability of Walter configuration

Calculation of the reliability or availability of a meshed structure – such as that described in Figure 1 – can be quite demanding, especially when all elements of the systems are distinct and the underlying graph representing the system is not of the series-parallel type. For a non-series-parallel graph, exact formula are often very complicated or cumbersome, with a number of terms that increases greatly with the number of components. Textbooks mostly limit themselves to the bridge structure (Kuo & Zuo, 2003; Rausand & Høyland, 2004). However, for a few recursive families of graphs, it is possible to obtain the results easily because of an inherent factorization of the reliability (Tanguy, 2007).

3.1. Recursive analysis of Walter configuration

It is possible to represent the graph of Figure 1 in a slightly different way, as displayed on Figure 2. The gist of the method is to reproduce the same network, by adding a few virtual links and nodes, the reliabilities of which are equal to 1 (they are perfect, and thus never fail). This means that all things considered, the behaviour of the whole system is not modified.

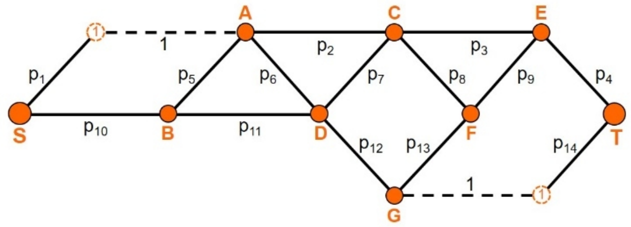

Figure 2. First modified representation of the graph of Figure 1. Note the addition of perfect nodes and perfect links between S and A, and G and T.

In a second step, we add extra nodes and links, the reliabilities of which are zero; they do *not* function (see Figure 3).

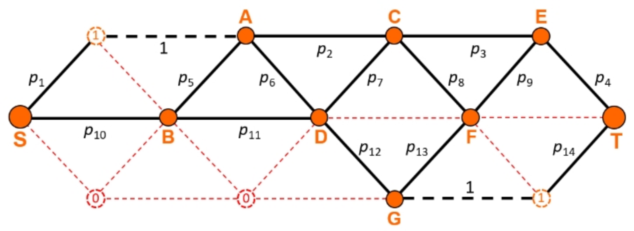

Figure 3. Second modified representation of the graph of Figure 1, after adding phantom nodes and links (in red). A structural recursiveness appears (Beichelt & Spross, 1989).

After the second step, one can observe the recursive structure of the graph (Beichelt & Spross, 1989; Prékopa et al., 1991). It turns out that this recursive architecture has been solved for arbitrary probabilities of operations for all elements (nodes *and* links). The two-terminal reliability between S and T is then obtained via a product of 15×15 matrices (Tanguy, 2009).

3.2. Complete analytical result

Using the general expression of these matrices with individual availabilities – ρ_S, ρ_T, etc. for the nodes, $p_1,\ldots,p_{14}$ for the links – we have been able to compute the exact two-terminal $\mathrm{Rel}_2(S \to T)$. Since the full expression contains 917 terms, we only report in the following the expressions for identical links or nodes, with only two variables p and ρ, not 23 as in the general case. The full expression may be obtained from the author.

3.3. Result for identical links and nodes

Considering identical nodes and identical links allows for a much simpler expression of the two-terminal availability between source S and destination T. We also give in equations (7) to (9) the results when either nodes or links are perfect:

$$\begin{aligned}\mathrm{Rel}_2(p,\rho) &= 3p^4\rho^5 + (8p^5 - 9p^6 + 2p^7)\rho^6 \\ &+ (13p^6 - 38p^7 + 25p^8 - 5p^9)\rho^7 \\ &+ (14p^7 - 89p^8 + 151p^9 - 94p^{10} + 20p^{11})\rho^8 \\ &+ (6p^8 - 78p^9 + 278p^{10} - 428p^{11} \\ &+ 326p^{12} - 122p^{13} + 18p^{14})\rho^9 \end{aligned} \qquad (7)$$

$$\begin{aligned}\mathrm{Rel}_2(p,\rho = 1) &= 3p^4 + 8p^5 + 4p^6 - 22p^7 \\ &- 58p^8 + 68p^9 + 184p^{10} - 408p^{11} \\ &+ 326p^{12} - 122p^{13} + 18p^{14}\end{aligned} \qquad (8)$$

$$\mathrm{Rel}_2(p = 1,\rho) = 3\rho^5 + \rho^6 - 5\rho^7 + 2\rho^8. \qquad (9)$$

It is worth noting that by contrast to what happens for the all-terminal reliability, $\mathrm{Rel}_2(p,\rho)$ does not factorize in functions of p and ρ (Eid, 2021; Tanguy, 2022a). Furthermore, all expressions simplify to 1 when links and nodes are perfect, as expected.

3.4. Steady-state unavailabilities

It is in general easier to deal with unavailabilities defined by $U = 1 - \mathrm{Rel}_2$, instead of availabilities. Considering the steady-state values for nodes and links given in equations (1) and (2), one gets (the subscript ∞ indicates a steady-state value)

$$(U_\infty)_{\text{total}} = 0.32302070840136, \qquad (10)$$

whereas if one considers perfect nodes ($\rho = 1$),

$$(U_\infty)_{\text{links}} = 0.06389109843798, \qquad (11)$$

while for perfect links ($p = 1$),

$$(U_\infty)_{\text{nodes}} = 0.20333435421525. \qquad (12)$$

It is important to stress that the sum of equations (11) and (12) does not give (10), because of correlations between the variables p and ρ in (7). Besides, these numerical values show that nodes contribute to a greater unavailability of the system at very long times. Before considering what happens at shorter times, we first investigate what can be said about the performance indices of each element, for which the complete expression of $\mathrm{Rel}_2(S \to T)$ is required.

4. Performance indices for end-to-end availability for Walter configuration

Various performance measures have been defined to assess the criticality of each element of the system to the latter's operation. The most popular are Birnbaum, Improvement Potential (IP), Risk Achievement Worth (RAW), Risk Reduction Worth (RRW), Criticality Importance (CI), Fussell-Vesely (FV), etc. (Rausand & Høyland, 2004). Their role is to *rank* or *sort* all elements by their relative importance. These performance indices depend on each element (node or link), through its individual availability as well as its lo-

cation in the system. It is worth stressing that several of these measures can be deduced directly from the structure function of the system, which is *formally* identical to $\mathrm{Rel}_2(S \rightarrow T)$ (Rausand & Høyland, 2004). In most studies, only the asymptotic availabilities are considered.
A similar study of the ranking of components of a system has been performed for the all-terminal availability of the Eid configuration (Eid, 2021; Tanguy, 2022a). In that case, the role of nodes and links were quite distinct, and all nodes had the same importance. In the present work, this is not true anymore when the two-terminal availability is under consideration.

4.1. Birnbaum importance factor

The Birnbaum importance factor — probably the most used index— is defined by the derivative of the system availability with respect to the availability of the component. The exact knowledge of $\mathrm{Rel}_2(S \rightarrow T)$ allows therefore to compute every Birnbaum factor $I^{(B)}$. Again, the full expressions are too lengthy to be given here. The numbers of terms are 917 for ρ_S and ρ_T (of course), but decrease to 607 for p_2 and 582 for p_6. Only after the differentiation has been performed can one again assume that all nodes and links are identical, and obtain simpler expressions of the Birnbaum index. They are:

$$\mathrm{I}^{(\mathrm{B})}(\rho_S) = 3p^4\rho^4 + (8p^5 - 9p^6 + 2p^7)\rho^5 + (13p^6 - 38p^7 + 25p^8 - 5p^9)\rho^6 + (14p^7 - 89p^8 + 151p^9 - 94p^{10} + 20p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8\} \quad (13a)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_T) = 3p^4\rho^4 + (8p^5 - 9p^6 + 2p^7)\rho^5 + (13p^6 - 38p^7 + 25p^8 - 5p^9)\rho^6 + (14p^7 - 89p^8 + 151p^9 - 94p^{10} + 20p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8\} \quad (13b)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_A) = 2p^4\rho^4 + (7p^5 - 9p^6 + 2p^7)\rho^5 + (10p^6 - 34p^7 + 25p^8 - 5p^9)\rho^6 + (12p^7 - 78p^8 + 138p^9 - 90p^{10} + 20p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13c)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_B) = p^4\rho^4 + (4p^5 - 6p^6 + 2p^7)\rho^5 + (9p^6 - 24p^7 + 18p^8 - 5p^9)\rho^6 + (11p^7 - 66p^8 + 106p^9 - 63p^{10} + 13p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13d)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_C) = p^4\rho^4 + (6p^5 - 4p^6)\rho^5 + (11p^6 - 36p^7 + 25p^8 - 5p^9)\rho^6 + (12p^7 - 82p^8 + 144p^9 - 92p^{10} + 20p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13e)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_D) = 2p^4\rho^4 + (5p^5 - 7p^6 + 2p^7)\rho^5 + (10p^6 - 31p^7 + 22p^8 - 5p^9)\rho^6 + (13p^7 - 84p^8 + 145p^9 - 92p^{10} + 20p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13f)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_E) = p^4\rho^4 + (4p^5 - 3p^6)\rho^5 + (9p^6 - 27p^7 + 17p^8 - 3p^9)\rho^6 + (11p^7 - 69p^8 + 117p^9 - 72p^{10} + 15p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13g)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_F) = (2p^5 - p^6)\rho^5 + (9p^6 - 17p^7 + 6p^8)\rho^6 + (14p^7 - 81p^8 + 128p^9 - 75p^{10} + 15p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13h)$$

$$\mathrm{I}^{(\mathrm{B})}(\rho_G) = 2p^4\rho^4 + (4p^5 - 6p^6 + 2p^7)\rho^5 + (7p^6 - 21p^7 + 12p^8 - 2p^9)\rho^6 + (11p^7 - 74p^8 + 128p^9 - 80p^{10} + 17p^{11})\rho^7 + (6p^8 - 78p^9 + 278p^{10} - 428p^{11} + 326p^{12} - 122p^{13} + 18p^{14})\rho^8 \quad (13i)$$

$$\mathrm{I}^{(\mathrm{B})}(p_1) = 2p^3\rho^5 + (5p^4 - 8p^5 + 2p^6)\rho^6 + (5p^5 - 26p^6 + 22p^7 - 5p^8)\rho^7 + (6p^6 - 50p^7 + 110p^8 - 82p^9 + 20p^{10})\rho^8 + (2p^7 - 35p^8 + 160p^9 - 299p^{10} + 264p^{11} - 111p^{12} + 18p^{13})\rho^9 \quad (13j)$$

$$\mathrm{I}^{(\mathrm{B})}(p_2) = p^3\rho^5 + (4p^4 - 4p^5)\rho^6 + (5p^5 - 25p^6 + 23p^7 - 5p^8)\rho^7 + (4p^6 - 44p^7 + 99p^8 - 78p^9 + 20p^{10})\rho^8 + (2p^7 - 46p^8 + 193p^9 - 333p^{10} + 280p^{11} - 114p^{12} + 18p^{13})\rho^9 \quad (13k)$$

$$\mathrm{I}^{(\mathrm{B})}(p_3) = p^3\rho^5 + (3p^4 - 3p^5)\rho^6 + (4p^5 - 22p^6 + 17p^7 - 3p^8)\rho^7 + (5p^6 - 37p^7 + 84p^8 - 63p^9 + 15p^{10})\rho^8 + (5p^7 - 53p^8 + 193p^9 - 325p^{10} + 273p^{11} - 112p^{12} + 18p^{13})\rho^9 \quad (13l)$$

$$I^{(B)}(p_4) = p^3\rho^5 + (4p^4 - 3p^5)\rho^6 + (8p^5 - 26p^6 + 17p^7 - 3p^8)\rho^7 + (8p^6 - 60p^7 + 109p^8 - 70p^9 + 15p^{10})\rho^8 + (3p^7 - 55p^8 + 224p^9 - 372p^{10} + 299p^{11} - 117p^{12} + 18p^{13})\rho^9 \quad (13m)$$

$$I^{(B)}(p_5) = (3p^4 - 5p^5 + 2p^6)\rho^6 + (5p^5 - 15p^6 + 15p^7 - 5p^8)\rho^7 + (7p^6 - 41p^7 + 72p^8 - 51p^9 + 13p^{10})\rho^8 + (5p^7 - 58p^8 + 205p^9 - 330p^{10} + 271p^{11} - 111p^{12} + 18p^{13})\rho^9 \quad (13n)$$

$$I^{(B)}(p_6) = p^3\rho^5 + (2p^4 - 6p^5 + 2p^6)\rho^6 + (5p^5 - 17p^6 + 18p^7 - 5p^8)\rho^7 + (7p^6 - 47p^7 + 95p^8 - 75p^9 + 20p^{10})\rho^8 + (3p^7 - 38p^8 + 155p^9 - 280p^{10} + 250p^{11} - 108p^{12} + 18p^{13})\rho^9 \quad (13o)$$

$$I^{(B)}(p_7) = (3p^4 - 2p^5)\rho^6 + (7p^5 - 25p^6 + 21p^7 - 5p^8)\rho^7 + (7p^6 - 53p^7 + 106p^8 - 79p^9 + 20p^{10})\rho^8 + (3p^7 - 38p^8 + 163p^9 - 296p^{10} + 260p^{11} - 110p^{12} + 18p^{13})\rho^9 \quad (13p)$$

$$I^{(B)}(p_8) = (2p^4 - p^5)\rho^6 + (6p^5 - 14p^6 + 6p^7)\rho^7 + (8p^6 - 59p^7 + 106p^8 - 69p^9 + 15p^{10})\rho^8 + (2p^7 - 41p^8 + 182p^9 - 322p^{10} + 273p^{11} - 112p^{12} + 18p^{13})\rho^9 \quad (13q)$$

$$I^{(B)}(p_9) = (p^4 - p^5)\rho^6 + (6p^5 - 10p^6 + 4p^7)\rho^7 + (9p^6 - 53p^7 + 84p^8 - 50p^9 + 10p^{10})\rho^8 + (4p^7 - 61p^8 + 226p^9 - 361p^{10} + 288p^{11} - 114p^{12} + 18p^{13})\rho^9 \quad (13r)$$

$$I^{(B)}(p_{10}) = p^3\rho^5 + (3p^4 - 5p^5 + 2p^6)\rho^6 + (8p^5 - 21p^6 + 16p^7 - 5p^8)\rho^7 + (8p^6 - 55p^7 + 93p^8 - 58p^9 + 13p^{10})\rho^8 + (4p^7 - 58p^8 + 223p^9 - 363p^{10} + 291p^{11} - 115p^{12} + 18p^{13})\rho^9 \quad (13s)$$

$$I^{(B)}(p_{11}) = p^3\rho^5 + (2p^4 - 4p^5 + 2p^6)\rho^6 + (5p^5 - 17p^6 + 15p^7 - 5p^8)\rho^7 + (7p^6 - 48p^7 + 85p^8 - 56p^9 + 13p^{10})\rho^8 + (3p^7 - 52p^8 + 210p^9 - 352p^{10} + 288p^{11} - 115p^{12} + 18p^{13})\rho^9 \quad (13t)$$

$$I^{(B)}(p_{12}) = 2p^3\rho^5 + (3p^4 - 6p^5 + 2p^6)\rho^6 + (3p^5 - 15p^6 + 10p^7 - 2p^8)\rho^7 + (5p^6 - 48p^7 + 103p^8 - 73p^9 + 17p^{10})\rho^8 + (3p^7 - 54p^8 + 213p^9 - 357p^{10} + 292p^{11} - 116p^{12} + 18p^{13})\rho^9 \quad (13u)$$

$$I^{(B)}(p_{13}) = p^4\rho^6 + (6p^5 - 12p^6 + 4p^7)\rho^7 + (11p^6 - 62p^7 + 99p^8 - 59p^9 + 12p^{10})\rho^8 + (6p^7 - 70p^8 + 247p^9 - 386p^{10} + 302p^{11} - 117p^{12} + 18p^{13})\rho^9 \quad (13v)$$

$$I^{(B)}(p_{14}) = 2p^3\rho^5 + (4p^4 - 6p^5 + 2p^6)\rho^6 + (5p^5 - 21p^6 + 12p^7 - 2p^8)\rho^7 + (6p^6 - 55p^7 + 114p^8 - 77p^9 + 17p^{10})\rho^8 + (3p^7 - 43p^8 + 186p^9 - 332p^{10} + 281p^{11} - 114p^{12} + 18p^{13})\rho^9. \quad (13w)$$

Here again we see that the expressions have mixed contributions from the two variables p and ρ which can be time-dependent.

We have ranked the 23 parameters according to the Birnbaum index when the availabilities have their steady-state values, deduced from (1) and (2): 0.85 for links and 0.91 for nodes. The results are given in Table 2; the list of components sorted by decreasing importance is (ρ_S and ρ_T are equivalent, of course). Different colours are used for nodes (blue) and links (violet).

$\rho_S, \rho_T, \rho_A, \rho_D, \rho_C, \rho_G, p_{14}, p_1, p_4, \rho_E, \rho_B, p_{10}, p_{12}, p_2, p_{11}, p_3, \rho_F, p_8, p_6, p_{13}, p_7, p_9, p_5$.

Table 2. Birnbaum importance index for steady-state availabilities (0.85 for links and 0.91 for nodes)

Component	Birnbaum importance factor
ρ_S	0.743933287471032626664
ρ_T	0.743933287471032626664
ρ_A	0.26333393407261324265
ρ_D	0.21733574589545803622
ρ_C	0.20664782846623041385
ρ_G	0.20441697276285038447
p_{14}	0.17391999915238775907
p_1	0.16454665926328098204
p_4	0.16022199561232842752
ρ_E	0.15905456597758160579
ρ_B	0.13943641196591197341
p_{10}	0.13611988634670647377
p_{12}	0.11423830510275209527
p_2	0.08777557939567974825
p_{11}	0.07419140283146072724
p_3	0.05752211328857445413
ρ_F	0.04955841196543736364
p_8	0.03097058413067667711
p_6	0.02987899928630222520
p_{13}	0.02939563654635550653
p_7	0.02766504542418655675
p_9	0.02221788403848708225
p_5	0.01796327868123634270

While the first six elements are nodes, one can observe that node F trails after eight links. Another interesting feature of the results of Table 2 is the huge difference between the Birnbaum indices for ρ_S (or ρ_T) and p_5: a factor larger than 40. That means that a greater attention should be exercised on the first elements of the above list, for the sake of maintenance and resilience policies in order to ensure or improve the system performance.
The variations of the Birnbaum factors have been plotted in Figure 4 for $\rho = \rho_\infty = 0.91$, while p may vary between 0 and 1. One can notice that the rankings do not vary very much when the availabilities remain larger than their asymptotic value $p_\infty = 0.85$.

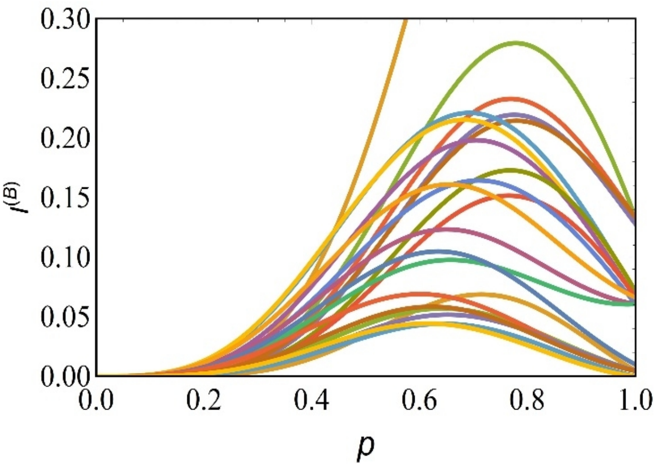

Figure 4. Birnbaum factors for $\rho = \rho_\infty = 0.91$.

The variations of the Birnbaum factors have also been plotted in Figure 5 for $p = p_\infty = 0.85$, while ρ may vary between 0.8 and 1. Again, the rankings do not change drastically when the availabilities remain larger than $\rho_\infty = 0.91$.

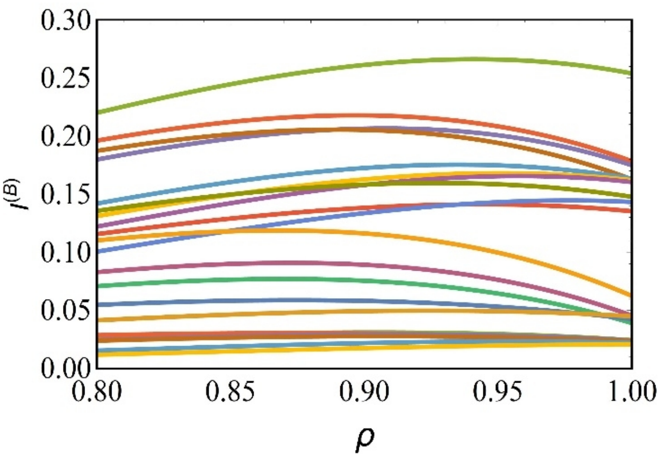

Figure 5. Birnbaum factors for $p = p_\infty = 0.85$.

4.2. Risk Achievement Worth

The Risk Achievement Worth (RAW) is proportional to the unavailability of the system when element i fails (Rausand & Høyland, 2004), so that (we omit here the common denominator $1 - \mathrm{Rel}_2(S \to T)$)

$$I^{(\mathrm{RAW})}(i) \propto 1 - \mathrm{Rel}_2(S \to T; 0_i). \quad (14)$$

For this importance measure, one can derive lengthy expressions again when individual availabilities are kept throughout the calculations. Returning to two families of nodes and links, simpler expressions similar to equations (13a to 13w) can be found.

$$I^{(\mathrm{RAW})}(\rho_S) \propto 1 \quad (15a)$$

$$I^{(\mathrm{RAW})}(\rho_T) \propto 1 \quad (15b)$$

$$I^{(\mathrm{RAW})}(\rho_A) \propto 1 - p^4\rho^5 - p^5\rho^6 + (-3p^6 + 4p^7)\rho^7 + (-2p^7 + 11p^8 - 13p^9 + 4p^{10})\rho^8 \quad (15c)$$

$$I^{(\mathrm{RAW})}(\rho_B) \propto 1 - 2p^4\rho^5 + (-4p^5 + 3p^6)\rho^6 + (-4p^6 + 14p^7 - 7p^8)\rho^7 + (-3p^7 + 23p^8 - 45p^9 + 31p^{10} - 7p^{11})\rho^8 \quad (15d)$$

$$I^{(\mathrm{RAW})}(\rho_C) \propto 1 - 2p^4\rho^5 + (-2p^5 + 5p^6 - 2p^7)\rho^6 + (-2p^6 + 2p^7)\rho^7 + (-2p^7 + 7p^8 - 7p^9 + 2p^{10})\rho^8 \quad (15e)$$

$$I^{(\mathrm{RAW})}(\rho_D) \propto 1 - p^4\rho^5 + (-3p^5 + 2p^6)\rho^6 + (-3p^6 + 7p^7 - 3p^8)\rho^7 + (-p^7 + 5p^8 - 6p^9 + 2p^{10})\rho^8 \quad (15f)$$

$$I^{(\mathrm{RAW})}(\rho_E) \propto 1 - 2p^4\rho^5 + (-4p^5 + 6p^6 - 2p^7)\rho^6 + (-4p^6 + 11p^7 - 8p^8 + 2p^9)\rho^7 + (-3p^7 + 20p^8 - 34p^9 + 22p^{10} - 5p^{11})\rho^8 \quad (15g)$$

$$I^{(\mathrm{RAW})}(\rho_F) \propto 1 - 3p^4\rho^5 + (-6p^5 + 8p^6 - 2p^7)\rho^6 + (-4p^6 + 21p^7 - 19p^8 + 5p^9)\rho^7 + (8p^8 - 23p^9 + 19p^{10} - 5p^{11})\rho^8 \quad (15h)$$

$$I^{(\mathrm{RAW})}(\rho_G) \propto 1 - p^4\rho^5 + (-4p^5 + 3p^6)\rho^6 + (-6p^6 + 17p^7 - 13p^8 + 3p^9)\rho^7 + (-3p^7 + 15p^8 - 23p^9 + 14p^{10} - 3p^{11})\rho^8 \quad (15i)$$

$$I^{(\mathrm{RAW})}(p_1) \propto 1 - p^4\rho^5 + (-3p^5 + p^6)\rho^6 + (-8p^6 + 12p^7 - 3p^8)\rho^7 + (-8p^7 + 39p^8 -$$

$$41p^9 + 12p^{10})\rho^8 + (-4p^8 + 43p^9 - 118p^{10} + 129p^{11} - 62p^{12} + 11p^{13})\rho^9 \quad (15j)$$

$$I^{(RAW)}(p_2) \propto 1 - 2p^4\rho^5 + (-4p^5 + 5p^6 - 2p^7)\rho^6 + (-8p^6 + 13p^7 - 2p^8)\rho^7 + (-10p^7 + 45p^8 - 52p^9 + 16p^{10})\rho^8 + (-4p^8 + 32p^9 - 85p^{10} + 95p^{11} - 46p^{12} + 8p^{13})\rho^9 \quad (15k)$$

$$I^{(RAW)}(p_3) \propto 1 - 2p^4\rho^5 + (-5p^5 + 6p^6 - 2p^7)\rho^6 + (-9p^6 + 16p^7 - 8p^8 + 2p^9)\rho^7 + (-9p^7 + 52p^8 - 67p^9 + 31p^{10} - 5p^{11})\rho^8 + (-p^8 + 25p^9 - 85p^{10} + 103p^{11} - 53p^{12} + 10p^{13})\rho^9 \quad (15l)$$

$$I^{(RAW)}(p_4) \propto 1 - 2p^4\rho^5 + (-4p^5 + 6p^6 - 2p^7)\rho^6 + (-5p^6 + 12p^7 - 8p^8 + 2p^9)\rho^7 + (-6p^7 + 29p^8 - 42p^9 + 24p^{10} - 5p^{11})\rho^8 + (-3p^8 + 23p^9 - 54p^{10} + 56p^{11} - 27p^{12} + 5p^{13})\rho^9 \quad (15m)$$

$$I^{(RAW)}(p_5) \propto 1 - 3p^4\rho^5 + (-5p^5 + 4p^6)\rho^6 + (-8p^6 + 23p^7 - 10p^8)\rho^7 + (-7p^7 + 48p^8 - 79p^9 + 43p^{10} - 7p^{11})\rho^8 + (-p^8 + 20p^9 - 73p^{10} + 98p^{11} - 55p^{12} + 11p^{13})\rho^9 \quad (15n)$$

$$I^{(RAW)}(p_6) \propto 1 - 2p^4\rho^5 + (-6p^5 + 3p^6)\rho^6 + (-8p^6 + 21p^7 - 7p^8)\rho^7 + (-7p^7 + 42p^8 - 56p^9 + 19p^{10})\rho^8 + (-3p^8 + 40p^9 - 123p^{10} + 148p^{11} - 76p^{12} + 14p^{13})\rho^9 \quad (15o)$$

$$I^{(RAW)}(p_7) \propto 1 - 3p^4\rho^5 + (-5p^5 + 7p^6 - 2p^7)\rho^6 + (-6p^6 + 13p^7 - 4p^8)\rho^7 + (-7p^7 + 36p^8 - 45p^9 + 15p^{10})\rho^8 + (-3p^8 + 40p^9 - 115p^{10} + 132p^{11} - 66p^{12} + 12p^{13})\rho^9 \quad (15p)$$

$$I^{(RAW)}(p_8) \propto 1 - 3p^4\rho^5 + (-6p^5 + 8p^6 - 2p^7)\rho^6 + (-7p^6 + 24p^7 - 19p^8 + 5p^9)\rho^7 + (-6p^7 + 30p^8 - 45p^9 + 25p^{10} - 5p^{11})\rho^8 + (-4p^8 + 37p^9 - 96p^{10} + 106p^{11} - 53p^{12} + 10p^{13})\rho^9 \quad (15q)$$

$$I^{(RAW)}(p_9) \propto 1 - 3p^4\rho^5 + (-7p^5 + 8p^6 - 2p^7)\rho^6 + (-7p^6 + 28p^7 - 21p^8 + 5p^9)\rho^7 + (-5p^7 + 36p^8 - 67p^9 + 44p^{10} - 10p^{11})\rho^8 + (-2p^8 + 17p^9 - 52p^{10} + 67p^{11} - 38p^{12} + 8p^{13})\rho^9 \quad (15r)$$

$$I^{(RAW)}(p_{10}) \propto 1 - 2p^4\rho^5 + (-5p^5 + 4p^6)\rho^6 + (-5p^6 + 17p^7 - 9p^8)\rho^7 + (-6p^7 + 34p^8 - 58p^9 + 36p^{10} - 7p^{11})\rho^8 + (-2p^8 + 20p^9 - 55p^{10} + 65p^{11} - 35p^{12} + 7p^{13})\rho^9 \quad (15s)$$

$$I^{(RAW)}(p_{11}) \propto 1 - 2p^4\rho^5 + (-6p^5 + 5p^6)\rho^6 + (-8p^6 + 21p^7 - 10p^8)\rho^7 + (-7p^7 + 41p^8 - 66p^9 + 38p^{10} - 7p^{11})\rho^8 + (-3p^8 + 26p^9 - 68p^{10} + 76p^{11} - 38p^{12} + 7p^{13})\rho^9 \quad (15t)$$

$$I^{(RAW)}(p_{12}) \propto 1 - p^4\rho^5 + (-5p^5 + 3p^6)\rho^6 + (-10p^6 + 23p^7 - 15p^8 + 3p^9)\rho^7 + (-9p^7 + 41p^8 - 48p^9 + 21p^{10} - 3p^{11})\rho^8 + (-3p^8 + 24p^9 - 65p^{10} + 71p^{11} - 34p^{12} + 6p^{13})\rho^9 \quad (15u)$$

$$I^{(RAW)}(p_{13}) \propto 1 - 3p^4\rho^5 + (-7p^5 + 9p^6 - 2p^7)\rho^6 + (-7p^6 + 26p^7 - 21p^8 + 5p^9)\rho^7 + (-3p^7 + 27p^8 - 52p^9 + 35p^{10} - 8p^{11})\rho^8 + (8p^9 - 31p^{10} + 42p^{11} - 24p^{12} + 5p^{13})\rho^9 \quad (15v)$$

$$I^{(RAW)}(p_{14}) \propto 1 - p^4\rho^5 + (-4p^5 + 3p^6)\rho^6 + (-8p^6 + 17p^7 - 13p^8 + 3p^9)\rho^7 + (-8p^7 + 34p^8 - 37p^9 + 17p^{10} - 3p^{11})\rho^8 + (-3p^8 + 35p^9 - 92p^{10} + 96p^{11} - 45p^{12} + 8p^{13})\rho^9. \quad (15w)$$

One can then evaluate the ranking when the steady-state availabilities are taken into account. The corresponding values are listed in Table 3. The variations of the RAW factors are also displayed in Figures 6 and 7, when either the nodes or links' availabilities take their steady-state values. As already seen for the Birnbaum index, the rankings do not change much when p and ρ change. There are simply groups of components of comparable importance.

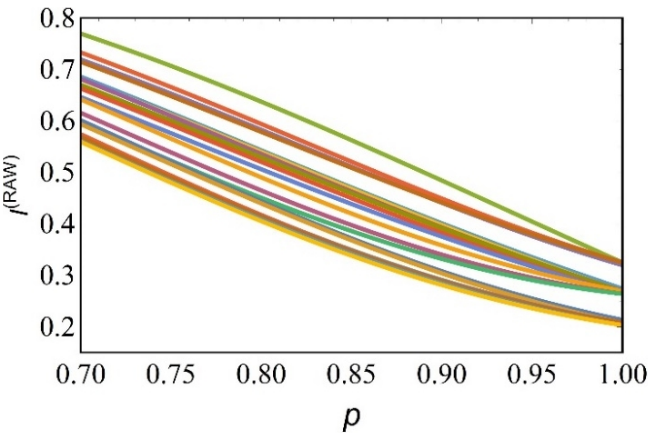

Figure 6. RAW factors for $\rho = \rho_\infty = 0.91$.

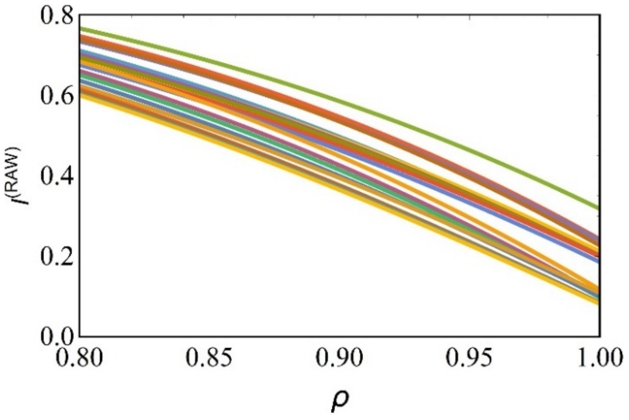

Figure 7. RAW factors for $p = p_\infty = 0.85$.

Table 3. Risk Achievement Worth index for steady-state availabilities (0.85 for links and 0.91 for nodes)

Component	RAW importance factor
ρ_S	1
ρ_T	1
ρ_A	0.56265458840743836058
ρ_D	0.52079623716622712272
ρ_C	0.51107023230562998637
ρ_G	0.50904015361555415963
p_{14}	0.47085270768088990497
ρ_E	0.46776036344095957103
p_1	0.46288536877514914449
p_4	0.45920940467183947316
ρ_B	0.44990784329034020556
p_{10}	0.43872261179606081247
p_{12}	0.42012326773869959074
p_2	0.39762995088768809577
p_{11}	0.38608340080810192791
p_3	0.37191450469664859577
ρ_F	0.36811886328990831067
p_8	0.34934570491243548530
p_6	0.34841785779471720118
p_{13}	0.34800699946576249031
p_7	0.34653599701191888300
p_9	0.34190590983407432967
p_5	0.33828949528041120105

One observes that the order of components is nearly the same as for the Birnbaum index, the node ρ_E gaining two places:

$\rho_S, \rho_T, \rho_A, \rho_D, \rho_C, \rho_G, p_{14}, \rho_E, p_1, p_4, \rho_B, p_{10}, p_{12}, p_2, p_{11}, p_3, \rho_F, p_8, p_6, p_{13}, p_7, p_9, p_5.$

4.3. Risk Reduction Worth

The Risk Reduction Worth (RRW) is inversely proportional to the unavailability of the system when element i works perfectly (Rausand & Høyland, 2004), so that (omitting the common prefactor $1 - \mathrm{Rel}_2(S \to T)$)

$$\mathrm{I}^{(\mathrm{RRW})}(i) \propto 1/(1 - \mathrm{Rel}_2(S \to T; 1_i)). \qquad (16)$$

One can derive again lengthy expressions when individual availabilities are kept throughout the calculations. With two families of identical nodes and links, simpler expressions similar to equations (13a) to (13w) are obtained; they read

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_S) \propto 1 - 3p^4\rho^4 + (-8p^5 + 9p^6 - 2p^7)\rho^5 + (-13p^6 + 38p^7 - 25p^8 + 5p^9)\rho^6 + (-14p^7 + 89p^8 - 151p^9 + 94p^{10} - 20p^{11})\rho^7 + (-6p^8 + 78p^9 - 278p^{10} + 428p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17a)$$

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_T) \propto 1 - 3p^4\rho^4 + (-8p^5 + 9p^6 - 2p^7)\rho^5 + (-13p^6 + 38p^7 - 25p^8 + 5p^9)\rho^6 + (-14p^7 + 89p^8 - 151p^9 + 94p^{10} - 20p^{11})\rho^7 + (-6p^8 + 78p^9 - 278p^{10} + 428p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17b)$$

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_A) \propto 1 - 2p^4\rho^4 + (-p^4 - 7p^5 + 9p^6 - 2p^7)\rho^5 + (-p^5 - 10p^6 + 34p^7 - 25p^8 + 5p^9)\rho^6 + (-3p^6 - 8p^7 + 78p^8 - 138p^9 + 90p^{10} - 20p^{11})\rho^7 + (-2p^7 + 5p^8 + 65p^9 - 274p^{10} + 428p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17c)$$

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_B) \propto 1 - p^4\rho^4 + (-2p^4 - 4p^5 + 6p^6 - 2p^7)\rho^5 + (-4p^5 - 6p^6 + 24p^7 - 18p^8 + 5p^9)\rho^6 + (-4p^6 + 3p^7 + 59p^8 - 106p^9 + 63p^{10} - 13p^{11})\rho^7 + (-3p^7 + 17p^8 + 33p^9 - 247p^{10} + 421p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17d)$$

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_C) \propto 1 - p^4\rho^4 + (-2p^4 - 6p^5 + 4p^6)\rho^5 + (-2p^5 - 6p^6 + 34p^7 - 25p^8 + 5p^9)\rho^6 + (-2p^6 - 10p^7 + 82p^8 - 144p^9 + 92p^{10} - 20p^{11})\rho^7 + (-2p^7 + p^8 + 71p^9 - 276p^{10} + 428p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17e)$$

$$1/\mathrm{I}^{(\mathrm{RRW})}(\rho_D) \propto 1 - 2p^4\rho^4 + (-p^4 - 5p^5 + 7p^6 - 2p^7)\rho^5 + (-3p^5 - 8p^6 + 31p^7 - 22p^8 + 5p^9)\rho^6 + (-3p^6 - 6p^7 + 81p^8 - 145p^9 + 92p^{10} - 20p^{11})\rho^7 + (-p^7 - p^8 + 72p^9 - 276p^{10} + 428p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \qquad (17f)$$

$$1/I^{(RRW)}(\rho_E) \propto 1 - p^4\rho^4 + (-2p^4 - 4p^5 + 3p^6)\rho^5 + (-4p^5 - 3p^6 + 25p^7 - 17p^8 + 3p^9)\rho^6 + (-4p^6 + 61p^8 - 115p^9 + 72p^{10} - 15p^{11})\rho^7 + (-3p^7 + 14p^8 + 44p^9 - 256p^{10} + 423p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \quad (17g)$$

$$1/I^{(RRW)}(\rho_F) \propto 1 + (-3p^4 - 2p^5 + p^6)\rho^5 + (-6p^5 - p^6 + 15p^7 - 6p^8)\rho^6 + (-4p^6 + 7p^7 + 62p^8 - 123p^9 + 75p^{10} - 15p^{11})\rho^7 + (2p^8 + 55p^9 - 259p^{10} + 423p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \quad (17h)$$

$$1/I^{(RRW)}(\rho_G) \propto 1 - 2p^4\rho^4 + (-p^4 - 4p^5 + 6p^6 - 2p^7)\rho^5 + (-4p^5 - 4p^6 + 21p^7 - 12p^8 + 2p^9)\rho^6 + (-6p^6 + 6p^7 + 61p^8 - 125p^9 + 80p^{10} - 17p^{11})\rho^7 + (-3p^7 + 9p^8 + 55p^9 - 264p^{10} + 425p^{11} - 326p^{12} + 122p^{13} - 18p^{14})\rho^8 \quad (17i)$$

$$1/I^{(RRW)}(p_1) \propto 1 + (-2p^3 - p^4)\rho^5 + (-5p^4 + 5p^5 - p^6)\rho^6 + (-5p^5 + 18p^6 - 10p^7 + 2p^8)\rho^7 + (-6p^6 + 42p^7 - 71p^8 + 41p^9 - 8p^{10})\rho^8 + (-2p^7 + 31p^8 - 117p^9 + 181p^{10} - 135p^{11} + 49p^{12} - 7p^{13})\rho^9 \quad (17j)$$

$$1/I^{(RRW)}(p_2) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-4p^4 + 5p^6 - 2p^7)\rho^6 + (-5p^5 + 17p^6 - 10p^7 + 3p^8)\rho^7 + (-4p^6 + 34p^7 - 54p^8 + 26p^9 - 4p^{10})\rho^8 + (-2p^7 + 42p^8 - 161p^9 + 248p^{10} - 185p^{11} + 68p^{12} - 10p^{13})\rho^9 \quad (17k)$$

$$1/I^{(RRW)}(p_3) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-3p^4 - 2p^5 + 6p^6 - 2p^7)\rho^6 + (-4p^5 + 13p^6 - p^7 - 5p^8 + 2p^9)\rho^7 + (-5p^6 + 28p^7 - 32p^8 - 4p^9 + 16p^{10} - 5p^{11})\rho^8 + (-5p^7 + 52p^8 - 168p^9 + 240p^{10} - 170p^{11} + 59p^{12} - 8p^{13})\rho^9 \quad (17l)$$

$$1/I^{(RRW)}(p_4) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-4p^4 - p^5 + 6p^6 - 2p^7)\rho^6 + (-8p^5 + 21p^6 - 5p^7 - 5p^8 + 2p^9)\rho^7 + (-8p^6 + 54p^7 - 80p^8 + 28p^9 + 9p^{10} - 5p^{11})\rho^8 + (-3p^7 + 52p^8 - 201p^9 + 318p^{10} - 243p^{11} + 90p^{12} - 13p^{13})\rho^9 \quad (17m)$$

$$1/I^{(RRW)}(p_5) \propto 1 - 3p^4\rho^5 + (-3p^4 + 2p^6)\rho^6 + (-5p^5 + 7p^6 + 8p^7 - 5p^8)\rho^7 + (-7p^6 + 34p^7 - 24p^8 - 28p^9 + 30p^{10} - 7p^{11})\rho^8 + (-5p^7 + 57p^8 - 185p^9 + 257p^{10} - 173p^{11} + 56p^{12} - 7p^{13})\rho^9 \quad (17n)$$

$$1/I^{(RRW)}(p_6) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-2p^4 + p^6)\rho^6 + (-5p^5 + 9p^6 + 3p^7 - 2p^8)\rho^7 + (-7p^6 + 40p^7 - 53p^8 + 19p^9 - p^{10})\rho^8 + (-3p^7 + 35p^8 - 115p^9 + 157p^{10} - 102p^{11} + 32p^{12} - 4p^{13})\rho^9 \quad (17o)$$

$$1/I^{(RRW)}(p_7) \propto 1 - 3p^4\rho^5 + (-3p^4 - 3p^5 + 7p^6 - 2p^7)\rho^6 + (-7p^5 + 19p^6 - 8p^7 + p^8)\rho^7 + (-7p^6 + 46p^7 - 70p^8 + 34p^9 - 5p^{10})\rho^8 + (-3p^7 + 35p^8 - 123p^9 + 181p^{10} - 128p^{11} + 44p^{12} - 6p^{13})\rho^9 \quad (17p)$$

$$1/I^{(RRW)}(p_8) \propto 1 - 3p^4\rho^5 + (-2p^4 - 5p^5 + 8p^6 - 2p^7)\rho^6 + (-6p^5 + 7p^6 + 18p^7 - 19p^8 + 5p^9)\rho^7 + (-8p^6 + 53p^7 - 76p^8 + 24p^9 + 10p^{10} - 5p^{11})\rho^8 + (-2p^7 + 37p^8 - 145p^9 + 226p^{10} - 167p^{11} + 59p^{12} - 8p^{13})\rho^9 \quad (17q)$$

$$1/I^{(RRW)}(p_9) \propto 1 - 3p^4\rho^5 + (-p^4 - 6p^5 + 8p^6 - 2p^7)\rho^6 + (-6p^5 + 3p^6 + 24p^7 - 21p^8 + 5p^9)\rho^7 + (-9p^6 + 48p^7 - 48p^8 - 17p^9 + 34p^{10} - 10p^{11})\rho^8 + (-4p^7 + 59p^8 - 209p^9 + 309p^{10} - 221p^{11} + 76p^{12} - 10p^{13})\rho^9 \quad (17r)$$

$$1/I^{(RRW)}(p_{10}) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-3p^4 + 2p^6)\rho^6 + (-8p^5 + 16p^6 + p^7 - 4p^8)\rho^7 + (-8p^6 + 49p^7 - 59p^8 + 23p^{10} - 7p^{11})\rho^8 + (-4p^7 + 56p^8 - 203p^9 + 308p^{10} - 226p^{11} + 80p^{12} - 11p^{13})\rho^9 \quad (17s)$$

$$1/I^{(RRW)}(p_{11}) \propto 1 + (-p^3 - 2p^4)\rho^5 + (-2p^4 - 2p^5 + 3p^6)\rho^6 + (-5p^5 + 9p^6 + 6p^7 - 5p^8)\rho^7 + (-7p^6 + 41p^7 - 44p^8 - 10p^9 + 25p^{10} - 7p^{11})\rho^8 + (-3p^7 + 49p^8 - 184p^9 + 284p^{10} - 212p^{11} + 77p^{12} - 11p^{13})\rho^9 \quad (17t)$$

$$1/I^{(RRW)}(p_{12}) \propto 1 + (-2p^3 - p^4)\rho^5 + (-3p^4 + p^5 + p^6)\rho^6 + (-3p^5 + 5p^6 + 13p^7 - 13p^8 + 3p^9)\rho^7 + (-5p^6 + 39p^7 - 62p^8 + 25p^9 + 4p^{10} - 3p^{11})\rho^8 + (-3p^7 + 51p^8 - 189p^9 + 292p^{10} - 221p^{11} + 82p^{12} - 12p^{13})\rho^9 \quad (17u)$$

$$1/I^{(RRW)}(p_{13}) \propto 1 - 3p^4\rho^5 + (-p^4 - 7p^5 + 9p^6 - 2p^7)\rho^6 + (-6p^5 + 5p^6 + 22p^7 - 21p^8 + 5p^9)\rho^7 + (-11p^6 + 59p^7 - 72p^8 + 7p^9 + 23p^{10} - 8p^{11})\rho^8 + (-6p^7 + 70p^8 -$$

$239p^9 + 355p^{10} - 260p^{11} + 93p^{12} - 13p^{13})\rho^9$ (17v)

$1/\mathrm{I}^{(\mathrm{RRW})}(p_{14}) \propto 1 + (-2p^3 - p^4)\rho^5 + (-4p^4 + 2p^5 + p^6)\rho^6 + (-5p^5 + 13p^6 + 5p^7 - 11p^8 + 3p^9)\rho^7 + (-6p^6 + 47p^7 - 80p^8 + 40p^9 - 3p^{11})\rho^8 + (-3p^7 + 40p^8 - 151p^9 + 240p^{10} - 185p^{11} + 69p^{12} - 10p^{13})\rho^9.$ (17w)

The corresponding values are listed in Table 4 when the steady-state availabilities are taken into account. The variations of the RRW factors are displayed in Figures 8 and 9, when one of the nodes or links' availabilities takes its steady-state value. Here again, the variation in rankings do not change much with p and ρ. Components of comparable importance come in groups.

Table 4. Risk Reduction Worth index for steady-state availabilities (0.85 for links and 0.91 for nodes)

Component	RRW importance factor
ρ_S	3.9052323127976881318
ρ_T	3.9052323127976881318
p_{14}	3.3677664038955527548
p_1	3.3518949037359805525
p_4	3.3446224479672253385
ρ_A	3.3408987502793014142
p_{10}	3.3046628992354096887
ρ_D	3.2953218912037175271
ρ_C	3.2849093476298736721
ρ_G	3.2827442652289020558
p_{12}	3.2692028773907447531
ρ_E	3.2393301590606858029
p_2	3.2273225489277682830
ρ_B	3.2209082675792041710
p_{11}	3.2062380775632912257
p_3	3.1807385526134531842
p_8	3.1409489458363941740
p_6	3.1393344117655980623
ρ_F	3.1391216199070682353
p_{13}	3.1386200129119401608
p_7	3.1360649034374827045
p_9	3.1280495962004320680
p_5	3.1218175353137047655

One observes that the order of components has changed again with respect to the Birnbaum and RAW indices, and that the values are closer to each other. A few links gain places:

$\rho_S, \rho_T, p_{14}, p_1, p_4, \rho_A, p_{10}, \rho_D, \rho_C, \rho_G, p_{12}, \rho_E, p_2, \rho_B, p_{11}, p_3, p_8, p_6, \rho_F, p_{13}, p_7, p_9, p_5.$

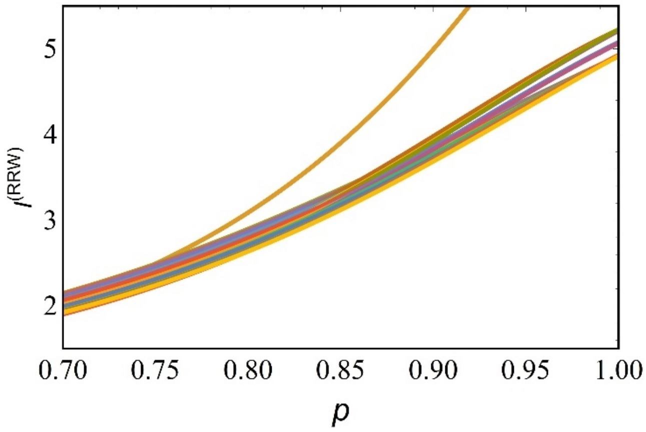

Figure 8. RRW factors for $\rho = \rho_\infty = 0.91$.

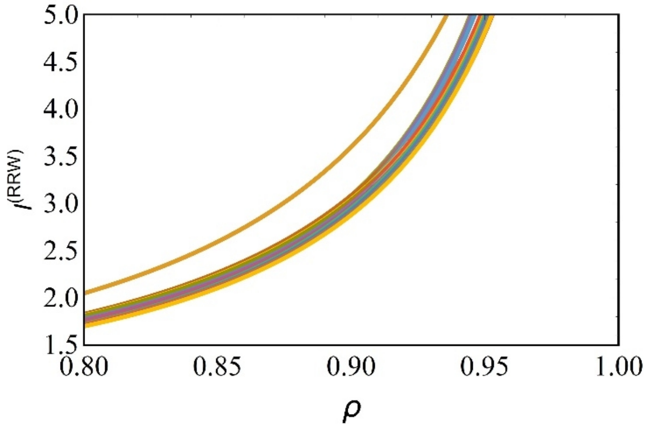

Figure 9. RRW factors for $p = p_\infty = 0.85$.

4.4. Time variation of performance indices

From the preceding Sections, one can observe that different performance indices may provide different results. However, within a given family (nodes or links), the rankings remain essentially unchanged, even when p and ρ change. One can therefore only expect a marginal influence of transient availabilities on the relative ranking of nodes and of links within their own categories. The relative influence of nodes and links will be addressed in the next Section.

4.5. Word of caution about uncertainties on effective failure and repair rates

Effective failure and repair rates are seldom known exactly. An uncertainty of 30% on the failure rates, and 15% on the repair rates would lead to uncertainties on the steady-state values given in (1) and (2); one should have instead

$$0.7875 \leq p_\infty \leq 0.9030, \quad (18)$$

$$0.8686 \leq \rho_\infty \leq 0.9432. \quad (19)$$

Considering uniform distributions of p and ρ in these intervals for each of the network's elements, one obtains performance indices in which the assumption of identical nodes/links is relaxed. For instance, the Birnbaum index's rank of the first link (p_1) is displayed in Figure 10 for a set of 50000 samples.

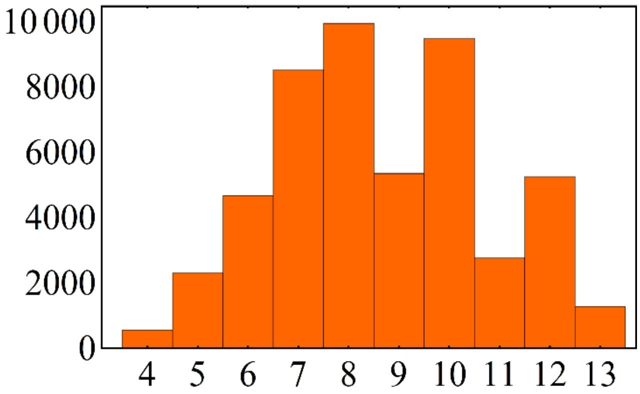

Figure 10. Histogram of the Birnbaum ranking for the first link (p_1) in the case of uncertainty over the steady-state availabilities (see (18) and (19)).

Even with such a limited uncertainty, it lies between 4 and 13. This should be kept in mind when proposing maintenance or resilience policies in network operations.

5. Relative influence of links and nodes on total unavailability

We have seen in the previous Sections that the ranks of nodes or edges of the network do not substantially vary for various performance indices. It does not mean, however, that transient effects cannot be crucial, as already shown in (Eid, 2021; Tanguy, 2022a). It is worthwhile to assess the relative importance of edges and nodes on the total unavailability, and investigate whether it varies with time. From equations (10) to (12), we have in steady-state

$$\frac{(U_\infty)_{\text{nodes}}}{(U_\infty)_{\text{total}}} = 0.62947776698761495023, \quad (20)$$

$$\frac{(U_\infty)_{\text{links}}}{(U_\infty)_{\text{total}}} = 0.19779257730620277452. \quad (21)$$

This shows that the main contribution to the total availability comes from nodes. We now have to check that such a claim is not invalidated at shorter times.

5.1. Case of exponential distributions

When all the failure and repair rates obey exponential distributions, $p(t)$ and $\rho(t)$ are given by (4) and (5). Using equations (7), (8), and (9), it is possible to plot the variations with time of the two ratios, displayed in Figure 11.

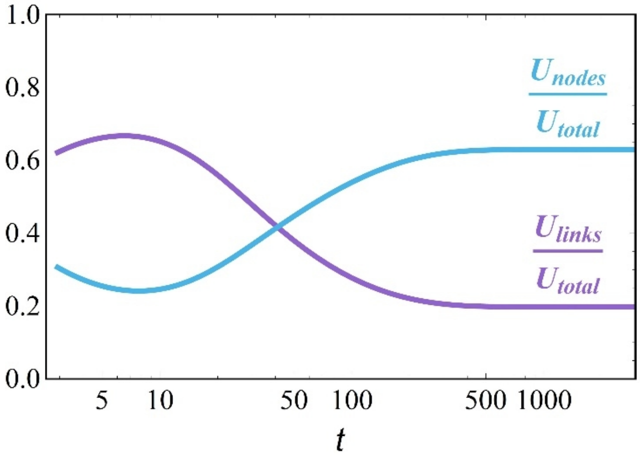

Figure 11. Time variation of the unavailability ratios for nodes (blue) and links (violet) for exponential distributions.

The striking feature of Figure 11 is that the ratios of unavailabilities strongly change during the mission time. The conclusion derived from the use of steady-state values may therefore be misleading: links actually may contribute more to the total unavailability.

5.2. Case of gamma ($\alpha = 2$) failure distributions

When all the failure times follow gamma distribution with $\alpha = 2$, it is possible to calculate analytically the exact values of $p(t)$ and $\rho(t)$:

$$p(t) = \frac{85}{100} + \frac{15}{100} e^{-\frac{29\,t}{200}} \cdot \left[\cos\left(\frac{\sqrt{119}\,t}{200}\right) + \frac{29}{\sqrt{119}} \sin\left(\frac{\sqrt{119}\,t}{200}\right)\right], \quad (22)$$

$$\rho(t) = \frac{91}{100} + \frac{9}{100} e^{-\frac{127\,t}{20000}} \cdot \left[20 \cosh\left(\frac{\sqrt{1729}\,t}{20000}\right) + \frac{127}{\sqrt{1729}} \sinh\left(\frac{\sqrt{1729}\,t}{20000}\right)\right]. \quad (23)$$

Using (22) and (23) in equations (7), (8), and (9), one can plot the variations with time of the two ratios, as shown in Figure 12.

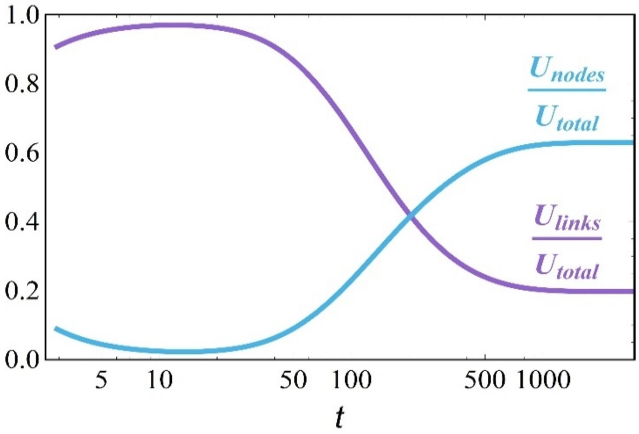

Figure 12. Time variation of the unavailability ratios for nodes (blue) and links (violet) for failure time gamma distribution with $\alpha = 2$.

5.3. Case of gamma ($\alpha_{\text{links}} = 20$, $\alpha_{\text{nodes}} = 10$) failure distributions

When the failure times follow gamma distribution with $\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$, it is still possible (Tanguy et al., 2019) to express analytically $p(t)$ and $\rho(t)$, now given by

$$p(t) = 0.85 + e^{-1.14336\,t}\,(0.310984\cos(0.0836738\,t) + 0.0473057\sin(0.0836738\,t)) + e^{-0.0285359\,t}\,(-0.293937\cos(0.15934\,t) + 0.103769\sin(0.15934\,t)) + e^{-1.09358\,t}\,(0.283138\cos(0.243529\,t) + 0.137739\sin(0.243529\,t)) + e^{-0.107054\,t}\,(-0.260862\cos(0.308935\,t) + 0.187246\sin(0.308935\,t)) + e^{-0.998411\,t}\,(0.229916\cos(0.381574\,t) + 0.216012\sin(0.381574\,t)) + e^{-0.225505\,t}\,(-0.199202\cos(0.432857\,t) + 0.253279\sin(0.432857\,t)) + e^{-0.86625\,t}\,(0.156045\cos(0.485426\,t) + 0.275241\sin(0.485426\,t)) + e^{-0.373866\,t}\,(-0.118065\cos(0.51825\,t) + 0.298392\sin(0.51825\,t)) + e^{-0.708737\,t}\,(0.0680901\cos(0.545738\,t) + 0.310262\sin(0.545738\,t)) + e^{-0.539697\,t}(-0.0261074\cos(0.55704\,t) + 0.318121\sin(0.55704\,t)) \qquad (24)$$

$$\rho(t) = 0.91 + e^{-0.0176533\,t}\,(0.172709\cos(0.00253813\,t) + 0.0507124\sin(0.00253813\,t)) + e^{-0.00143016\,t}\,(-0.151426\cos(0.00487063\,t) + 0.0973163\sin(0.00487063\,t)) + e^{-0.0149088\,t}\,(0.117875\cos(0.00680855\,t) + 0.136036\sin(0.00680855\,t)) + e^{-0.00526658\,t}\,(-0.0747747\cos(0.00819487\,t) + 0.163735\sin(0.00819487\,t)) + e^{-0.0102912\,t}\,(0.0256167\cos(0.0089173\,t) + 0.17817\sin(0.0089173\,t)) \qquad (25)$$

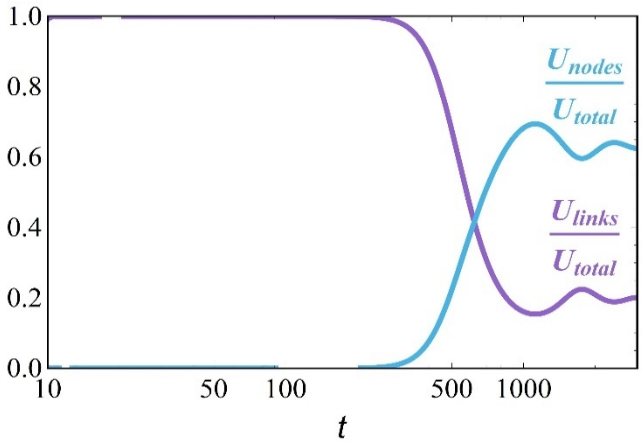

Figure 13. Time variation of the unavailability ratios for nodes (blue) and links (violet) for failure time gamma distribution ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$).

5.4. Time behaviors of aggregate unavailabilities

As Figure 13 exhibits oscillations, it is worth investigating the time dependence of all three unavailabilities defined in (7), (8), and (9). The various configurations corresponding to Sections 5.1, 5.2, and 5.3 are displayed in Figures 14 to 16. While in the first two cases the unavailabilities increase with time and reach their long-time limits from below, in the last case, there are overshoots. This means that the steady-state values of U_{total}, U_{links}, and U_{nodes} underestimate the true one during the mission time.

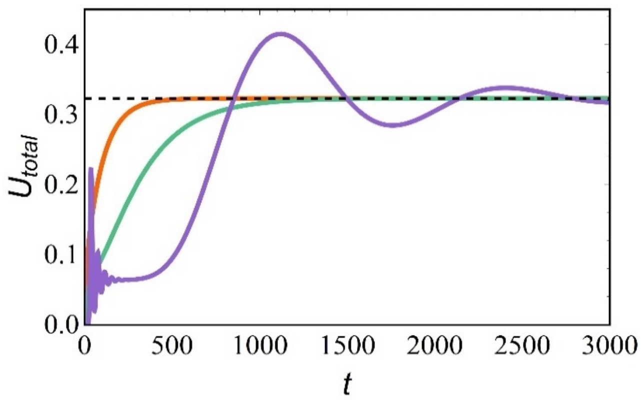

Figure 14. Time variation of U_{total} for different configurations: exponentials (orange); gamma ($\alpha = 2$) (green); ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$) (violet); the steady-state limit is the black dashed line.

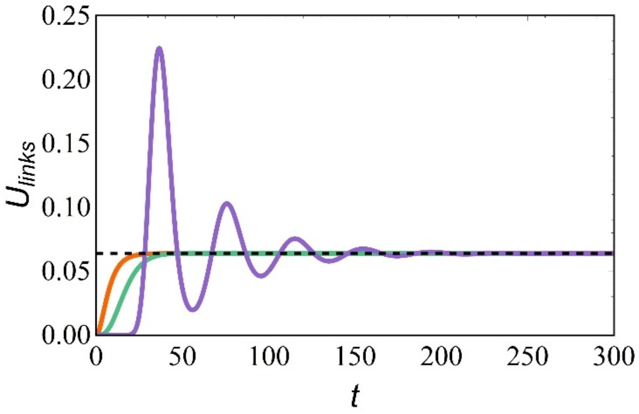

Figure 15. Time variation of U_{links} for different configurations: exponentials (orange); gamma (α = 2) (green); (α_{links} = 20 and α_{nodes} = 10) (violet); the steady-state limit is the black dashed line.

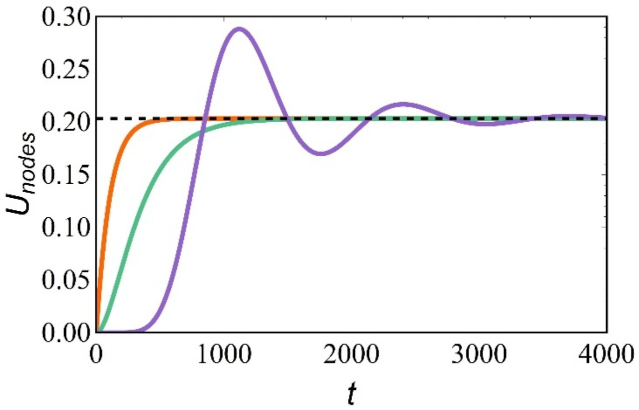

Figure 16. Time variation of U_{nodes} for different configurations: exponentials (orange); gamma (α = 2) (green); (α_{links} = 20 and α_{nodes} = 10) (violet); the steady-state limit is the black dashed line.

The lesson to be learned is that it is not always safe to consider the steady-state availabilities of components to assess the end-to-end availability of a system, all the more so in the first phases of operation (the mission time).

6. All-terminal reliability of Walter configuration

For the sake of comparison with the two-terminal availability results given in the preceding Sections, we now consider the all-terminal availability for the same configuration shown in Figure 1. Similar studies on another configuration have already been performed (Eid, 2021; Tanguy, 2022a), where the all-terminal availability has been shown to attain values smaller than its steady state-limit.

6.1. All-terminal availabilities

The probability that all nodes of the systems are connected implies that all nodes must be operational. This means that the all-terminal reliability or availability can be factorized as follows

$$\text{Rel}_\text{A}(\text{total}) = \text{Rel}_\text{A}(\text{links})\,\text{Rel}_\text{A}(\text{nodes}) \qquad (26)$$

$\text{Rel}_\text{A}(\text{nodes})$ is very simple, since it corresponds to a series system

$$\text{Rel}_\text{A}(\text{nodes}) = \rho_S\,\rho_T\,\rho_A\,\rho_B\,\rho_C\,\rho_D\,\rho_E\,\rho_F\,\rho_G. \qquad (27)$$

The calculation of $\text{Rel}_\text{A}(\text{links})$ is more complicated. When individual availabilities are considered (from p_1 to p_{14}), the final expression totals 2856 terms. Assuming that all links are identical leads to a much simpler formula

$$\text{Rel}_\text{A}(\text{links}) = 647p^8 - 2862p^9 + 5360p^{10} - 5430p^{11} + 3134p^{12} - 976p^{13} + 128p^{14} \qquad (28)$$

For identical nodes, (27) reduces to

$$\text{Rel}_\text{A}(\text{nodes}) = \rho^9. \qquad (29)$$

6.2. Steady-state unavailabilities

Here again it may be simpler to deal with the all-terminal unavailability $U_A = 1 - \text{Rel}_\text{A}$, instead of Rel_A. Considering the steady-state values for nodes and links given in (1) and (2), we find (the subscripts ∞ indicate the steady-state value)

$$\left(U_{A,\infty}\right)_{\text{total}} = 0.60498385816355262, \qquad (30)$$

whereas

$$\left(U_{A,\infty}\right)_{\text{links}} = 0.076913686037659746, \qquad (31)$$

$$\left(U_{A,\infty}\right)_{\text{nodes}} = 0.5720701998702116. \qquad (32)$$

These values, larger than those in (10)–(12) since it is more difficult to ensure that all nodes are connected instead of a single pair of them, show that nodes contribute again to a greater unavailability of the system at very long times.

6.3. Birnbaum importance factor

For the all-terminal availability, the Birnbaum importance factor is necessarily the same for identical nodes, because of (26) and (27). Consequently, only the links' ranking will be considered in what follows. From the exact expression of the all-terminal availability, one gets the Birnbaum importance factors, the list of which is given below for the sake of completeness and for comparison with equations (13) in Section 4.1. Note that the common factor ρ^9 has been omitted, since it does not affect the relative ranking of links.

$$I^{(B)}(p_1) = 400p^7 - 1948p^8 + 3983p^9 - 4377p^{10} + 2726p^{11} - 912p^{12} + 128p^{13} \quad (33a)$$

$$I^{(B)}(p_2) = 367p^7 - 1845p^8 + 3863p^9 - 4315p^{10} + 2714p^{11} - 912p^{12} + 128p^{13} \quad (33b)$$

$$I^{(B)}(p_3) = 364p^7 - 1819p^8 + 3796p^9 - 4239p^{10} + 2674p^{11} - 904p^{12} + 128p^{13} \quad (33c)$$

$$I^{(B)}(p_4) = 440p^7 - 2115p^8 + 4261p^9 - 4608p^{10} + 2822p^{11} - 928p^{12} + 128p^{13} \quad (33d)$$

$$I^{(B)}(p_5) = 306p^7 - 1574p^8 + 3384p^9 - 3894p^{10} + 2530p^{11} - 880p^{12} + 128p^{13} \quad (33e)$$

$$I^{(B)}(p_6) = 295p^7 - 1536p^8 + 3335p^9 - 3866p^{10} + 2524p^{11} - 880p^{12} + 128p^{13} \quad (33f)$$

$$I^{(B)}(p_7) = 312p^7 - 1611p^8 + 3466p^9 - 3979p^{10} + 2572p^{11} - 888p^{12} + 128p^{13} \quad (33g)$$

$$I^{(B)}(p_8) = 343p^7 - 1733p^8 + 3655p^9 - 4123p^{10} + 2626p^{11} - 896p^{12} + 128p^{13} \quad (33h)$$

$$I^{(B)}(p_9) = 351p^7 - 1760p^8 + 3689p^9 - 4142p^{10} + 2630p^{11} - 896p^{12} + 128p^{13} \quad (33i)$$

$$I^{(B)}(p_{10}) = 400p^7 - 1948p^8 + 3983p^9 - 4377p^{10} + 2726p^{11} - 912p^{12} + 128p^{13} \quad (33j)$$

$$I^{(B)}(p_{11}) = 365p^7 - 1834p^8 + 3843p^9 - 4300p^{10} + 2710p^{11} - 912p^{12} + 128p^{13} \quad (33k)$$

$$I^{(B)}(p_{12}) = 416p^7 - 2042p^8 + 4178p^9 - 4566p^{10} + 2814p^{11} - 928p^{12} + 128p^{13} \quad (33l)$$

$$I^{(B)}(p_{13}) = 377p^7 - 1878p^8 + 3903p^9 - 4336p^{10} + 2718p^{11} - 912p^{12} + 128p^{13} \quad (33m)$$

$$I^{(B)}(p_{14}) = 440p^7 - 2115p^8 + 4261p^9 - 4608p^{10} + 2822p^{11} - 928p^{12} + 128p^{13}. \quad (33n)$$

We can rank the 14 links according to the Birnbaum index when the availabilities have their steady-state values, deduced from (1), 0.85. The list, by decreasing importance, is

$p_{14}, p_4, p_{10}, p_1, p_{12}, p_{13}, p_3, p_9, p_{11}, p_2, p_8, p_5, p_7, p_6.$

Note that the ordering of the fourteen links is not the same as for the two-terminal availability case in Section 4.1. However, the changes are marginal, since the same groups of links aggregate at the beginning and at the end of the lists.

6.4. RAW and RRW performance indices

It is possible to compute the Risk Achievement Worth and the Risk Reduction Worth without any difficulty. Their expression are not given here, it turns out that the ranking of all links is exactly the same as that for the Birnbaum index.

6.5. Fussell-Vesely performance index

By definition, the Fussell-Vesely performance index is the probability that at least one minimal cutset containing element i fails ($\Pr(D_i)$), provided that the whole system fails. In the notation of (Rausand & Høyland, 2004), namely $\Pr(C) \equiv U_A$,

$$I^{(FV)}(p_i) = \frac{\Pr(D_i)}{\Pr(C)}. \quad (34)$$

The calculations of these indices are more involved, since they cannot be directly obtained from the availability expression (*formally* similar to the structure function). One must first start to identify the minimal cutsets, which is another potentially complex task, especially when the number of system elements increases. Here, there are 75 minimal cutsets:

$\{p_1, p_{10}\}, \{p_4, p_{14}\}, \{p_1, p_5, p_{11}\}, \{p_5, p_{10}, p_{11}\}, \{p_2, p_6, p_{11}\}, \{p_2, p_7, p_{12}\}, \{p_3, p_8, p_{12}\}, \{p_3, p_4, p_9\}, \{p_3, p_9, p_{14}\}, \{p_8, p_9, p_{13}\}, \{p_4, p_{12}, p_{13}\}, \{p_{12}, p_{13}, p_{14}\}, \{p_1, p_2, p_5, p_6\}, \{p_2, p_5, p_6, p_{10}\}, \{p_2, p_3, p_7, p_8\}, \{p_2, p_4, p_7, p_{13}\}, \{p_2, p_7, p_{13}, p_{14}\}, \{p_6, p_7, p_{11}, p_{12}\}, \{p_3, p_4, p_8, p_{13}\}, \{p_3, p_8, p_{13}, p_{14}\},$

$\{p_3, p_9, p_{12}, p_{13}\}, \{p_4, p_8, p_9, p_{12}\}, \{p_8, p_9, p_{12}, p_{14}\},$
$\{p_1, p_5, p_6, p_7, p_{12}\}, \{p_5, p_6, p_7, p_{10}, p_{12}\},$
$\{p_2, p_3, p_7, p_9, p_{13}\}, \{p_2, p_4, p_7, p_8, p_9\},$
$\{p_2, p_7, p_8, p_9, p_{14}\}, \{p_3, p_6, p_7, p_8, p_{11}\},$
$\{p_4, p_6, p_7, p_{11}, p_{13}\}, \{p_6, p_7, p_{11}, p_{13}, p_{14}\},$
$\{p_1, p_3, p_5, p_6, p_7, p_8\}, \{p_1, p_4, p_5, p_6, p_7, p_{13}\},$
$\{p_1, p_5, p_6, p_7, p_{13}, p_{14}\}, \{p_3, p_5, p_6, p_7, p_8, p_{10}\},$
$\{p_4, p_5, p_6, p_7, p_{10}, p_{13}\}, \{p_5, p_6, p_7, p_{10}, p_{13}, p_{14}\},$
$\{p_3, p_6, p_7, p_9, p_{11}, p_{13}\}, \{p_4, p_6, p_7, p_8, p_9, p_{11}\},$
$\{p_6, p_7, p_8, p_9, p_{11}, p_{14}\}, \{p_1, p_3, p_4, p_5, p_6, p_7, p_8\},$
$\{p_1, p_3, p_5, p_6, p_7, p_8, p_{14}\},$
$\{p_1, p_3, p_5, p_6, p_7, p_8, p_{13}\},$
$\{p_1, p_3, p_5, p_6, p_7, p_8, p_9\},$
$\{p_1, p_3, p_4, p_5, p_6, p_7, p_{13}\},$
$\{p_1, p_3, p_5, p_6, p_7, p_{13}, p_{14}\},$
$\{p_1, p_3, p_5, p_6, p_7, p_9, p_{13}\},$
$\{p_1, p_4, p_5, p_6, p_7, p_8, p_{13}\},$
$\{p_1, p_5, p_6, p_7, p_8, p_{13}, p_{14}\},$
$\{p_1, p_4, p_5, p_6, p_7, p_8, p_9\},$
$\{p_1, p_5, p_6, p_7, p_8, p_9, p_{14}\},$
$\{p_1, p_4, p_5, p_6, p_7, p_9, p_{13}\},$
$\{p_1, p_5, p_6, p_7, p_9, p_{13}, p_{14}\},$
$\{p_1, p_3, p_6, p_7, p_9, p_{11}, p_{13}\},$
$\{p_1, p_4, p_6, p_7, p_8, p_9, p_{11}\},$
$\{p_1, p_6, p_7, p_8, p_9, p_{11}, p_{14}\},$
$\{p_3, p_4, p_5, p_6, p_7, p_8, p_{10}\},$
$\{p_3, p_5, p_6, p_7, p_8, p_{10}, p_{14}\},$
$\{p_3, p_5, p_6, p_7, p_8, p_{10}, p_{13}\},$
$\{p_3, p_5, p_6, p_7, p_8, p_9, p_{10}\},$
$\{p_3, p_4, p_5, p_6, p_7, p_{10}, p_{13}\},$
$\{p_3, p_5, p_6, p_7, p_{10}, p_{13}, p_{14}\},$
$\{p_3, p_5, p_6, p_7, p_9, p_{10}, p_{13}\},$
$\{p_4, p_5, p_6, p_7, p_8, p_{10}, p_{13}\},$
$\{p_5, p_6, p_7, p_8, p_{10}, p_{13}, p_{14}\},$
$\{p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\},$
$\{p_5, p_6, p_7, p_8, p_9, p_{10}, p_{14}\},$
$\{p_4, p_5, p_6, p_7, p_9, p_{10}, p_{13}\},$
$\{p_5, p_6, p_7, p_9, p_{10}, p_{13}, p_{14}\},$
$\{p_3, p_6, p_7, p_9, p_{10}, p_{11}, p_{13}\},$
$\{p_4, p_6, p_7, p_8, p_9, p_{10}, p_{11}\},$
$\{p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{14}\},$
$\{p_3, p_5, p_6, p_7, p_9, p_{11}, p_{13}\},$
$\{p_4, p_5, p_6, p_7, p_8, p_9, p_{11}\},$
$\{p_5, p_6, p_7, p_8, p_9, p_{11}, p_{14}\}.$

The numbers of cutsets to which the link corresponding to p_i occurs are respectively 23, 10, 29, 24, 41, 53, 57, 36, 33, 23, 19, 10, 35, and 24 for $p_1, p_2, \ldots, p_{14}$. One then has to compute the probabilities $\Pr(D_i)$, which are given by the polynomials in equations (35a) to (35n):

$$I^{(FV)}(p_1) \propto 1 - p - 5p^3 + 9p^4 + p^5 - 11p^6 - 4p^7 + 37p^8 - 47p^9 + 15p^{10} + 22p^{11} - 27p^{12} + 12p^{13} - 2p^{14} \quad (35a)$$

$$I^{(FV)}(p_2) \propto 1 - p - p^2 + p^5 + 8p^7 - 9p^8 - 8p^9 + 5p^{10} + 20p^{11} - 26p^{12} + 12p^{13} - 2p^{14} \quad (35b)$$

$$I^{(FV)}(p_3) \propto 1 - p - 4p^3 + 5p^4 + 3p^5 - 10p^6 + 13p^7 - 2p^8 - 20p^9 + 17p^{10} + 10p^{11} - 21p^{12} + 11p^{13} - 2p^{14} \quad (35c)$$

$$I^{(FV)}(p_4) \propto 1 - p - 7p^4 + 15p^5 - 10p^6 + 12p^8 - 20p^9 + 3p^{10} + 24p^{11} - 27p^{12} + 12p^{13} - 2p^{14} \quad (35d)$$

$$I^{(FV)}(p_5) \propto 1 - p - p^2 - 4p^3 + 11p^4 - 3p^5 - 8p^6 - 2p^7 + 33p^8 - 58p^9 + 45p^{10} - 6p^{11} - 15p^{12} + 10p^{13} - 2p^{14} \quad (35e)$$

$$I^{(FV)}(p_6) \propto 1 - p - 2p^2 + p^3 + 2p^4 - p^5 + 2p^6 + 4p^7 - 14p^8 + 3p^9 + 9p^{10} + 4p^{11} - 16p^{12} + 10p^{13} - 2p^{14} \quad (35f)$$

$$I^{(FV)}(p_7) \propto 1 - p - p^2 - p^3 + 5p^5 + 3p^6 - 6p^7 - 6p^8 + p^9 + 9p^{10} + 4p^{11} - 16p^{12} + 10p^{13} - 2p^{14} \quad (35g)$$

$$I^{(FV)}(p_8) \propto 1 - p - 5p^3 + 7p^4 + 3p^5 - 11p^6 + 13p^7 - 8p^8 - 5p^9 + 3p^{10} + 16p^{11} - 22p^{12} + 11p^{13} - 2p^{14} \quad (35h)$$

$$I^{(FV)}(p_9) \propto 1 - p - 4p^3 + 2p^4 + 15p^5 - 25p^6 + 13p^7 + 12p^8 - 28p^9 + 14p^{10} + 14p^{11} - 22p^{12} + 11p^{13} - 2p^{14} \quad (35i)$$

$$I^{(FV)}(p_{10}) \propto 1 - p - 5p^3 + 9p^4 + p^5 - 11p^6 - 4p^7 + 37p^8 - 47p^9 + 15p^{10} + 22p^{11} - 27p^{12} + 12p^{13} - 2p^{14} \quad (35j)$$

$$I^{(FV)}(p_{11}) \propto 1 - p - p^2 - p^3 + 3p^4 - 3p^6 + 6p^7 + p^8 - 16p^9 + 7p^{10} + 20p^{11} - 26p^{12} + 12p^{13} - 2p^{14} \quad (35k)$$

$$I^{(FV)}(p_{12}) \propto 1 - p - p^3 - 2p^4 + p^5 + 7p^6 + p^7 - 7p^8 - 8p^9 + 5p^{10} + 20p^{11} - 26p^{12} + 12p^{13} - 2p^{14} \quad (35l)$$

$$I^{(FV)}(p_{13}) \propto 1 - p - p^3 - 7p^4 + 18p^5 - 10p^6 - p^7 + 6p^8 - 17p^9 + 13p^{10} + 11p^{11} - 21p^{12} + 11p^{13} - 2p^{14} \quad (35m)$$

$$I^{(FV)}(p_{14}) \propto 1 - p - 7p^4 + 15p^5 - 10p^6 + 12p^8 - 20p^9 + 3p^{10} + 24p^{11} - 27p^{12} + 12p^{13} - 2p^{14}. \quad (35n)$$

The values of the $\Pr(D_i)/(1-p)^2$ are displayed in Figure 17. The modifications of ranks when p is modified are marginal, apart from those of p_{10} and p_1, the importance of which decreases when p goes below 0.6.

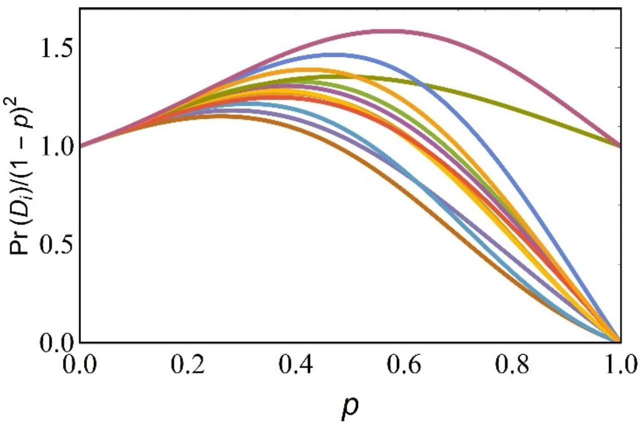

Figure 17. $\Pr(D_i)/(1-p)^2$ values for $\rho = 0.91$.

Another noteworthy result is that, by contrast to what occurs for the two-terminal availability, the importance list of the links when $p = 0.85$ is exactly the same as that of Birnbaum, RAW, and RRW given in Section 6.3. This makes the cumbersome calculations of the Fussell-Vesely index of performance appear of limited interest, all the more so in not so small systems.

6.6. Relative influence of links and nodes on total all-terminal unavailability

One can perform for the all-terminal availability the same calculations of the relative roles of nodes and links on the total availability, as performed in Section 5. The results are displayed in Figures 18 to 20, and the all-terminal unavailabilities are again denoted by U_{total}, U_{links}, and U_{nodes} in order to shorten the notations. Note that the asymptotic limits are well separated at 0.127133451578582 for the links, and 0.945595807476167 for the nodes. When the failure and repair time distributions are exponentials (Figure 18), there is no crossing between the two curves. When gamma distributions are considered, the behaviours observed for the two-terminal availability (see Figures 12 and 13) reappear. Our conclusion is that for the two- or all-terminal availabilities, great caution should be exercised when identifying weak elements of the system based only on p_∞ and ρ_∞.

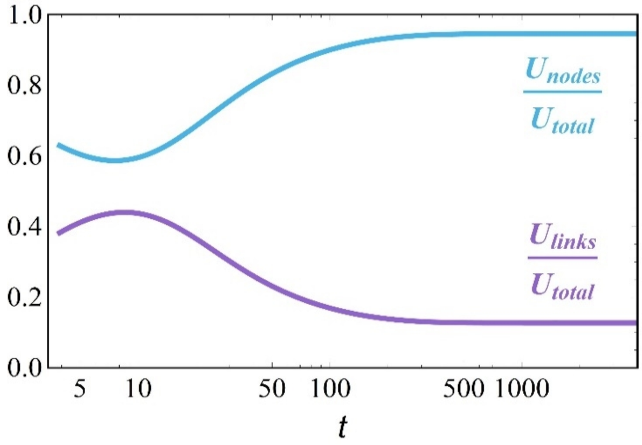

Figure 18. Time variation of the unavailability ratios for nodes (blue) and links (violet) for exponential distributions and the all-terminal availability case.

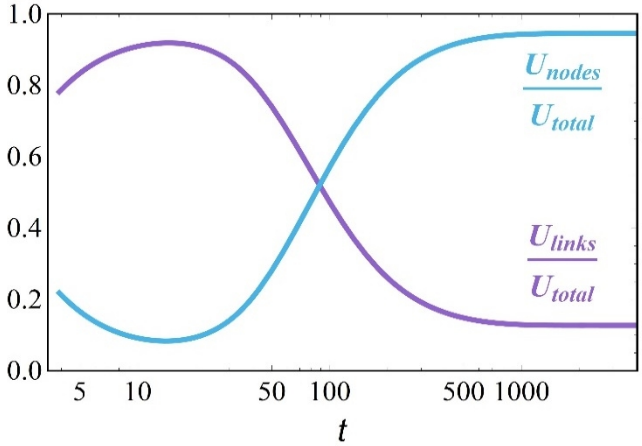

Figure 19. Time variation of the unavailability ratios for nodes (blue) and links (violet) for failure time gamma distribution with $\alpha = 2$ and the all-terminal availability case.

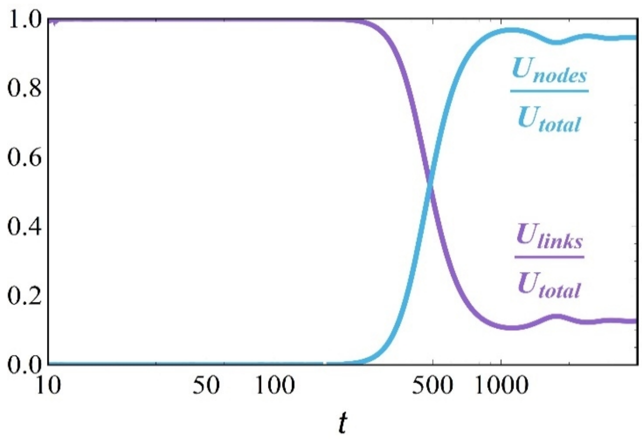

Figure 20. Time variation of the unavailability ratios for nodes (blue) and links (violet) for failure time gamma distribution ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$) and the all-terminal availability case.

6.7. Time behaviours of aggregate unavailabilities

In this Section, we study, as in Section 5.4, how the stationary regime is reached for various failure time distributions. The results are displayed on Figures 21 to 23.

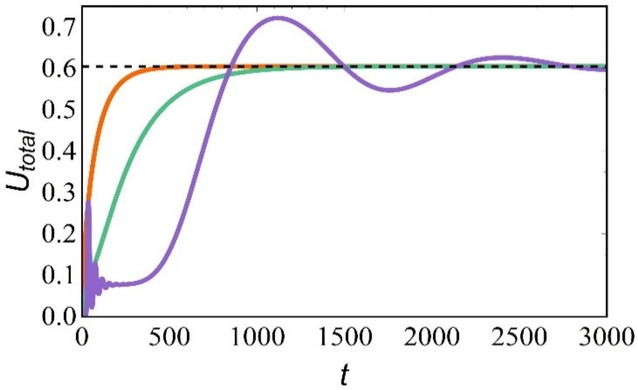

Figure 21. Time variation of U_{total} for different configurations: exponentials (orange); gamma ($\alpha = 2$) (green); ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$) (violet); the steady-state limit is the black dashed line.

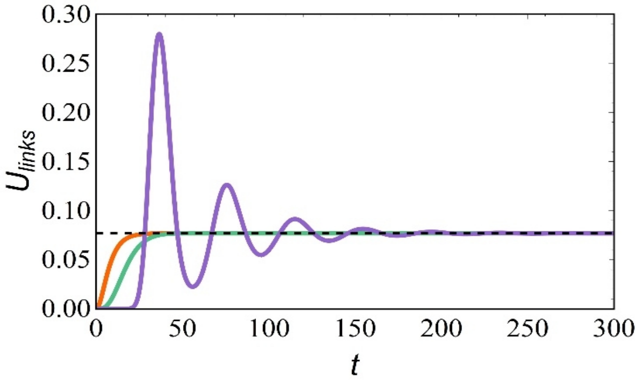

Figure 22. Time variation of U_{links} for different configurations: exponentials (orange); gamma ($\alpha = 2$) (green); ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$) (violet); the steady-state limit is the black dashed line.

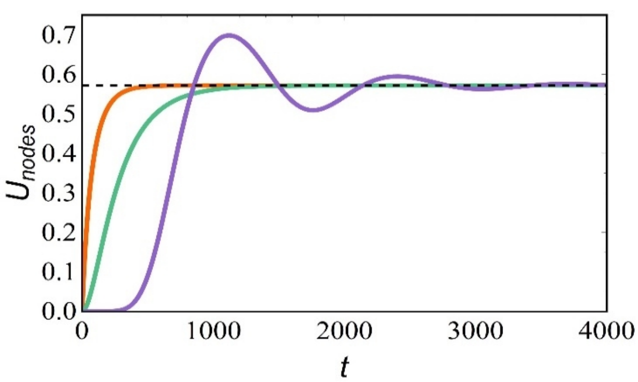

Figure 23. Time variation of U_{nodes} for different configurations: exponentials (orange); gamma ($\alpha = 2$) (green); ($\alpha_{\text{links}} = 20$ and $\alpha_{\text{nodes}} = 10$) (violet); the steady-state limit is the black dashed line.

One can observe the same behaviours as those observed for the two-terminal availability. The conclusions to be drawn are therefore the same: transient effects may be important and should not be cursorily overlooked. They might indeed lead to underestimates to the true unavailability of systems.

7. Conclusion

We have considered a meshed network proposed by Walter, Esch, and Limbourg (Walter et al., 2008), for which the performance measure is the end-to-end availability between a source and a destination. The exact expression of this availability has been obtained for arbitrary components' availabilities, which allowed the assessment of several performance indices such as Birnbaum, Risk Achievement Worth, Risk Reduction Worth, and Fussell-Vesely in order to know on which element(s) of the system maintenance and resilience practitioners should focus their attention.

Our most important result has been to demonstrate that transient effects can be significant. Maintenance and resilience studies often use the steady-state values of each element's availability. In the cases of non-steady-state availabilities and non-exponential failure distributions, the connection unavailability may be temporarily higher than its steady-state value. At different instants of the mission time, nodes and links could alternatively be held responsible for a majority of outages. The lesson learned from the present study is that there is no fixed guilty party. Great caution should therefore be exercised when trying to optimize the operation of a system without considering transient effects, which should not be overlooked.

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