

ON ASSOCIATIVE RATIONAL FUNCTIONS WITH ADDITIVE GENERATORS

KATARZYNA DOMAŃSKA

ABSTRACT

We consider the class of rational functions defined by the formula

$$F(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where φ is a homographic function and we describe all associative functions of the above form.

1. INTRODUCTION

The functional equation of the form

$$f(x + y) = F(f(x), f(y)),$$

where F is an associative rational function is called an addition formula. For the rational two-place real-valued function F given by

$$F(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where φ is a homographic function (such F is called a function with an additive generator), the addition formula has the form

$$\varphi(f(x + y)) = \varphi(f(x)) + \varphi(f(y))$$

and it is a conditional functional equation if the domain of φ is not equal to \mathbb{R} . Solutions of the above conditional equation are functions of the homographic type. Some results on such equations can be found in the article [2]. It seems to be interesting which homographic functions lead F of above form to be associative.

The following lemma will be useful in the sequel.

Lemma. *Let $A, B, C, D \in \mathbb{R}$ be given and let $AD \neq BC, C \neq 0$. For φ given by*

$$\varphi(x) = \frac{Ax + B}{Cx + D},$$

holds

$$\varphi^{-1}(\varphi(x) + \varphi(y)) = \frac{C(BC - 2AD)xy - AD^2(x + y) - BD^2}{AC^2xy + BC^2(x + y) + (2BC - AD)D}.$$

Proof. We have

$$\varphi(x) + \varphi(y) = \frac{2ACxy + (AD + BC)(x + y) + 2BD}{C^2xy + CD(x + y) + D^2}$$

and

$$\varphi^{-1}(x) = \frac{-Dx + B}{Cx - A}.$$

A simple calculation shows that the above equation holds true. \square

2. MAIN RESULTS

We proceed with a description of the class of rational functions with additive generators.

Theorem 1. *The following functions (with natural domains in question) are the only associative members of the class of rational functions with additive generators:*

$$F(x, y) = \frac{xy}{\alpha xy + x + y}, \quad \alpha \in \mathbb{R};$$

$$F(x, y) = \frac{x + y + 2\lambda xy}{1 - \lambda^2 xy}, \quad \lambda \neq 0;$$

$$F(x, y) = \frac{(1 - 2\alpha\beta)xy - \alpha\beta^2(x + y) - \beta^2}{\alpha xy + x + y + 2\beta - \alpha\beta^2}, \quad \alpha \in \mathbb{R}, \beta \neq 0.$$

Proof. Assume that F is associative and has the additive generator

$$\varphi(x) = \frac{Ax + B}{Cx + D},$$

where $A, B, C, D \in \mathbb{R}$ and $AD \neq BC$, $C \neq 0$.

We infer from Lemma that

$$F(x, y) = \frac{C(BC - 2AD)xy - AD^2(x + y) - BD^2}{AC^2xy + BC^2(x + y) + (2BC - AD)D} \quad (1)$$

First assume that $D = 0$. If $B = 0$ then $AD = BC$ in contradiction to the assumption. Hence $B \neq 0$. Putting $D = 0$ in (1) we obtain

$$F(x, y) = \frac{BC^2xy}{AC^2xy + BC^2(x + y)}$$

Consequently putting

$$\alpha = \frac{A}{B},$$

we infer that

$$F(x, y) = \frac{xy}{\alpha xy + x + y}, \quad \alpha \in \mathbb{R}.$$

Let now $D \neq 0$. We get

$$\varphi(x) = \frac{\hat{A}x + \hat{B}}{\hat{C}x + 1},$$

where

$$\hat{A} = \frac{A}{D}, \hat{B} = \frac{B}{D}, \hat{C} = \frac{C}{D}.$$

Replacing \hat{A} by A , \hat{B} by B and \hat{C} by C we have by (1)

$$F(x, y) = \frac{C(BC - 2A)xy - A(x + y) - B}{AC^2xy + BC^2(x + y) + 2BC - A}. \quad (2)$$

From Theorem 1. (proved in article [1]) we obtain that every F of the above form is associative. Consequently, in the case $B = 0$ (if $B = 0$ then $A \neq 0$) we have

$$F(x, y) = \frac{-2ACxy - A(x + y)}{AC^2xy - A} = \frac{x + y + 2\lambda xy}{1 - \lambda^2 xy}$$

with $\lambda = C$.

In the case $B, C \neq 0$ in (2) we have

$$\begin{aligned} F(x, y) &= \frac{(1 - 2\frac{A}{B} \cdot \frac{1}{C})xy - \frac{A}{B} \cdot \frac{1}{C^2}(x + y) - \frac{1}{C^2}}{\frac{A}{B}xy + x + y + 2\frac{1}{C} - \frac{A}{B} \cdot \frac{1}{C^2}} = \\ &= \frac{(1 - 2\alpha\beta)xy - \alpha\beta^2(x + y) - \beta^2}{\alpha xy + x + y + 2\beta - \alpha\beta^2} \end{aligned}$$

with $\alpha = \frac{A}{B}$, $\beta = \frac{1}{C}$.

It is easy to check (see Theorem 2 or Theorem 1 in [1]) that each of the function above yields a rational associative function. Thus the proof has been completed. \square

Now we indicate homographic functions φ which by the formula

$$F(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$$

lead to associative functions F .

Theorem 2. *For following homographic functions (with natural domains in question) we obtain all rational associative functions with an additive generator*

$$\begin{aligned} \varphi(x) &= \frac{1}{x} \\ \varphi(x) &= \frac{\alpha x + 1}{\alpha x} \end{aligned}$$

$$\begin{aligned}\varphi(x) &= \frac{\lambda}{x + \lambda} \\ \varphi(x) &= \frac{\lambda x}{\lambda x + 1} \\ \varphi(x) &= \beta \frac{\alpha x + 1}{x + \beta},\end{aligned}$$

where $\alpha, \beta, \lambda \in \mathbb{R} \setminus \{0\}$ are arbitrary constants.

Proof. It is easy to check that each of the function above yields a generator to the rational function which is associative. Moreover they are generators of

$$\begin{aligned}F(x, y) &= \frac{xy}{x + y}; \\ F(x, y) &= \frac{xy}{\alpha xy + x + y}, \quad \alpha \neq 0; \\ F(x, y) &= \frac{xy - \lambda^2}{x + y + 2\lambda}, \quad \lambda \neq 0; \\ F(x, y) &= \frac{x + y + 2\lambda xy}{1 - \lambda^2 xy}, \quad \lambda \neq 0; \\ F(x, y) &= \frac{(1 - 2\alpha\beta)xy - \alpha\beta^2(x + y) - \beta^2}{\alpha xy + x + y + 2\beta - \alpha\beta^2}, \quad \alpha, \beta \neq 0.\end{aligned}$$

respectively. From Theorem 1 it is the assertion of Theorem 2. Thus the proof has been completed. \square

REFERENCES

- [1] K. Domańska, *An analytic description of the class of rational associative functions*, Annales Universitatis Paedagogicae Cracovienis Studia Mathematica 11 (2012), 111-122.
- [2] K. Domańska, *On some addition formulas for homographic type functions*, Scientific Issues, Jan Długosz University in Częstochowa, Mathematics XVII, (2012) 17-24.

Katarzyna Domańska

JAN DŁUGOSZ UNIVERSITY,

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE,

42-200 CZĘSTOCHOWA, AL. ARMII KRAJOWEJ 13/15, POLAND

E-mail address: k.domanska@ajd.czest.pl