

# The pointwise completeness and the pointwise degeneracy of fractional descriptor discrete-time linear systems

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**Abstract.** The Drazin inverse of matrices is applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor linear discrete-time systems. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the fractional descriptor linear discrete-time systems are established. It is shown that every fractional descriptor linear discrete-time systems is not pointwise complete and it is pointwise degenerated in one step (for  $i = 1$ ).

**Key words:** fractional, descriptor, pointwise completeness, pointwise degeneracy.

## 1. Introduction

A dynamical system described by homogenous equation is called pointwise complete if every final state of the system can be reached by suitable choice of its initial state. A system, which is not pointwise complete is called pointwise degenerated. The pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays have been investigated in [2, 3, 9, 15, 17], the pointwise completeness of linear discrete-time cone systems with delays in [18] and of fractional linear systems are presented in [1, 9, 10]. The pointwise completeness and pointwise degeneracy of standard and positive hybrid systems described by the general model have been analyzed in [7] and of positive linear systems with state-feedbacks in [8]. Some new results in fractional systems have been given in [4, 13, 14 16].

The Drazin inverse of matrices has been applied to analysis of the pointwise completeness and of the pointwise degeneracy of the descriptor linear continuous-time and discrete-time systems in [5] and for fractional standard and descriptor linear continuous-time systems in [12].

In this paper the Drazin inverse of matrices will be applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor discrete-time linear systems.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the fractional descriptor linear discrete-time systems and the Drazin inverse of matrices are recalled. The pointwise completeness of the fractional descriptor linear discrete-time systems is investigated in Section 3 and the pointwise degeneracy in Section 4. Concluding remarks are given in section 5. The considerations are illustrated by numerical example of fractional linear discrete-time system.

The following notation will be used:  $\mathfrak{R}$  – the set of real numbers,  $\mathfrak{R}^{n \times m}$  – the set of  $n \times m$  real matrices,  $\mathfrak{R}_+^{n \times m}$  – the set of  $n \times m$  real matrices with nonnegative entries and  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ,  $I_n$  – the  $n \times n$  identity matrix.  $\text{Im } P$  is the image of the operator (matrix)  $P$ .

## 2. Fractional autonomous descriptor discrete-time linear systems and their solutions

Consider the fractional autonomous descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i, \quad i \in Z_+ = \{0, 1, \dots\}, \quad (1)$$

where,  $x_i \in \mathfrak{R}^n$  is the state vector  $E, A \in \mathfrak{R}^{n \times n}$  and

$$\Delta^\alpha x_i = \sum_{j=0}^i (-1)^j \binom{\alpha}{j} x_{i-j} \quad (2a)$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1)\dots\alpha(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots \end{cases} \quad (2b)$$

is the fractional  $\alpha \in \mathfrak{R}$  order difference of  $x_i$ .

Substituting (2) into (1) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} c_j Ex_{i-j+1}, \quad (3a)$$

where

$$A_\alpha = A + E\alpha, \quad c_j = (-1)^j \binom{\alpha}{j}. \quad (3b)$$

It is assumed that  $\det E = 0$  and

$$\det[Ez - A_\alpha] \neq 0 \quad \text{for some } z \in \mathbb{C}, \quad (4)$$

where  $\mathbb{C}$  is the field of complex numbers.

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Assuming that for some chosen  $c \in \mathbb{C}$ ,  $\det[Ec - A_\alpha]^{-1} \neq 0$  and premultiplying (3a) by  $[Ec - A_\alpha]^{-1}$ , we obtain

$$\bar{E}x_{i+1} = \bar{A}_\alpha x_i + \sum_{j=2}^{i+1} c_j \bar{E}x_{i-j+1}, \quad (5a)$$

where

$$\bar{E} = [Ec - A_\alpha]^{-1}E, \quad \bar{A}_\alpha = [Ec - A_\alpha]^{-1}A_\alpha. \quad (5b)$$

Note that the equations (3a) and (5a) have the same solution  $x_i, i \in \mathbb{Z}_+$ .

**Definition 1.** [6, 11] The smallest nonnegative integer  $q$  is called the index of the matrix  $\bar{E} \in \mathbb{R}^{n \times n}$  if

$$\text{rank } \bar{E}^q = \text{rank } \bar{E}^{q+1}. \quad (6)$$

**Definition 2.** [6, 11] A matrix  $\bar{E}^D$  is called the Drazin inverse of  $\bar{E} \in \mathbb{R}^{n \times n}$  if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \quad (7a)$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \quad (7b)$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \quad (7c)$$

where  $q$  is the index of  $\bar{E}$  defined by (6).

The Drazin inverse  $\bar{E}^D$  of a square matrix  $\bar{E}$  always exists and is unique [5, 6]. If  $\det \bar{E} \neq 0$  then  $\bar{E}^D = \bar{E}^{-1}$ . Some methods for computation of the Drazin inverse are given in [5, 9].

**Theorem 1.** The matrices  $\bar{E}$  and  $\bar{A}_\alpha$  defined by (5b) satisfy the following equalities

$$1. \quad \bar{A}_\alpha \bar{E} = \bar{E} \bar{A}_\alpha \quad \text{and} \quad \bar{A}_\alpha^D \bar{E} = \bar{E} \bar{A}_\alpha^D, \quad \bar{E}^D \bar{A}_\alpha = \bar{A}_\alpha \bar{E}^D, \quad (8a)$$

$$\bar{A}_\alpha^D \bar{E}^D = \bar{E}^D \bar{A}_\alpha^D,$$

$$2. \quad \ker \bar{A}_\alpha \cap \ker \bar{E} = \{0\}, \quad (8b)$$

$$3. \quad \bar{E} = T \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1}, \quad \bar{E}^D = T \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad (8c)$$

$\det T \neq 0, J \in \mathbb{R}^{n_1 \times n_1}$  is nonsingular,  $N \in \mathbb{R}^{n_2 \times n_2}$  is nilpotent,  $n_1 + n_2 = n$ ,

Proof is given in [6].

**Theorem 2.** Let

$$P = \bar{E}\bar{E}^D, \quad (9a)$$

$$Q = \bar{E}^D\bar{A}_\alpha. \quad (9b)$$

Then:

$$1) \quad P^k = P \quad \text{for } k = 2, 3, \dots, \quad (10)$$

$$2) \quad PQ = QP = Q, \quad (11)$$

$$3) \quad P\bar{E}^D = \bar{E}^D. \quad (12)$$

Proof is given in [6].

**Theorem 3.** The solution to the equation (5a) is given by

$$x_i = (Q^i + c_2Q^{i-2} + c_3Q^{i-3} + \dots + 2c_{i-1}Q + I_n c_i)Pw = T_i x_0, \quad i \in \mathbb{Z}_+, \quad (13a)$$

$$T_i = Q^i + c_2Q^{i-2} + c_3Q^{i-3} + \dots + 2c_{i-1}Q + I_n c_i, \quad x_0 \in \text{Im } P, \quad (13b)$$

where  $Q$  and  $P$  are defined by (9), coefficient  $c_j$  can be computed using (3b) and  $w \in \mathbb{R}^n$  is arbitrary.

Proof is given in [6].

**Theorem 4.** Let

$$\Phi_0(i) = Q^i + \sum_{k=2}^i c_k \bar{A}_\alpha P, \quad (14)$$

where  $Q$  and  $P$  are defined by (9).

Then

$$P\Phi_0(i) = \Phi_0(i). \quad (15)$$

Proof is given in [5].

### 3. Pointwise completeness of fractional descriptor discrete-time linear systems

In this section conditions for the pointwise completeness of fractional descriptor linear discrete-time linear systems will be established.

**Definition 3.** The fractional descriptor discrete-time linear system (1) is called pointwise complete for  $i = q$  if for final state  $x_f \in \mathbb{R}^n$ , there exists an initial condition  $x_0 \in \text{Im } P$  such that

$$x_f = x_q \in \text{Im } P \quad (16)$$

where  $P$  is defined by (9a).

**Theorem 5.** The fractional descriptor discrete-time system (1) is pointwise complete for any  $i = q$  and every  $x_f \in \mathbb{R}^n$  if and only if

$$\text{rank } T_q = n, \quad (17)$$

where  $T_q$  is defined by (13b) for  $i = q$ .

**Proof.** Note that there exists the inverse matrix  $T_q^{-1}$  if and only if the condition (17) is satisfied. In this case from (13a) for  $i = q$  we obtain

$$x_0 = T_q^{-1}x_f. \tag{18}$$

Therefore, for every  $x_f$  there exists  $x_0 \in \text{Im } P$  such that  $x_q = x_f$  if and only if the condition (17) is satisfied.  $\square$

**Theorem 6.** Every fractional descriptor linear discrete-time system (1) is not pointwise complete for  $q = 1$ .

**Proof.** From the assumption  $\det E = 0$  it follows that

$$\det \bar{E} = \det \{ [Ec - A]^{-1}E \} = \det [Ec - A]^{-1} \det E = 0$$

and this implies  $\det E^D = 0$ . Using (9b) we obtain

$$\det Q = \det [\bar{E}^D \bar{A}_\alpha] = \det \bar{E}^D \det \bar{A}_\alpha = 0. \tag{19}$$

From (13b) for  $i = 1$  we have  $T_1 = Q$  and  $\det T_1 = \det Q = 0$ . Therefore, by Theorem 5 every fractional descriptor linear discrete-time system is not pointwise complete for  $q = 1$ .  $\square$

**Example 1.** Consider the fractional descriptor linear system (1) with  $\alpha = 0.6$  and the matrices

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1.6 & 0 & -1.6 \\ 0 & -1.6 & -1.6 \\ 1 & 1 & -1 \end{bmatrix}. \tag{20}$$

Note that the matrix  $A_\alpha$  for  $A$  and  $E$  given by (20) has the form

$$A_\alpha = A + E\alpha = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \tag{21}$$

and it is nonsingular. Therefore, we choose in (4)  $z = c = 0$  and we obtain

$$\det [Ec - A_\alpha] = \det [-A_\alpha] = 3 \tag{22}$$

and

$$\bar{E} = [-A_\alpha]^{-1}E = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \tag{23}$$

$$\bar{A}_\alpha = [-A_\alpha]^{-1}A_\alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that  $\det \bar{E} = 0$  and  $\det \bar{A}_\alpha \neq 0$ .

The Drazin inverse of the matrix  $\bar{E}$  given by (23) has the form

$$\bar{E}^D = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \bar{E} \tag{24}$$

$$P = \bar{E} \bar{E}^D = \bar{E}^2 = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{25}$$

$$Q = \bar{E}^D \bar{A}_\alpha = \frac{1}{3} \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}. \tag{26}$$

Note that  $\det P = \det \bar{E}^2 = 0$  and  $\det Q = \det \bar{E}^D \det A_\alpha = 0$  since  $\det \bar{E}^D = 0$ .

Therefore, by Theorem 6 the fractional descriptor system with  $\alpha = 0.6$  and the matrices (20) is not pointwise complete for  $q = 1$ , since  $\det T_1 = \det Q = 0$ .

Using (13) and (26) we obtain

$$T_2 = Q^2 + c_2 I_3 = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 0.12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \tag{27}$$

$$= \begin{bmatrix} 0.5467 & -0.3333 & 0.3333 \\ -0.3333 & 0.5467 & 0.3333 \\ 0.3333 & 0.3333 & 0.5467 \end{bmatrix}$$

and

$$\det T_2 = 0.0929. \tag{28}$$

Therefore, by Theorem 5 the descriptor system for  $q = 2$  is pointwise complete.

$$T_3 = Q^3 + 2c_2 Q + c_3 I_3 = \left( \frac{1}{3} \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \right)^3 - \tag{29}$$

$$- \frac{0.24}{3} \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} - 0.056 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -0.5627 & 0.2533 & -0.2533 \\ 0.2533 & -0.5627 & -0.2533 \\ -0.2533 & -0.2533 & -0.5627 \end{bmatrix}$$

and

$$\det T_3 = -0.0373. \quad (30)$$

Therefore, by Theorem 5 the descriptor system for  $q = 3$  is also pointwise complete.

#### 4. Pointwise degeneracy of fractional descriptor discrete-time linear systems

In this section conditions for the pointwise degeneracy of fractional descriptor linear discrete-time linear systems will be established.

**Definition 4.** The fractional descriptor discrete-time linear system (1) is called pointwise degenerated in the direction  $v \in \mathfrak{R}^n$  for  $i = q$  if there exists a nonzero vector  $v$  such that for all initial conditions  $x_0 \in \text{Im } P$  the solution  $x_i$  of the system (1) for  $i = q$  satisfies the condition

$$v^T x_q = 0. \quad (31)$$

**Theorem 7.** The fractional descriptor discrete-time linear system (1) is pointwise degenerated in the direction  $v \in \mathfrak{R}^n$  for  $i = q$  if and only if

$$\det T_q = 0 \quad (32)$$

where  $T_q$  is defined by (13b) for  $i = q$ .

**Proof.** From (31) and (13a) for  $i = q$  we have

$$v^T T_q x_0 = 0. \quad (33)$$

Note that there exists nonzero vector  $v \in \mathfrak{R}^n$  such that the condition (33) is satisfied for all  $x_0 \in \text{Im } P$  if and only if the matrix  $T_q$  is singular. Therefore, the fractional descriptor system (1) is pointwise degenerated in the direction  $v \in \mathfrak{R}^n$  for  $i = q$  if and only if the condition (32) is satisfied.  $\square$

**Theorem 8.** Every fractional descriptor linear discrete-time system (1) is pointwise degenerated for  $q = 1$ .

**Proof.** From the assumption  $\det E = 0$  it follows that  $\det \bar{E} = 0$  and  $\det \bar{E}^D = 0$  this implies that  $\det Q = 0$  (see (19)). Taking into account that  $T_1 = Q$  we obtain  $\det T_1 = 0$ . Therefore, by Theorem 7 every fractional descriptor discrete-time system (1) is pointwise degenerated for  $q = 1$ .  $\square$

**Example 2.** (Continuation of Example 1). Consider the fractional descriptor system (1) with  $\alpha = 0.6$  and the matrices  $E$ ,  $A$  given by (20).

In this case the matrix  $Q$  is given by (26) and

$$\det T_1 = \det Q = \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = 0. \quad (34)$$

Therefore, by Theorem 8 the fractional descriptor system for  $q = 1$  is pointwise degenerated.

The vector  $v \in \mathfrak{R}^3$  in which the system is pointwise degenerated can be computed from the equation

$$Q^T v = \frac{1}{3} \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (35)$$

and its solution is  $v^T = [v_1, v_2, v_3]^T = [-a, -a, a]$  for any number  $a$ .

#### 5. Concluding remarks

The Drazin inverse of matrices has been applied to analysis of the pointwise completeness and of the pointwise degeneracy of the fractional descriptor linear discrete-time systems. Necessary and sufficient conditions for the pointwise completeness and the pointwise degeneracy of the fractional descriptor linear discrete-time systems have been established (Theorems 5 and 7). It is shown that every fractional descriptor linear discrete-time systems is not pointwise complete and it is pointwise degenerated for  $i = 1$  (Theorems 6 and 8).

The considerations have been illustrated by numerical example of the fractional descriptor linear discrete-time system. The considerations can be extended to the fractional different orders linear continuous-time and discrete-time linear systems.

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