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PERFORMANCE ANALYSIS IN THE PRESENCE OF BOUNDED, DISCRETE, AND FLEXIBLE MEASURES

SOHRAB KORDROSTAMI*, MONIREH JAHANI SAYYAD NOVEIRI

Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, P.O. Box 1616, Iran

In conventional data envelopment analysis (DEA) models, the relative efficiency of decision-making units (*DMU*s) is evaluated while all measures with certain input and/or output status are considered as continuous data without upper and/or lower bounds. However, there are occasions in real-world applications that the efficiency of firms must be assessed while bounded elements, discrete values, and flexible measures are present. For this purpose, the current study proposes DEA-based approaches to estimate the relative efficiency of *DMU*s where bounded factors, integer values, and flexible measures exist. To illustrate it, radial models based on two aspects, individual and aggregate, are introduced to measure the performance of entities and to handle the status of the flexible measure such that there are bounded components and discrete data. Applications of approaches proposed in the areas of quality management, highway maintenance patrols, and university performance measurement are given to clarify the issue and to show their practicability. It was found that the introduced procedure can determine practical projection points for bounded measures and integer values (from the individual *DMU* viewpoint) and can classify flexible measures along with evaluation of *DMU*s relative efficiency.

Keywords: *data envelopment analysis, efficiency, bounded data, discrete data, flexible measure*

1. Introduction

Data envelopment analysis (DEA) is a well-known non-parametric mathematical programming approach, firstly introduced by Charnes et al. [8] to evaluate the relative efficiency of decision-making units (*DMU*s) with multiple inputs and outputs. In DEA, it is usually supposed that the input/output status of measures is known while continuous and unbounded input-output measures are included. Nevertheless, there are situations in reality that bounded values and discrete measures are presented whilst some measures can play either input or output roles. In the DEA literature, measures with undetermined input and/or output status before the efficiency estimation are called flexible measures.

*Corresponding author, email address: sohrabkordrostami@gmail.com
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Beasley [5] as a pioneer considers the research income as a measure with roles of input and output in measuring the efficiency of university departments. Bala and Cook [4] design a two-stage process for incorporating expert knowledge in the form of the classification of *DMU*s within DEA framework. Afterwards, Cook and Zhu [15] introduce individual *DMU* and aggregate models to handle flexible measures. Amirteimoori and Emrouznejad [2] state that Cook ad Zhu's approach [15] overestimates the efficiency and propose an alternative DEA-based approach for this purpose. Toloo [35] considers alternative optimal solutions to handle flexible measures that are not addressed in [2] and [15]. Kordrostami and Jahani Sayyad Noveiri [26] suggest an approach based on DEA to deal with the relative efficiency of *DMU*s in the presence of flexible and negative measures. Moreover, a method to estimate the relative efficiency of *DMU*s in the presence of the interval and flexible measures is provided by Kordrostami and Noveiri [27]. Indeed, their approach is used for situations in which there are imprecise and flexible measures. After that, Toloo et al. [37] also address interval efficiency values from optimistic and pessimistic aspects where interval inputs, outputs, and dual-role measures are presented and they provide a fuzzy approach based on fuzzy max-min criterion. In addition, some studies [20, 25, 30] investigate the relative efficiency of organisations where there are flexible and fuzzy variables. Amirteimoori et al. [3] develop a slack-based measure to classify flexible measures. Tohidi and Matroud [34] focus on an alternative non-oriented model based on variable returns to scale (VRS) to classify flexible measures and to describe the returns to scale status. Toloo et al. [36] develop a non-radial directional distance method to classify inputs and outputs and apply it to the bank industry. Kiyadeh et al. [33] integrate flexible measures into the Russell measure to estimate the maximum efficiency and to detect the role of flexible measures. They apply a mixed-integer second-order conic problem to deal with flexibility. Kordrostami et al. [24] propose a slacks-based measure of efficiency to analyse the performance of firms in the presence of integer and flexible measures. Recently, Boda [7] provides a substitute approach to the flexible slacks-based measure model evolved by Amirteimoori et al. [3] that leads to different projection points for inefficient units and disparate classification.

Nevertheless, the aforementioned studies have not incorporated bounds or bounded integer factors, including Likert scales, as shown in Table 1. However, there are real situations in which bounded or discrete bounded data exist. Some researchers considered the performance of organisations in the presence of bounded or discrete bounded factors. Cook et al. [13] incorporate ordinal data in the DEA structure. Zhu [40] review and compare some methods to address unknown measures such as bounded, ordinal, and ratio bounded ones. Cooper et al. [16] provide bounded additive models by incorporating lower bounds for inputs and upper bounds for outputs under any returns to scale assumption. Toloo and Mensah [38] address the robust optimisation problem with non-negative decision variables and render a reduced robust DEA model with uncertain input-output measures. Chen et al. [10] incorporate bounded and discrete data as well Likert scales in DEA models. Afterwards, Chen et al. [11] evaluate the efficiency of

National Basketball Association (NBA) players using bounded integer DEA. Chen et al. [9] develop a hybrid DEA approach, that is an input-oriented-bounded-and-discrete-data DEA model and context-dependent DEA to assess the efficiency of college undergraduate students. Kazemi Matin and Emrouznejad [22] extend the axiomatic foundation of integer-valued DEA models for addressing bounded outputs. Zhang et al. [39] provide a DEA method with assurance region (AR) constraints and including bounded and discrete data.

Table 1. Comparative investigations

	Model	Flexible	Bounded	Integer	Likert scale
Cook and Zhu [15]	radial	+	–	–	–
Amirteimoori et al. [3]	radial	+	–	–	–
Amirteimooi and Emrouznejad [2]	non-radial	+	–	–	–
Toloo [35]	radial	+	–	–	–
Toloo [36]	non-radial	+	–	–	–
Tohidi and Matroud [34]	radial	+	–	–	–
Kordrostami et al. [24]	non-radial	+	–	+	–
Kiyadeh et al. [33]	non-radial	+	–	–	–
Boda [7]	non-radial	+	–	–	–
Present study	radial	+	+	+	+

+ means that the mentioned type of data has been included, – that it is not contained.

Whilst some research has been carried out on bounded factors or discrete and bounded values, no studies which analyse the performance of *DMU*s in the presence of bounded, discrete and flexible measures have been found. Indeed, there exist conditions in real-world problems in which the performance of organisations must be addressed while flexible measures, bounded elements and discrete data are present. For instance, consider hospitals. Factors such as quality of nursing care and satisfaction of patients are bounded and they can be shown by Likert scales or the rate of survival is a bounded factor. Nurse trainees and medical interns can also be treated as integer flexible measures in the measurement of hospital efficiency. According to [12], the number of nurse trainees on staff is a discrete output for a hospital, but it is a key component of the hospital's total staff complement, thus it is a discrete input. As another example, a factor such as the number of customers in banks can be assumed as either input or output (i.e., flexible measure) that is a discrete value. As argued by Cook and Zhu [15], it can play the role of proxy for future investment, therefore, it can be treated as an output. However, it can be considered as an environmental input that aids the branch in creating its existent investment portfolio. As Lozano and Villa [29] mention, the influence of general rounding input-output targets of large integer values is small; however, it is not correct for small integer measures. But, practically rounding the continuous targets of integer measures is not sensible in addition to many situations in which small integer measures are presented or/and the importance of performance measures is large. Moreover, service quality and customer satisfaction in banks are bounded and can be measured by Likert scales. Average pavement rating can also be deemed as the bounded flexible measure in assessing highway maintenance patrols. As Cook and Zhu [15] mention, on the one hand, it can be considered as the input that influences the outputs and, on the other hand, it can be treated as the output that has an impact through the level of the annual maintenance cost.

Therefore, this paper provides radial DEA-based approaches to assess the relative efficiency of *DMU*s where flexible, bounded, and discrete measures are present. To illustrate, individual *DMU* and aggregate models based on the constant returns to scale (CRS) assumption are introduced to classify flexible measures and to measure the performance of *DMU*s with bounded factors and with or without integer values, although the proposed approaches can be extended under the VRS assumption. Furthermore, projection points for *DMU*s with bounded and discrete values are accurately investigated and identified when the individual *DMU* model is used. Several examples are provided to address the applicability of the approaches presented herein.

The paper is organised as follows. Preliminaries appear in Section 2. Models to evaluate the relative efficiency of *DMU*s in the presence of bounded, discrete, and flexible measures are introduced in Section 3. Practical applications are given in Section 4 to illustrate and validate the approaches proposed herein. Conclusions and remarks are included in Section 5.

2. Preliminaries

In this section, at first, Amirteimoori and Emrouznejad's approach [2] to classify flexible measures is briefly explained. Then, an overview of the research of Chen et al. [10] about bounded, discrete data, and Likert scales in the DEA is presented.

2.1. Classifying performance measures

Suppose there are n *DMU*s, $DMU_j \ (j=1,...,n)$, with m inputs $x_{ij} \ (i=1,...,m)$ and s outputs $y_{rj} \ (r=1,...,s)$. Charnes et al. [8] propose the following CRS radial model (called the CCR model) to estimate the relative efficiency of *DMU*s:

$$
\begin{aligned}
&\min \theta \\
\text{s.t. } &\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \ i=1,2,...,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \ r=1,2,...,s \\
&\lambda_j \ge 0, \ j=1,...,n
\end{aligned} \tag{1}
$$

in which λ_j $(j=1,...,n)$ are intensity variables, and also x_{io} and y_{ro} are *i*th input and *r*th output of the unit under consideration, DMU_o. Besides, the term θ is used to refer to the efficiency variable.

In the presence of *K* flexible measures w_{kj} $(k=1,...,K)$, Amirteimoori and Emrouznejad [2] modify model (1) and introduce model (2) for evaluating the relative efficiency of *DMU*s and classifying flexible measures.

$$
\begin{aligned}
&\min \theta \\
\text{s.t.}\ &\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io},\ i=1,2,...,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro},\ r=1,2,...,s \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \le \theta w_{ko} + Md_k,\ k=1,2,...,K \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \ge w_{ko} - M(1-d_k),\ k=1,2,...,K \\
&d_k \in \{0,1\},\ k=1,...,K,\ \lambda_j \ge 0,\ j=1,...,n
\end{aligned}
\tag{2}
$$

in which *M* is a large positive number, and d_k $(k=1,...,K)$ are binary variables. If $d_k=1$, then the flexible measure *k* is output and if $d_k=0$, it is deemed as input. The majority rule is used to classify flexible measures. Furthermore, alternative optimal solutions are not taken into account in the overall classification of inputs and outputs as mentioned by Toloo [35]. To illustrate, incorporating alternative optimal solutions can make incorrect results.

2.2. The DEA model with bounded and discrete data and Likert scales

Chen et al. [10] with the purpose of the performance measurement of *DMU*s in the presence of bounded and discrete data and Likert scales proposed the following model, which evaluates the relative efficiency of DMU_o and determines the projection points:

$$
\begin{aligned}
&\min \frac{1}{m}\sum_{i=1}^{m} \alpha_i \\
\text{s.t.}\ &\sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{x}_{io},\ i=1,2,...,m
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{x}_{io} \le \alpha_i x_{io}, i = 1, 2, ..., m \\
&L_{i_{\text{Bnd}}} \le \tilde{x}_{i_{\text{Bnd}}o} \le U_{i_{\text{Bnd}}}, i_{\text{Bnd}} \in I_{\text{Bnd}} \\
&\tilde{x}_{i_{\text{Int}}o} \text{ integer}, i_{\text{Int}} \in I_{\text{Int}} \\
&1 \le \tilde{x}_{i_{\text{Lik}}o} \le L, i_{\text{Lik}} \in I_{\text{Lik}} \\
&\tilde{x}_{i_{\text{Lik}}o} \text{ integer}, i_{\text{Lik}} \in I_{\text{Lik}} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, r = 1, 2, ..., s \\
&\lambda_j \ge 0, j = 1, ..., n
\end{aligned}
\tag{3}
$$

in which inputs x_{ij} $(i = 1, ..., m)$ may be integer, non-integer, bounded, unbounded, or Likert scale variables. Thus, inputs divided to integer and continuous measures denoted by $x_{i_{\text{Int}}j\,\text{Int}}$ (the subscript i_{Int} indicates the inputs in the subset I_{Int}) and $x_{i_{\text{cont}}j}$ (the subscript i_{cont} indicates the inputs in the subset I_{cont}) that $I_{\text{Int}} \cup \text{I}_{\text{cont}} = \{1, ..., m\}$, bounded and unbounded inputs shown by $x_{i_{\text{Bnd}}j}$ (the subscript i_{Bnd} indicates the inputs in the subset I_{Bnd}) and $x_{i_{\text{Unb}}j}$ (the subscript i_{Unb} indicates the inputs in the subset I_{Unb}), while $I_{\text{Bnd}} \cup I_{\text{Unb}} = \{1, ..., m\}$, or Likert scale variables such that I_{Lik} indicates the subset of inputs declared by Likert scales, and L indicates the number of the levels of Likert scale data. Furthermore, outputs are shown by y_{rj} $(r = 1, ..., s)$ and $\tilde{x}_{i_{\text{Int}}o}$, $i_{\text{Int}} \in I_{\text{Int}}$ are integer variables. Moreover, $\tilde{x}_{i_{\text{Bnd}}o}$, $i_{\text{Bnd}} \in I_{\text{Bnd}}$ and $\tilde{x}_{i_{\text{Lik}}o}$, $i_{\text{Lik}} \in I_{\text{Lik}}$ are applied to indicate bounded and Likert scale variables, respectively.

It appears from the aforementioned investigations that flexible measures and bounded and discrete data have not been simultaneously handled in the existing DEA literature. To illustrate it, no attempt has been made to analyse the performance of *DMU*s in the presence of flexible, bounded, and discrete measures in the same way as bounded flexible factors. Accordingly, models are proposed in the following section to deal with the relative efficiency of *DMU*s with flexible measures and bounded and discrete data, and also where bounded flexible variables are presented in the system under consideration.

3. The proposed approach

The section below describes the approaches introduced in this investigation. The first part deals with the relative efficiency of *DMU*s from the individual *DMU* aspect, and the second part moves on to describe it from the aggregate aspect. For this purpose, $\tilde{x}_{io}, \tilde{x}_{i_{\mathrm{Int}}o}, \tilde{x}_{i_{\mathrm{Bnd}}o}, \tilde{x}_{i_{\mathrm{Lik}}o}, \tilde{y}_{ro}, \tilde{y}_{r_{\mathrm{Int}}o}, \tilde{y}_{r_{\mathrm{Bnd}}o}, \tilde{y}_{r_{\mathrm{Lik}}o}, \tilde{w}_{ko}, \tilde{w}_{k_{\mathrm{Int}}o}, \tilde{w}_{k_{\mathrm{Bnd}}o}$, and $\tilde{w}_{k_{\mathrm{Lik}}o}$ are treated as unknown variables and other notations used in this section are similar to the previous section. Also, the inputs are portioned into continuous and integer inputs; that is I_{cont} and I_{int} are used to denote the subsets of continuous and integer inputs. Further, $I_{\mathrm{int}} \cup I_{\mathrm{cont}} = \{1, \ldots, m\}$. Similarly, outputs and flexible measures are divided into R_{int} and R_{cont}, and also K_{Int} and K_{cont}, respectively. Thus, $x_{i_{\mathrm{Int}}j}$ (the subscript i_{Int} indicates the inputs in the subset I_{Int}, i.e, $i_{\mathrm{Int}} \in I_{\mathrm{Int}}$), $y_{r_{\mathrm{Int}}j}$ ($r_{\mathrm{Int}} \in R_{\mathrm{Int}}$) and $w_{k_{\mathrm{Int}}j}$ ($k_{\mathrm{Int}} \in K_{\mathrm{Int}}$) show integer--valued inputs, outputs, and flexible measures. Subscripts r_{Bnd} and r_{Unb} are likewise used to indicate bounded outputs in the subset R_{Bnd} and unbounded outputs in the subset R_{Unb} where $R_{\mathrm{Bnd}} \cup R_{\mathrm{Unb}} = \{1, \ldots, s\}$. Therefore, we suppose $L_{r_{\mathrm{Bnd}}}$ and $U_{r_{\mathrm{Bnd}}}$ show lower and upper bounds for the output r_{Bnd}. It means that we have $L_{r_{\mathrm{Bnd}}} \le y_{r_{\mathrm{Bnd}}j} \le U_{r_{\mathrm{Bnd}}}$ for DMU_j. In the same vein, we assume $L_{i_{\mathrm{Bnd}}} \le x_{i_{\mathrm{Bnd}}j} \le U_{i_{\mathrm{Bnd}}}$ that subscript i_{Bnd} is used to show bounded inputs in the subset I_{Bnd} and $L_{i_{\mathrm{Bnd}}}$ and $U_{i_{\mathrm{Bnd}}}$ describe lower and upper bounds for the input i_{Bnd}, respectively. Therefore, we should obtain $\tilde{x}_{i_{\mathrm{Bnd}}j}$ and $\tilde{y}_{r_{\mathrm{Bnd}}j}$ within intervals $[L_{i_{\mathrm{Bnd}}}, U_{i_{\mathrm{Bnd}}}]$ and $[L_{r_{\mathrm{Bnd}}}, U_{r_{\mathrm{Bnd}}}]$, respectively. $L_{k_{\mathrm{Bnd}}}$ and $U_{k_{\mathrm{Bnd}}}$ also show lower and upper bounds of the bounded flexible measure *k*. Subscript k_{Bnd} is used to indicate bounded flexible measures in the subset K_{Bnd}. The terms R_{Lik}, I_{Lik} and K_{Lik} are also utilised to represent the subsets of outputs, inputs and flexible measures indicated by Likert scale data that $r_{\mathrm{Lik}} \in R_{\mathrm{Lik}}$, $i_{\mathrm{Lik}} \in I_{\mathrm{Lik}}$ and $k_{\mathrm{Lik}} \in K_{\mathrm{Lik}}$. Moreover, *L* provides the number of the levels of Likert scale data.

3.1. The individual aspect

With regards to aforementioned notations, we propose the radial model (6) that is from the individual *DMU* perspective to classify the flexible measures and to evaluate the relative efficiency of *DMU*s in the presence of flexible measures and discrete and bounded data. According to the description of radial measures of the efficiency and model orientation, the proportionate reduction of inputs and flexible measures deemed as inputs is dealt with. However, the approach can be extended to investigate the proportionate expansion of outputs and flexible measures treated as outputs and also to address non-radial measure of efficiency.

At first, we consider the following either-or constraints to investigate flexible measures:

$$\begin{cases} \text{either} \\ \sum_{j=1}^{n} \lambda_j w_{kj} \le \tilde{w}_{ko}, \ k=1,2,...,K \\ \tilde{w}_{ko} \le \theta w_{ko}, \ k=1,2,...,K \\ \text{or} \\ \sum_{j=1}^{n} \lambda_j w_{kj} \ge \tilde{w}_{ko}, \ k=1,2,...,K \\ \tilde{w}_{ko} \ge w_{ko}, \ k=1,2,...,K \end{cases} \tag{4}$$

Statements (4) are equal to the following:

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j w_{kj} \le \tilde{w}_{ko} + Md_k, \ k=1,2,...,K && (5.1) \\ &\tilde{w}_{ko} \le \theta w_{ko} + Md_k, \ k=1,2,...,K && (5.2) \\ &\sum_{j=1}^{n} \lambda_j w_{kj} \ge \tilde{w}_{ko} - M(1-d_k), \ k=1,2,...,K && (5.3) \\ &\tilde{w}_{ko} \ge w_{ko} - M(1-d_k), \ k=1,2,...,K && (5.4) \\ &d_k \in \{0,1\}, \ k=1,...,K \end{aligned} \tag{5}$$

in which M is a large positive number and d_k $(k=1,...,K)$ are binary variables. If $d_k = 0$, then expressions (5.1) and (5.2) are binding and others are redundant and also the kth flexible measure is considered as the input. Otherwise, if $d_k = 1$, then constraints (5.3) and (5.4) are active, while other constraints are inactive and the flexible measure k is treated as the output. Therefore, we introduce model (6) from the individual *DMU* aspect:

$$\begin{aligned} &\min \theta \\ \text{s.t.} \ &\sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{x}_{io}, \ i=1,2,...,m \\ &\tilde{x}_{io} \le \theta x_{io}, \ i=1,...,m \\ &\tilde{x}_{i_{\text{Int}}o} \ \text{integer}, \ \text{i}_{\text{Int}} \in I_{\text{Int}} \\ &L_{i_{\text{Bnd}}} \le \tilde{x}_{i_{\text{Bnd}}o} \le U_{i_{\text{Bnd}}}, \ i_{\text{Bnd}} \in I_{\text{Bnd}} \\ &1 \le \tilde{x}_{i_{\text{Lik}}o} \le L, \ i_{\text{Lik}} \in I_{\text{Lik}} \end{aligned}$$

$$
\begin{aligned}
&\tilde{x}_{i_{\text{Lik}}o} \ \text{ integer}, \ i_{\text{Lik}} \in I_{\text{Lik}} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{y}_{ro}, \ r = 1, 2, \ldots, s \\
&\tilde{y}_{ro} \geq y_{ro}, \ r = 1, 2, \ldots, s \\
&L_{r_{\text{Bnd}}} \leq \tilde{y}_{r_{\text{Bnd}}o} \leq U_{r_{\text{Bnd}}}, \ r_{\text{Bnd}} \in R_{\text{Bnd}} \\
&\tilde{y}_{r_{\text{Int}}o} \ \text{ integer}, \ r_{\text{Int}} \in R_{\text{Int}} \\
&1 \leq \tilde{y}_{r_{\text{Lik}}o} \leq L, \ r_{\text{Lik}} \in R_{\text{Lik}} \\
&\tilde{y}_{r_{\text{Lik}}o} \ \text{ integer}, \ r_{\text{Lik}} \in R_{\text{Lik}} \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \leq \tilde{w}_{ko} + Md_k, \ k = 1, 2, \ldots, K \\
&\tilde{w}_{ko} \leq \theta w_{ko} + Md_k, \ k = 1, 2, \ldots, K \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \geq \tilde{w}_{ko} - M(1 - d_k), \ k = 1, 2, \ldots, K \\
&\tilde{w}_{ko} \geq w_{ko} - M(1 - d_k), k = 1, 2, \ldots, K \\
&\tilde{w}_{k_{\text{Int}}o} \ \text{ integer}, \ k_{\text{Int}} \in K_{\text{Int}} \\
&L_{k_{\text{Bnd}}} \leq \tilde{w}_{k_{\text{Bnd}}o} \leq U_{k_{\text{Bnd}}}, \ k_{\text{Bnd}} \in K_{\text{Bnd}} \\
&1 \leq \ \tilde{w}_{k_{\text{Lik}}o} \leq L, \ k_{\text{Lik}} \in K_{\text{Lik}} \\
&\tilde{w}_{k_{\text{Lik}}o} \ \text{ integer}, \ k_{\text{Lik}} \in K_{\text{Lik}} \\
&d_k \in \{0, 1\}, \ k = 1, \ldots, \ K, \lambda_j \geq 0, \ j = 1, \ldots, n
\end{aligned}
\tag{6}
$$

Note that in addition to integer and bounded input-output factors and Likert scale input-output measures, integer and non-integer flexible measures, bounded flexible measures, and Likert scale flexible measures have been incorporated in model (6).

Theorem 1. Model (6) is always feasible.

Proof. Supposing M as a user-defined sufficient large positive number and $d_k = 0$ (or $d_k = 1$), there is a feasible solution $(\theta, \lambda, d, \tilde{x}_{io}, \tilde{y}_{ro}, \tilde{w}_{ko})$ of model (6) while $\theta = 1$, $\lambda_o = 1, \lambda_j = 0, j \neq o$, $\tilde{x}_{io} = x_{io}$, $\tilde{y}_{ro} = y_{ro}$ and $\tilde{w}_{ko} = w_{ko}$. Thus, the proof is completed. □

Theorem 2. The optimal objective function value of model (6) is obtained between zero and one; i.e., $0 < \theta^* \leq 1$.

Proof. As a result of Theorem 1 that $(\theta, \lambda, d, \tilde{x}_{io}, \tilde{y}_{ro}, \tilde{w}_{ko})$ with $\theta = 1$ is a feasible solution, the optimal objective function value of model (6) is not more than one. Also, as based on semi-positive performance measures, the vector $\boldsymbol{\lambda}$ cannot be zero because of satisfying the following constraints of model (6):

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{y}_{ro}, \; r = 1, 2, ..., s$$

$$\tilde{y}_{ro} \geq y_{ro}, \; r = 1, 2, ..., s$$

Moreover, examining the next constraints of model (6)

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \tilde{x}_{io}, \; i = 1, 2, ..., m$$

$$\tilde{x}_{io} \leq \theta x_{io}, \; i = 1, ..., m$$

θ cannot be zero or less than zero. Therefore, $0 < \theta^* \leq 1$. □

Definition 1. The unit under evaluation, DMU_o, is said to be efficient if and only if the optimal objective function value of model (6) is equal to one, i.e., $\theta_o^* = 1$. Otherwise, it is said to be inefficient.

$(\tilde{x}_{io}, \tilde{y}_{ro}, \tilde{w}_{ko})$ also indicates the projection point of the unit *o*.

According to [15], the majority choice rule is used to classify the flexible measures as inputs or outputs. However, alternative solutions are not considered in the choice similar to [35]. Actually, for the k^{th} flexible measure, we have:

1. If $d_k^* = 0$ or 1, the *k*th flexible measure can play the role of either input or output that it can occur when the *DMU* under evaluation is efficient. It means that there is an alternative optimal solution. As Toloo [35] states, specifying the presence of alternative solution is not easy, but due to the formulation of model (6), it can be found. For more illustration, from $\theta^* = \min\{\theta_a^*, \theta_b^*\} = 1$ in model (6) and considering this issue that the optimal objective function value of model (6) is not more than one based on Theorem 2, $\theta^* = \max\{\theta_a^*, \theta_b^*\} = 1$ which θ_a^* and θ_b^* are obtained by considering d as a predefined form $d = 0$ and $d = 1$ in model (6), respectively. Accordingly, there is no contrast to identify the role of the flexible measure as the input or output in this case.

2. If $d_k^* = 0$, then the kth flexible measure is deemed as the input,

3. If $d_k^* = 1$, then the kth flexible measure is treated as the output.

If the number of *DMU*s that satisfy the case 2 (i.e., the flexible measure k assumed as the input) is greater than that which fulfills the case 3 (i.e., the flexible measure k assumed as the output), the role of the flexible measure k is considered as the input; otherwise it is deemed as the output if the number of firms satisfying the case 3 is over it accomplishing the case 2. Notice that *DMU*s with the condition 1 is not included in the classification. When the number of *DMU*s with meeting conditions 2 and 3 are equal, an alternative approach, that is aggregate approach, can be used to determine the role of flexible measures. It means that if the number of *DMU*s that determine a flexible measure as the input is equal to the number of *DMU*s that identify it as the output, the status of the flexible measure can be found out from the decision maker's perspective, using an alternative approach explained in the following subsection.

3.2. The aggregate aspect

For calculating the aggregate efficiency of the set of *DMU*s, we introduce model (7). In this case, the input/output status of the flexible measure is specified by solving the model and without using the majority choice rule. Actually, this model examines the performance and the role of flexible measures from the manager's perspective of the set of entities. According to [15, 36], such approach would be valuable when ties are met in the investigation from the individual aspect.

$$
\begin{aligned}
& \min \theta \\
\text{s.t. } & \sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{x}_i, \; i = 1, 2, \ldots, m \\
& \tilde{x}_i \le \theta \sum_{j=1}^{n} x_{ij}, \; i = 1, \ldots, m \\
& \tilde{x}_{i_{\text{Int}}} \;\; \text{integer}, \;\; \mathrm{i}_{\text{Int}} \in I_{\text{Int}} \\
& \sum_{j=1}^{n} L_{i_{\text{Bnd}} j} \le \tilde{x}_{i_{\text{Bnd}}} \le \sum_{j=1}^{n} U_{i_{\text{Bnd}} j}, \; i_{\text{Bnd}} \in I_{\text{Bnd}} \\
& n \le \tilde{x}_{i_{\text{Lik}}} \le nL, \; i_{\text{Lik}} \in I_{\text{Lik}} \\
& \tilde{x}_{i_{\text{Lik}}} \;\; \text{integer}, i_{\text{Lik}} \in I_{\text{Lik}} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge \tilde{y}_r, \; r = 1, 2, \ldots, s \\
& \tilde{y}_r \ge \sum_{j=1}^{n} y_{rj}, \; r = 1, 2, \ldots, s
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n} L_{r_{\text{Bnd}}j} \le \tilde{y}_{r_{\text{Bnd}}} \le \sum_{j=1}^{n} U_{r_{\text{Bnd}}j},\ r_{\text{Bnd}} \in R_{\text{Bnd}} \\
&\tilde{y}_{r_{\text{Int}}} \ \text{ integer},\ r_{\text{Int}} \in R_{\text{Int}} \\
&n \le \tilde{y}_{r_{\text{Lik}}} \le nL,\ r_{\text{Lik}} \in R_{\text{Lik}} \\
&\tilde{y}_{r_{\text{Lik}}} \ \text{ integer},\ r_{\text{Lik}} \in R_{\text{Lik}} \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \le \tilde{w}_k + Md_k,\ k = 1, 2, ..., K \\
&\tilde{w}_k \le \theta \sum_{j=1}^{n} w_{kj} + Md_k,\ k = 1, 2, ..., K \\
&\tilde{w}_{k_{Int}} \ \text{ integer},\ k_{Int} \in K_{Int} \\
&\sum_{j=1}^{n} \lambda_j w_{kj} \ge \tilde{w}_k - M(1-d_k),\ k = 1, 2, ..., K \qquad (7) \\
&\tilde{w}_k \ge \sum_{j=1}^{n} w_{kj} - M(1-d_k),\ k = 1, 2, ..., K \\
&\sum_{j=1}^{n} L_{k_{\text{Bnd}}j} \le \tilde{w}_{k_{\text{Bnd}}} \le \sum_{j=1}^{n} U_{k_{\text{Bnd}}j},\ k_{\text{Bnd}} \in K_{\text{Bnd}} \\
&n \le \tilde{w}_{k_{\text{Lik}}} \le nL,\ k_{\text{Lik}} \in K_{\text{Lik}} \\
&\tilde{w}_{k_{\text{Lik}}} \ \text{ integer},\ k_{\text{Lik}} \in K_{\text{Lik}} \\
&d_k \in \{0, 1\},\ k = 1, ..., K,\ \lambda_j \ge 0,\ j = 1, ..., n
\end{aligned}
$$

that $\tilde{x}_i, \tilde{x}_{i_{\text{Int}}}, \tilde{x}_{i_{\text{Bnd}}}, \tilde{x}_{i_{\text{Lik}}}, \tilde{y}_r, \tilde{y}_{r_{\text{Int}}}, \tilde{y}_{r_{\text{Bnd}}}, \tilde{y}_{r_{\text{Lik}}}, \tilde{w}_k, \tilde{w}_{k_{\text{Int}}}, \tilde{w}_{k_{\text{Bnd}}}$ and $\tilde{w}_{k_{\text{Lik}}}$ are treated as unknown variables. The flexible measure k is treated as the input if the optimal value d_k (i.e., d_k^*) equals to zero $(d_k^* = 0)$ in model (7). It is also deemed as the output provided that $d_k^* = 1$.

Theorem 3. Model (7) is always feasible.

Proof. Considering M as a sufficiently large positive number and $d_k = 0$ (or $d_k = 1$), there is a feasible solution $(\theta, \lambda, d, \tilde{x}_i, \tilde{y}_r, \tilde{w}_k)$ of model (7), whereas $\theta = 1$, $\lambda_1 = ... = \lambda_j = 1$, $\tilde{x}_i = \sum_{j=1}^{n} x_{ij}$, $\tilde{y}_r = \sum_{j=1}^{n} y_{rj}$ and $\tilde{w}_k = \sum_{j=1}^{n} w_{kj}$. Thus, the proof is completed. □

Theorem 4. The optimal objective function value of model (7) is obtained between zero and one, i.e., $0 < \theta^* \le 1$.

Proof. Due to Theorem 3 that $(\theta, \lambda, d, \tilde{x}_i, \tilde{y}_r, \tilde{w}_k)$ with $\theta = 1$ is a feasible solution, the optimal objective function value of model (7) is not more than one. Also, according to semi-positive performance measures, the vector $\boldsymbol{\lambda}$ cannot be zero because of satisfying the following constraints of model (7):

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{y}_r, \; r = 1, 2, ..., s$$

$$\tilde{y}_r \geq \sum_{j=1}^{n} y_{rj}, \; r = 1, 2, ..., s$$

Moreover, considering the subsequent constraints of model (7),

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \tilde{x}_i, \; i = 1, 2, ..., m$$

$$\tilde{x}_i \leq \theta \sum_{j=1}^{n} x_{ij}, \; i = 1, ..., m$$

θ cannot be zero or less than zero. Therefore, $0 < \theta^* \leq 1$. □

Definition 2. The set of *DMU*s in model (7) is said to be aggregate efficient if and only if $\theta^* = 1$; otherwise it is called aggregate inefficient.

From the aggregate perspective, $(\tilde{x}_i, \tilde{y}_r, \tilde{w}_k)$ can be considered as the projection point of the set of *DMU*s.

Notice that once model (7) is computed, the aggregate efficiency is obtained from the standpoint of manager of the collection of *DMU*s, and the status of flexible measure is identified as solved, while the relative efficiency score and the role of flexible measure are determined for each *DMU* by calculating model (6) and the majority rule through *DMU*s is used to make a general decision about flexible measures.

Also, as can be found, the model introduced by Amirteimoori and Emrouznejad [2] determines the status of flexible measures and the efficiency values, while all performance measures are assumed as continuous values and without bound. Furthermore, the approach by Chen et al. [10] estimates the efficiency scores and finds targets of bounded and discrete data as well as Likert scales, while the role of performance measures are known. But, the approaches rendered in this research address all the cases dealt with in the two aforementioned studies at once and, moreover, measure the relative efficiency of systems in the presence of bounded (integer or non-integer) flexible measure. The models discussed in this section are input-oriented; however, they can be extended for output-oriented version.

In the section that follows, the suggested models, models (6) and (7), are applied to measure the relative efficiency and to classify the flexible measure into three case studies. Also, the results are compared to some existing basic models.

4. Applications

To demonstrate the potential of approaches and their suitability for the application, firstly, the relative efficiency of 20 companies is analysed, while bounded, discrete, and flexible measures are present in the case under evaluation. Afterwards, the performance of highway maintenance patrols with the bounded flexible measure is examined. Finally, the relative efficiency of 15 branches of one of the universities in Iran is assessed in the presence of bounded, discrete, and flexible factors. To explain, in the first case, the continuous flexible measure is presented and inputs and outputs are integer, non-integer and bounded continuous. In the second case, there are continuous input-output factors and bounded flexible measures. The third case includes integer, non-integer, bounded integer input, and output variables and continuous flexible measures. The provided approaches have been examined for different situations.

4.1. Quality management performance analysis

Data applied in this case can be found in [28]. It consists of 20 firms with two inputs, four outputs, and one flexible measure. The data set is shown in Table 2 and the factors used are represented in Table 3.

As can be seen from Table 3, the number of employees is the discrete measure. Furthermore, the quality cost is the factor that can play either input or output role. To illustrate the problem, the efficiency is increased when it increases, thus it is deemed as the output, and it is a significant indicator of the firm. Therefore, it is assumed as the input. Its role is determined due to different perspectives.

To evaluate the relative efficiency of firms and to determine the role of quality costs, model (6) is firstly utilized. The results obtained from model (6) are given in Table 4. The relative efficiency scores of firms are displayed in column 2. It is easy to see that 5 firms are estimated as efficient, and quality costs can be assumed as input or output for these firms. As shown in column 3, 12 firms assess the role of quality costs as the input and 3 firms regard it as the output. Therefore, the role of the flexible measure is identified as the input just as the majority rule is. Furthermore, integer projections are attained for integer-valued measures and projection points of bounded measures have been obtained between the intended interval as shown in columns 4–8.

Table 2. Data set

Firm	x_1	x_2	y_1	y_2	y_3	y_4	w_1
1	898	28	92	98	88	50	108
2	115	27	92	91	86	47	34
3	882	40	80	88	87	19	71
4	325	34	94	86	80	20	50
5	456	32	98	92	82	14	53
6	649	58	93	98	91	20	113
7	758	70	97	78	99	47	75
8	196	28	95	96	88	21	37
9	135	40	96	98	96	19	39
10	840	36	90	99	88	14	55
11	776	43	96	97	91	19	69
12	384	26	95	97	88	41	43
13	404	57	92	96	86	43	49
14	180	41	88	91	90	38	40
15	157	55	96	95	87	11	51
16	117	54	96	92	92	24	58
17	708	58	95	89	84	30	81
18	934	42	89	93	95	42	64
19	958	53	90	95	76	12	98
20	448	37	97	91	87	19	57

Table 3. Description of factors

Symbol	Measures	Type
Input		
x_1	total employees	integer (number)
x_2	employees related to quality	bounded and continuous (per cent)
Output		
y_1	quality products	bounded and continuous (per cent)
y_2	satisfaction rate of customer	bounded and continuous (per cent)
y_3	on time delivery	bounded and continuous (per cent)
y_4	revenue	continuous
Flexible		
w_1	quality costs	continuous (million USD)

Now, model (7) is applied to estimate the aggregate efficiency score of firms and to identify the role of quality costs from the manager's viewpoint. The efficiency score calculated from model (7) is equal to 0.6248 and the status of quality costs is determined as the input. Notice that the status of flexible measure has been determined by model (6). The objective of calculating model (7) was only to specify the performance and flexible measure's status from manager's point of view.

Table 4. Results of model (6)

Supplier	Efficiency	d^*	x_1^*	x_2^*	y_1^*	y_2^*	y_3^*
1	1	0 or 1	898	28.00	92	98	88
2	1	0 or 1	115	27.00	92	91	86
3	0.64	0	567	25.71	80	88	87
4	0.78	0	254	26.62	94	86	80
5	0.84	0	383	26.91	98	92	82
6	0.47	0	307	27.44	93	98	91
7	0.52	0	324	31.08	97	78	99
8	1	0 or 1	194	28.00	95	96	88
9	0.94	1	127	35.86	96	98	96
10	0.76	0	619	27.38	90	99	88
11	0.63	0	485	27.09	96	97	91
12	1	0 or 1	384	26.00	95	97	88
13	0.56	1	227	32.04	92	96	86
14	0.80	1	124	32.85	88	91	90
15	0.77	0	121	32.45	96	95	87
16	1	0 or 1	117	54.00	96	92	92
17	0.47	0	323	27.20	95	89	84
18	0.68	0	415	28.62	89	93	95
19	0.48	0	460	25.46	90	95	76
20	0.73	0	327	27.07	97	91	87

Table 5. Results of the CCR model

Firm	Efficiency	x_1^*	x_2^*	y_1^*	y_2^*	y_3^*
1	1	898.00	28.00	92.00	98.00	88.00
2	1	115.00	27.00	92.00	91.00	86.00
3	0.64	379.64	25.70	93.92	95.90	87.00
4	0.78	254.42	26.62	94.00	94.54	87.45
5	0.84	383.48	26.91	98.00	99.92	90.82
6	0.47	307.01	27.44	97.95	98.99	91.00
7	0.52	132.38	31.08	105.91	104.76	99.00
8	1	196.00	28.00	95.00	96.00	88.00
9	0.97	128.37	30.14	102.70	101.58	96.00
10	0.76	323.67	27.38	97.32	99.00	90.15
11	0.63	364.54	27.09	98.13	99.83	91.00
12	1	384.00	26.00	95.00	97.00	88.00
13	0.73	121.32	28.48	97.05	96.00	90.73
14	0.89	120.35	28.26	96.28	95.23	90.00
15	0.76	120.05	28.19	96.04	95.00	89.78
16	1	117.00	54.00	96.00	92.00	92.00
17	0.47	215.29	27.20	95.00	95.07	88.51
18	0.68	323.61	28.62	102.26	103.39	95.00
19	0.48	376.08	25.46	93.04	95.00	86.19
20	0.73	318.13	27.07	97.00	98.20	90.08

To exhibit the propriety of the proposed approach, the results obtained from model (6) are now compared with the previous studies, the CCR model, and model (2). Note that the CCR model is computed while the quality cost is considered as the input. The findings obtained from prior surveys are described in Tables 5 and 6.

Table 6. Results of model (2)

Firm	Efficiency	d^*	x_1^*	x_2^*	y_1^*	y_2^*	y_3^*
1	1	0 or 1	898.00	28.00	92.00	98.00	88.00
2	1	0 or 1	115.00	27.00	92.00	91.00	86.00
3	0.64	0	379.64	25.70	93.92	95.90	87.00
4	0.78	0	254.42	26.62	94.00	94.54	87.45
5	0.84	0	383.48	26.91	98.00	99.92	90.82
6	0.47	0	307.01	27.44	97.95	98.99	91.00
7	0.52	0	132.38	31.08	105.91	104.76	99.00
8	0.99	1	193.98	27.71	96.20	96.00	89.71
9	0.94	1	126.60	37.51	101.99	100.01	96.00
10	0.76	0	323.67	27.38	97.32	99.00	90.15
11	0.63	0	364.54	27.09	98.13	99.83	91.00
12	1	0 or 1	384.00	26.00	95.00	97.00	88.00
13	0.56	1	226.59	31.97	95.94	96.00	90.28
14	0.76	1	137.67	31.36	95.93	94.84	90.00
15	0.76	0	120.05	28.19	96.04	95.00	89.78
16	1	0 or 1	117.00	54.00	96.00	92.00	92.00
17	0.47	0	215.29	27.20	95.00	95.07	88.51
18	0.68	0	323.61	28.62	102.26	103.39	95.00
19	0.48	0	376.08	25.46	93.04	95.00	86.19
20	0.73	0	318.13	27.07	97.00	98.20	90.08

A comparison of the results reveals that projection points of bounded measures might be found out of defined bounds in the CCR model and in model (2) as indicated in Tables 5 and 6, whilst they are established within specific boundaries using model (6). Also, it is possible that the CCR model and model (2) obtain non-integer projections for discrete measures as shown in column 3 of Table 5 and column 4 of Table 6, while model (6) finds integer-valued points for them. To illustrate it in more details, consider firm 18. $x_2^* = 28.62$, $y_1^* = 89$, $y_2^* = 93$ and $y_3^* = 95$ that have been estimated using model (6) within the defined bound, while $y_1^* = 102.26$ and $y_2^* = 103.39$ in the CCR model and model (2) that are over the specific boundary. Also, $x_1^* = 415$ in the suggested approach, whilst this amount reaches the non-integer value 323.61 in both the CCR model and in model (2). Furthermore, 5 firms are efficient in the CCR model, as shown in column 2 of Table 5. This amount reaches to 4 firms in model (2).

According to the majority rule, the role of quality costs in model (2) is also determined as the input, as can be seen in column 3 of Table 6. To illustrate, 12 firms considered it as input and 4 firms deemed it as outputs, while 4 firms are indifferent or unbiased related to input or output roles. Therefore, it is considered as the input.

To sum up, an obvious advantage of using the suggested individual *DMU* approach is finding appropriate projection points for bounded factors and integer measures.

4.2. An example of highway maintenance patrols

In this subsection, we assume there are 14 crews with one continuous input, total expenditure (x_1), two continuous outputs, assignment size factor (y_1) and average traffic served (y_2), and one bounded flexible measure, average pavement rating (w_1) (on the range from 0 to 100). Data are partially taken from [14]. It means that the amounts of average pavement condition rating are considered as different for representing this subject that there are situations that the projection points of bounded measures may not stand in the given bounds, and the efficiency results may be irrational and incorrect in some existing models in contrast to the proposed approach. The data set is given in Table 7.

Table 7. Patrol data and results

Crew	y_1	y_2	x_1	w_1	CCR	w_1^*	Model (2)	w_1^*	d	Model (6)	w_1^*	d
1	696	39	751	67	1.00	67.00	0.93	114.21	1	0.95	100.00	1
2	616	26	611	98	0.97	94.98	0.94	111.52	1	0.95	100.00	1
3	456	17	538	80	0.83	66.37	0.79	88.18	1	0.79	88.18	1
4	616	31	584	99	1.00	99.00	1.00	99.00	0 or 1	1.00	99.00	0 or1
5	560	16	665	95	0.83	79.03	0.78	109.27	1	0.79	100.00	1
6	446	16	445	75	0.95	71.37	0.93	87.02	1	0.93	87.02	1
7	517	26	554	76	0.93	70.70	0.88	83.13	1	0.88	83.13	1
8	492	18	457	96	1.00	96.00	1.00	96.00	0 or 1	1.00	96.00	0 or 1
9	558	23	582	74	0.97	71.87	0.90	102.40	1	0.90	100.00	1
10	407	18	556	64	0.76	48.40	0.69	71.63	1	0.69	71.63	1
11	402	33	590	78	0.88	68.67	0.88	68.67	0	0.88	68.67	0
12	350	88	1074	98	1.00	98.00	1.00	98.00	0 or 1	1.00	98.00	0 or 1
13	581	64	1072	74	1.00	74.00	0.86	113.99	1	0.87	100.00	1
14	413	24	696	97	0.62	60.44	0.62	60.44	0	0.62	60.44	0

To measure the performance and to compare the results, the CCR model and models (2) and (6) are applied at this stage. To estimate the relative efficiency using the CCR model, the flexible measure is assumed as the input.

The results of the CCR model are provided in columns 6 and 7 of Table 7 when the flexible measure is considered as an unbounded input. 5 patrols are determined as efficient in this case, while three patrols, 4, 8, and 12, are efficient in models (2) and (6).

A comparison of the projection points of average pavement rating (w_1^*) found from three approaches shows that model (2) may determine the projection points out of the bound as shown in column 9. Projection points are among the defined bound in the CCR model; however, the flexible measure has been considered as the input in this case. But model (6) involves the flexible measure and projection points stand in the given bound. According to the majority rule, both models (2) and (6) detect the role of the flexible measure as the output. Moreover, the role of average pavement rating from the manager's viewpoint is determined as the output with the efficiency score of 0.8283 obtained from model (7).

Therefore, the introduced approach can be used to analyse the relative efficiency of *DMU*s in the presence of bounded flexible measures.

4.3. An application to evaluate university branches

The proposed models are developed herein to measure the relative efficiency of 15 branches of a university in Iran and find projection points of measures. The relative efficiency analysis in education sectors, such as universities, addressed by many researchers, is an essential and beneficial affair for both society and individuals. Table 8 lists some of these research studies based on DEA.

As can be seen in Table 8, some previous studies on efficiency analysis of higher education sectors consist of bounded and integer measures. Furthermore, some measures, such as research income can play both input role and output role. However, a few authors have analysed the performance of higher education sectors by taking into account the kind of measures into evaluation and incorporating flexible measures. Following the literature review presented in Table 8 and as based on data availability, the factors described in Table 9 are used in this study.

As can be seen, the number of staff, the number of papers and the number of graduations are integer factors; therefore, we expect their projection point to be integer values. Furthermore, level of manager satisfaction is an ordinal data. To illustrate, 5-point Likert scale is used here, that is a bounded and integer measure. Note that research income has been taken as a flexible measure. Some authors consider it as the input due to its role in producing further outputs, and others treat it as the output because it is acquired by institutions. The data used in this research are given in Table 10 for a particular year.

Due to presence of flexible, bounded, and discrete measures, model (6) is calculated to analyse the performance of branches. The results are shown in Table 11. As can be seen in column 2, 5 branches are efficient with the efficiency score one. For these branches, research income can play either input or output role, as shown in column 3.

7 branches consider research income as the output, while branches 5, 8, and 13 take it into account as the input. Therefore, according to the majority rule, research income is considered as the output. Moreover, the projection points of integer measures and also the bounded integer indicator are provided in columns 4-7. It is clear that integer projection values are obtained for integer-valued variables and the projection point of the bounded integer-valued measure is resulted within the determined bound.

Table 8. Some DEA studies in the higher education sector

Reference	Input/output variables
Hamdi, Lotfi, and Moghaddas [19]	inputs: space, equipment, income, number of employees outputs: satisfactory, number of professors, number of students, number of fields of study, number of infringement, number of turning over to the committee of peculiar cases
Flegg, Allen, Field, Thurlow [17]	inputs: number of staff, number of undergraduate students, number of postgraduate students, aggregate departmental expenditure outputs: income from research and consultancy, number of undergraduate degrees awarded, adjusted for quality, number of postgraduate degrees awarded
García-Aracil, Palomares-Montero [18]	inputs: total expenditure, number of academic staff, number of non-academic staff outputs: number of graduates, number of publications, applied research
Abbott, Doucouliagos [1]	inputs: total number of academic staff, number of non-academic staff, expenditures on all other inputs than labour, value of non-current assets outputs: number of equivalent full-time students, research quantum allocation
Kempkes, Pohl [23]	inputs: number of research personnel, number of technical personnel, current expenditures outputs: number of graduates, amount of research grants
Beasley [5]	inputs: general expenditure, equipment expenditure, research income outputs: number of undergraduates, number of taught postgraduates, number of research postgraduates, research income, four research rating categories (star, above average, average and below average)
Beasley [6]	inputs: general expenditure, equipment expenditure, research income outputs: number of undergraduates, number of taught postgraduates, number of research postgraduates, research income, four research rating categories (star, above average, average and below average)
Katharaki, Katharakis [21]	inputs: number of academic staff, number of non-academic staff, number of active registered students, operating expenses other than labour inputs outputs: number of graduates, research income
Olariu, Brad [31]	inputs: number of academic staff, number of non-academic staff, the number of accredited programs in universities outputs: total number of undergraduate enrolment, the total number of graduate enrolment, the amount of money received from the state for basic institutional funding

Sagarra, Mar-Molinero, Agasisti) [32]	inputs:	full-time equivalent faculty, total enrolment, first joining graduates
	outputs:	scopus papers, graduates

Table 9. Variables description

Symbol	Measure	Type
Input		
x_1	expenditures	continuous
x_2	number of staff	integer
Output		
y_1	number of papers	integer
y_2	number of graduations	integer
y_3	level of manager satisfaction	bounded and integer (Likert scale)
Flexible		
w_1	research income	continuous

Table 10. Branches data

Branch	Inputs		Outputs			Flexible
	Expenditures	No. of staff	No. of papers	No. of graduations	Level of manager satisfaction	Research income
1	4939563	85	92	600	4	1485315
2	1736777	15	50	504	3	931847
3	5983527	76	24	2714	2	2057569
4	1473233	22	5	200	3	657430
5	27095070	311	330	3709	4	15342642
6	2046463	24	17	5	4	902708
7	2700994	32	39	494	2	409280
8	1851397	18	6	230	4	3179612
9	4549481	34	7	281	3	783669
10	16676280	256	376	1723	4	2129568
11	3030234	42	36	393	3	1314672
12	3288686	6	36	69	4	15004530
13	1247634	7	2	158	2	708878
14	1927629	7	23	626	2	708878
15	349579	6	0	113	3	298468

Afterwards, model (7) is solved to assess the aggregate efficiency of branches from the aggregate viewpoint. By computing the aforementioned model, the role of research income is distinguished as the input with an efficiency score 0.49. The comparison of results found by models (6) and (7) shows the different role detected for the research income in two models. According to Cook and Zhu [15], it is possible due to the fact that the aggregate model could be overly sensitive to extreme units or possibly to the larger units.

Table 11. Results of model (6)

Branch	Efficiency of model (6)	d^*	Projection points				Efficiency of model (3)	Projection point
			x_2^*	y_1^*	y_2^*	y_3^*		x_2^*
1	0.65	1	54	92	600	4	0.69	40
2	1	0 or 1	15	50	504	3	1.00	15
3	1	0 or 1	76	24	2714	2	1.00	76
4	0.43	1	9	5	200	3	0.44	7
5	0.45	0	110	330	3709	4	0.40	100
6	0.48	1	11	17	5	4	0.54	12
7	0.58	1	18	39	494	2	0.97	17
8	0.5	0	9	6	230	4	0.35	9
9	0.21	1	7	7	281	3	0.32	7
10	0.78	1	200	376	1723	4	1.00	256
11	0.48	1	18	36	393	3	0.44	13
12	1	0 or 1	6	36	69	4	1.00	15
13	0.71	0	5	2	158	2	0.49	5
14	1	0 or 1	7	23	626	2	1.00	7
15	1	0 or 1	6	0	113	3	1.00	6

Table 12. Results of the CCR model and model (2)

Branch	Efficiency of the CCR model	Projection points				Efficiency of model (2)	d^*	Projection points			
		x_2^*	y_1^*	y_2^*	y_3^*			x_2^*	y_1^*	y_2^*	y_3^*
1	0.65	27.6	92	927.36	5.52	0.65	1	27.6	92	927.36	5.52
2	1	15	50	504	3	1	0 or 1	15	50	504	3
3	1	76	24	2714	2	1	0 or 1	76	24	2714	2
4	0.43	8.25	5	200	3	0.43	1	8.25	5	200	3
5	0.51	111.73	330	3709	21.06	0.45	0	109.65	330	3709	19.89
6	0.48	10.89	17	276.18	4	0.48	1	10.89	17	276.18	4
7	0.58	14.51	39	494	2.36	0.58	1	14.51	39	494	2.36
8	0.65	10.21	7.55	230	4	0.47	0	8.48	6	230	4
9	0.2	6.97	7.77	281	3	0.2	1	6.97	7.77	281	3
10	1	104.28	376	3268.75	25.82	0.78	1	112.8	376	3790.08	22.56
11	0.48	12.5	36	393	3	0.48	1	12.5	36	393	3
12	1	6	36	69	4	1	0 or 1	6	36	69	4
13	0.72	5.01	7.89	158	2	0.62	1	4.31	7.05	158	2
14	1	7	23	626	2	1	0 or 1	7	23	626	2
15	1	6	0	113	3	1	0 or 1	6	0	113	3

To compare the findings gained from the proposed approach with the previous studies, we firstly regard the CCR model and model (2). Notice that the CCR model is solved whilst research income is considered as the output. The results obtained from them are presented in Table 12. Column 2 shows the relative efficiency score of the CCR model and the projection points of integer measures, and the integer bounded factor are provided in columns 3–6. As can be seen, there are 6 branches determined as efficient in the CCR model and there are some non-integer projections for integer measures. Also, for the bounded integer factor y_3^*, some non-integer projections out of the specified bound are obtained. See, to illustrate it, branches 1, 5, and 10. In model (2), 5 branches are evaluated as efficient, which is displayed in column 7. The status of research income is determined as the output by 8 branches through model (2), as shown in column 8; thus, according to the majority rule, the role of flexible measure is estimated as the output. Also, some non-integer values are achieved for integer factors, as is evident from columns 9–12. Furthermore, non-integer and beyond projections y_3^* are found for branches 1, 5, and 10, while integer projections inside the defined interval are obtained for bounded and integer variables in the proposed approach.

Finally, we compute model (3) while research income is deemed as the input. The results are presented in columns 8–9 of Table 11. As can be seen, there are differences between the efficiency scores and the projection points of the integer input resulting from models (3) and (6). However, the projection points of outputs in the two models are similar.

The most interesting aspect of the proposed approach is obtaining bounded and integer projection points for bounded and integer factors, alongside handling the status of flexible measures, while non-integer and outside points might be attained for integer measures and bounded factors in the CCR model and model (2). Furthermore, the comparison of the findings resulting from models (3) and (6) shows relative efficiency scores may be different in these models because model (3) considers the role of flexible measure known before the relative efficiency analysis, while they are unknown in model (6) and are obtained after the calculation. Notice that in these examples, the role of flexible measures has been determined according to the results of model (6) and there is no emphasis to calculate model (7). Nevertheless, it has been solved to detect its results. For more illustration, model (7) can be applied if the role of flexible measures as either input or output cannot be identified using model (6).

In summary, the assessment of the relative efficiency of *DMU*s and the exploration of strengths and weaknesses is substantial to progress and make plans. In many instances, determining the role of the performance measure as the input or the output is not simple. Also, bounded measures and integer values are presented in many investigations. Therefore, the proposed approach can be beneficial and valuable to evaluate the relative efficiency of firms, to classify performance measures and to specify projection points of variables where, in addition to bounded and discrete data, flexible measures are present and also in situations that bounded flexible measures are available.

5. Conclusions

Addressing the correct nature of the data has a significant role in the realistic and rational measurement of the firm performance and the investigation of the projection points of factors. There are situations in real-world applications in which the performance of firms should be evaluated when bounded, discrete, and flexible measures are present. To accomplish this aim, the current study employs approaches based on DEA to measure the relative efficiency of firms and to classify the flexible measures in the presence of bounded and discrete factors. It also analyses the performance in the presence of bounded flexible measures. To illustrate more, models from two viewpoints, including individual and aggregate ones, are designed to address the role of flexible measures. In addition, to estimate the relative efficiency and to determine the status of flexible measure, the suggested individual *DMU* approach can properly determine bounded and integer projection points of bounded and integer-valued performance measures (and the projection point of the set of firms has been identified from the aggregate viewpoint). To emphasise the advantages and explanatory power of our two models, the approaches suggested are applied to analyse the performance of quality management, highway maintenance patrols and university branches as three investigations. Also, the results are compared with the previous approaches.

The findings reveal that the introduced model from an individual aspect can accurately assess projection points of bounded and discrete measures. In contrast, the previous models under consideration might obtain incorrect projection points for discrete and bounded factors.

It is recommended that further research be undertaken in the following areas:

- investigating the performance of firms in the presence of discrete and negative data;
- estimating the efficiency of systems with imprecise discrete, bounded, and flexible measures;
- addressing the status of flexible performance measures and the efficiency analysis when discrete, bounded, and undesirable factors are presented.

References

[1] ABBOTT M., DOUCOULIAGOS C., *The efficiency of Australian universities: a data envelopment analysis*, Econ. Educ. Rev., 2003, 22, 89–97.

[2] AMIRTEIMOORI A., EMROUZNEJAD A., *Flexible measures in production process: a DEA-based approach*, RAIRO Oper. Res., 2011, 45, 63–74.

[3] AMIRTEIMOORI A., EMROUZNEJAD A., KHOSHANDAM L., *Classifying flexible measures in data envelopment analysis: a slack-based measure*, Measurement, 2013, 46, 4100–4107.

[4] BALA K., COOK W.D., *Performance measurement with classification information: an enhanced additive DEA model*, Omega, 2003, 31, 439–450.

[5] BEASLEY J.E., *Comparing university departments*, Omega, 1990, 18, 171–183.

[6] BEASLEY J.E., *Determining teaching and research efficiencies*, J. Oper. Res. Soc., 1995, 46, 441–452.

[7] BOĎA M., *Classifying flexible measures in data envelopment analysis: A slacks-based measure. A comment*, Measurement, 2020, 150, 107045.

[8] CHARNES A., COOPER W.W., RHODES E., *Measuring the efficiency of decision making units*, Eur. J. Oper. Res., 1978, 2, 429–444.

[9] CHEN Y., CHEN Y., OZTEKIN A., *A hybrid data envelopment analysis approach to analyse college graduation rate at higher education institutions*, INFOR, 2017, 55, 188–210.

[10] CHEN Y., COOK W.D., DU J., HU H., ZHU J., *Bounded and discrete data and Likert scales in data envelopment analysis: application to regional energy efficiency in China*, Ann. Oper. Res., 2017, 255, 347–366.

[11] CHEN Y., GONG Y., LI X., *Evaluating NBA player performance using bounded integer data envelopment analysis*, INFOR, 2017, 55, 38–51.

[12] COOK W.D., GREEN R.H., ZHU J., *Dual-role factors in data envelopment analysis*, IIE. Trans., 2006, 38, 105–115.

[13] COOK W.D., KRESS M., SEIFORD L.M., *On the use of ordinal data in data envelopment analysis*, J. Oper. Res. Soc., 1993, 44, 133–140.

[14] COOK W.D., ROLL Y., KAZAKOV A., *A DEA model for measuring the relative eeficiency of highway maintenance patrols*, INFOR, 1990, 28, 113–124.

[15] COOK W.D., ZHU J., *Classifying inputs and outputs in data envelopment analysis*, Eur. J. Oper. Res., 2007, 180, 692–699.

[16] COOPER W.W., PASTOR J.T., BORRAS F., APARICIO J., PASTOR D., *BAM: a bounded adjusted measure of efficiency for use with bounded additive models*, J. Prod. Anal., 2011, 35, 85–94.

[17] FLEGG A., ALLEN D., FIELD K., THURLOW T., *Measuring the efficiency and productivity of British universities: an application of DEA and the Malmquist approach*, University of the West of England, Department of Economics, series Discussion Papers, 2003, 304.

[18] GARCÍA-ARACIL A., PALOMARES-MONTERO D., *Evaluation of Spanish universities: efficiency, technology and productivity change*, Prime-Latin America Conference at Mexico City, September 2008, 24–26.

[19] HAMDI K., LOTFI F.H., MOGHADDAS Z., *An application of DEA in efficiency evaluation of universities*, Int. J. Math. Oper. Res., 2014, 6, 550–566.

[20] JOULAEI M., MIRBOLOUKI M., BAGHERZADEH-VALAMI H., *Classifying fuzzy flexible measures in data envelopment analysis*, J. Intell. Fuzzy. Syst., 2019, 36, 3791–3800.

[21] KATHARAKI M., KATHARAKIS G., *A comparative assessment of Greek universities' efficiency using quantitative analysis*, Int. J. Educ. Res., 2010, 49, 115–128.

[22] KAZEMI MATIN R., EMROUZNEJAD A., *An integer-valued data envelopment analysis model with bounded outputs*, Int. Trans. Oper. Res., 2011, 18, 741–749.

[23] KEMPKES G., POHL C., *The efficiency of German universities – some evidence from nonparametric and parametric methods*, Appl. Econ., 2010, 42, 2063–2079.

[24] KORDROSTAMI S., AMIRTEIMOORI A., JAHANI SAYYAD NOVEIRI M., *Inputs and outputs classification in integer-valued data envelopment analysis*, Measurement, 2019, 139, 317–325.

[25] KORDROSTAMI S., FARAJPOUR G., JAHANI SAYYAD NOVEIRI M., *Evaluating the efficiency and classifying the fuzzy data: A DEA based approach*, Int. J. Ind. Math., 2014, 6, 321–327.

[26] KORDROSTAMI S., JAHANI SAYYAD NOVEIRI M., *Evaluating the efficiency of decision making units in the presence of flexible and negative data*, Indian J. Sci. Technol., 2012, 5, 3776–3782.
[27] KORDROSTAMI S., JAHANI SAYYAD NOVEIRI M., *Evaluating the performance and classifying the interval data in data envelopment analysis*, Int. J. Manage. Sci. Eng. Manage., 2014, 9, 243–248.
[28] KUAH C.T., WONG K.Y., BEHROUZI F., *Application of data envelopment analysis to assess quality management efficiency*, Int. J. Ind. Manuf. Eng., 2010, 4, 1066–1071.
[29] LOZANO S., VILLA G., *Data envelopment analysis of integer-valued inputs and outputs*, Comp. Oper. Res., 2006, 33, 3004–3014.
[30] NABAHAT M., SANGARI F.E., *Classifying inputs and outputs with fuzzy data*, J. Math. Res., 2012, 4, 46–50.
[31] OLARIU G.V., BRAD S., *Efficiency assessment of universities with DEA method based on public data*. Balkan Region Conference on Engineering and Business Education De Gruyter Open, 2017, 106–114.
[32] SAGARRA M., MAR-MOLINERO C., AGASISTI T., *Exploring the efficiency of Mexican universities: integrating data envelopment analysis and multidimensional scaling*, Omega, 2017, 67, 123–133.
[33] SEDIGHI HASSAN KIYADEH M., SAATI S., KORDROSTAMI S., *Improvement of models for determination of flexible factor type in data envelopment analysis*, Measurement, 2019, 137, 49–57.
[34] TOHIDI G., MATROUD F., *A new non-oriented model for classifying flexible measures in DEA*, J. Oper. Res. Soc., 2017, 68, 1019–1029.
[35] TOLOO M., *Alternative solutions for classifying inputs and outputs in data envelopment analysis*, Comp. Math. Appl., 2012, 63, 1104–1110.
[36] TOLOO M., ALLAHYAR M., HANČLOVÁ J., *A non-radial directional distance method on classifying inputs and outputs in DEA: Application to banking industry*, Expert. Syst. Appl., 2018, 92, 495–506.
[37] TOLOO M., KESHAVARZ E., HATAMI-MARBINI A., *Dual-role factors for imprecise data envelopment analysis*, Omega, 2018, 77, 15–31.
[38] TOLOO M., MENSAH E.K., *Robust optimization with nonnegative decision variables: a DEA approach*, Comput. Ind. Eng., 2019, 127, 313–325.
[39] ZHANG B., FENG C., YANG M., XIE J., CHEN Y., *Bounded and discrete data in data envelopment analysis with assurance regions: application to design performance evaluation of gear shaping machines*, J. Model. Manage., 2020, 15, 1017–1036.
[40] ZHU J., *Imprecise data envelopment analysis (IDEA): a review and improvement with an application*, Eur. J. Oper. Res., 2003, 144, 513–529.