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Stability of fractional positive nonlinear systems

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The conditions for positivity and stability of a class of fractional nonlinear continuous-time systems are established. It is assumed that the nonlinear vector function is continuous, satisfies the Lipschitz condition and the linear part is described by a Metzler matrix. The stability conditions are established by the use of an extension of the Lyapunov method to fractional positive nonlinear systems.

Key words: positive, fractional, nonlinear, system, stability.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [5, 8].

Positive linear systems with different fractional orders have been addressed in [10, 14–17]. Stability of positive linear systems has been investigated in [1, 5, 8] and of fractional linear systems in [2–4]. The problem of preservation of positivity by approximation the continuous-time linear systems by corresponding discrete-time linear systems has been addressed in [6, 7]. The approximation of positive stable continuous-time linear systems by positive stable discrete-time linear systems has been considered in [6]. Descriptor (singular) fractional linear systems have been analyzed in [11–13]. The stability of a class of nonlinear fractional-order systems has been analyzed in [18]. In this paper the positivity and asymptotic stability of a class of fractional nonlinear

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systems will be addressed. Necessary and sufficient conditions for the positivity and sufficient conditions for the asymptotic stability will be established.

The paper is organized as follows. In section 2 necessary and sufficient conditions for a class of fractional nonlinear continuous-time systems are established. The asymptotic stability of the nonlinear fractional system by the use of an extension of the Lyapunov method is addressed in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices and $\Re^{n}=\Re^{n\times 1}$, $\Re_{+}^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_{+}^{n}=\Re_{+}^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. Positivity of a class of fractional nonlinear systems

Consider the fractional nonlinear continuous-time system

$$ {}_0D_t^{\alpha}x(t)=\frac{d^{\alpha}x(t)}{dt^{\alpha}}=Ax(t)+f(x(t)),\quad 0<\alpha<1 \tag{1}$$

where

$$ {}_0D_t^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}}d\tau,\ \ \dot{x}(\tau)=\frac{dx(\tau)}{d\tau} \tag{2}$$

is the Caputo fractional derivative of the order α of the state vector $x(t)\in\Re^{n}$ and

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt,\ \ \mathrm{Re}(z)>0 \tag{3}$$

is the Euler gamma function, $A\in\Re^{n\times n}$ and $f(x(t)\ \in\Re^{n}$ is the continuous vector function of $x(t)$.

It is assumed that the nonlinear equation (1) has a solution $x(t)$ and the Lipschitz condition

$$|f_i(t,x_1',...,x_n')-f_i(t,x_1'',...,x_n'')|<c\{|x_1'-x_1''|+...+|x_n'-x_n''|\}\geq 0,\quad t\geq 0,\quad i=1,2,...,n \tag{4}$$

is satisfied for some constant c and x_k',x_k'' for $k=1,2,...,n$ are some values of the state vector $x_k(t)$.

Definition 1 *The fractional nonlinear system* (1) *is called (internally) positive if* $x(t)\in\Re_{+}^{n}$ *for all initial conditions* $x_0=x(0)\in\Re_{+}^{n}$.

Theorem 1. *The fractional nonlinear system* (1) *is positive if*

$$A\in M_n \text{ and } f(x(t)\ \in\Re_{+}^{n} \text{ for } x(t)\in\Re_{+}^{n} \tag{5}$$

Proof It is well-known [14] that if $f(x(t) = 0$ than $x(t) \in \Re_+^n$, $t \geq 0$ if and only if $A \in M_n$ and $x_0 \in \Re_+^n$. By assumption the equation (1) has a solution and the condition (4) is satisfied. Using the Picard method it can be shown that the equation has a solution $x(t) \in \Re_+^n$ if the condition (5) are met. □

6. Stability of continuous-time nonlinear systems

Consider the positive continuous-time nonlinear system

$$\dot{x} = Ax + f(x), \tag{6}$$

where $x = x(t) \in \Re^n$, $A \in M_n$, $f(x) \in \Re_+^n$ is a continuous and bounded vector function and $f(0) = 0$.

Definition 2 *The positive continuous-time nonlinear system* (6) *is called asymptotically stable in the region* $D \in \Re_+^n$ *if* $x(t) \in \Re_+^n$, $t \geq 0$ *and*

$$\lim_{t \to \infty} x(t) = 0 \text{ for any finite } x_0 \in D \in \Re_+^n. \tag{7}$$

To test the asymptotic stability of the positive system (6) the Lyapunov method will be used. As a candidate of Lyapunov function we choose

$$V(x) = c^T x > 0 \text{ for } x = x(t) \in \Re_+^n,\ t \geq 0 \tag{8}$$

where $c \in \Re_+^n$ is a vector with strictly positive components $c_k > 0$ for $k = 1,...,n$.

Using (8) and (6) we obtain

$$\dot{V}(x) = c^T \dot{x} = c^T [Ax + f(x)] < 0 \tag{9}$$

for

$$Ax + f(x) < 0 \text{ for } x \in D \in \Re_+^n,\ t \geq 0 \tag{10}$$

since $c \in \Re_+^n$ is strictly positive vector.

Therefore, the following theorem has been proved. □

Theorem 2 *The positive continuous-time nonlinear system* (6) *is asymptotically stable in the region* $D \in \Re_+^n$ *if the condition* (10) *is satisfied.*

Example 1 Consider the nonlinear system (6) with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad f(x) = \begin{bmatrix} x_1 x_2 \\ x_2^2 \end{bmatrix}. \tag{11}$$

The nonlinear system (6) with (11) is positive since $A \in M_2$ and $f(x) \in \Re_+^2$ for all $x \in \Re_+^2$, $t \geq 0$.

In this case the condition (9) is satisfied in the region D defined by

$$D := \{x_1, x_2\} = \begin{bmatrix} -2x_1 + x_2 + x_1 x_2 \\ x_1 - 3x_2 + x_2^2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

From (12) we have

$$x_1(2 - x_2) > x_2 > 0 \text{ and } 0 \leq x_1 < (3 - x_2)x_2 . \tag{13}$$

The region D is shown on the Fig. 1.

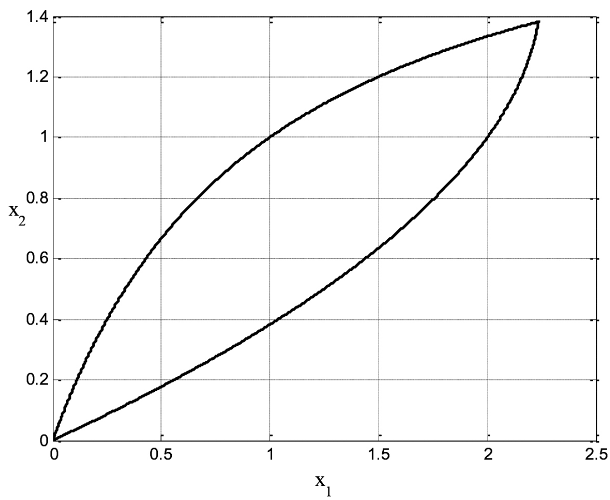

Fig. 1. Stability region (inside the curved line).

By Theorem 2 the positive nonlinear system (6) with (11) is asymptotically stable in the region (12).

4. Concluding remarks

The positivity and stability of a class of fractional nonlinear continuous-time systems described by the equation (1) has been addressed. Necessary and sufficient conditions for the positivity (Theorem 1) and sufficient conditions for the asymptotic stability (Theorem 2) have been established. The considerations have been illustrated by numerical example of nonlinear fractional system. An open problem is an extension of the results to descriptor fractional continuous-time nonlinear systems.

References

[1] A. BERMAN and R.J. PLEMMONS: Nonegative Matrices in the Mathematical Sciences. SIAM, 1994.

[2] M. BUSŁOWICZ: Stability of linear continuous-time fractional order systems with delays of the retarded type. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, **56**(4), (2008), 319-324.

[3] M. BUSŁOWICZ: Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, **60**(2), (2012), 279-284.

[4] M. BUSŁOWICZ and T. KACZOREK: Simple conditions for practical stability of positive fractional discrete-time linear systems, *Int. J. of Applied Mathematics and Computer Science*, **19**(2), (2009), 263-169.

[5] L. FARINA and S. RINALDI: Positive Linear Systems; Theory and Applications. J. Wiley, New York, 2000.

[6] T. KACZOREK: Approximation of positive stable continuous-time linear systems by positive stable discrete-time systems. *Pomiary Automatyka Robotyka*, **2**, (2013), 359–364.

[7] T. KACZOREK: Comparison of approximation methods of positive stable contiunuous-time linear systems by positive stable discrete-time systems. Computer Applications in Electrical Engeenering, Pub. House of Poznan University of Tehnology, Poznań, 2013, 1-10.

[8] T. KACZOREK: Positive 1D and 2D Systems. Springer Verlag, London, 2002.

[9] T. KACZOREK: Linear Control Systems. **1**, Research Studies Press J. Wiley, New York, 1992.

[10] T. KACZOREK: Positive linear systems with different fractional orders. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, **58**(3), (2010), 453-458.

[11] T. KACZOREK: Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils. *Int. J. of Applied Mathematics and Computer Science*, **23**(1), (2013), 29-34.

[12] T. KACZOREK: Reduction and decomposition of singular fractional discrete-time linear systems. *Acta Mechanica et Automatica*, **5**(4), (2011), 62-66.

[13] T. KACZOREK: Singular fractional discrete-time linear systems. *Control and Cybernetics*, **40**(3), (2011), 753-761.

[14] T. KACZOREK: Selected Problems of Fractional System Theory, Springer Verlag, Berlin, 2011.

[15] T. KACZOREK: Fractional positive continuous-time linear systems and their reachability. *Int. J. of Applied Mathematics and Computer Science*, **18**(2), (2008), 223-228.

[16] T. KACZOREK: Positive linear systems consisting of *n* subsystems with different fractional orders. *IEEE Trans. on Circuits and Systems*, **58**(7), (2011), 1203-1210.

[17] T. KACZOREK: Positive fractional continuous-time linear systems with singular pencils. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, **60**(1), (2012), 9-12.

[18] W. XIANG-JUN, W. ZHENG-MAO and L. JUN-GUO: Stability analysis of a class of nonlinear fractional-order systems. *IEEE Trans. Circuits and Systems-II, Express Briefs*, **55**(11), (2008), 1178-1182.