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FREE VIBRATION ANALYSIS OF POINT SUPPORTED RECTANGULAR PLATES USING QUADRATURE ELEMENT METHOD

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In this study, the hybrid approach of the Quadrature Element Method (QEM) has been employed to generate solutions for point supported isotropic plates. The Hybrid QEM technique consists of a collocation method with the Galerkin finite element technique to combine the high accurate and rapid converging of Differential Quadrature Method (DQM) for efficient solution of differential equations. To present the validity of the solutions, the results have been compared with other known solutions for point supported rectangular plates. In addition, different solutions are carried out for different type boundary conditions, different locations and number of point supports. Results for the first vibration modes of plates are also tested using a commercial finite element code, and it is shown that they are in good agreement with literature.

Keywords: Quadrature Element Method, point support, plates, free vibration

1. Introduction

In the applications of modern structures, i.e. carousers, building floors, bridge decks, solar panels, aircraft and ship industries, bolted, riveted or spot-welded plate bodies are used. Designers have to know how these components change the dynamic characteristic of the structures. These types of engineering problems are known as point supported plate problems and they are frequently encountered in practice. Both analytical and numerical methods have been developed for the analysis of these problems. Although there are no exact solutions for these problems, various numerical approaches have been utilized. For example, Cox and Boxer (1960) used a finite difference method, Damle and Feeser (1972) used the finite element method, Fan and Cheung (1984) used the spline finite strip method, Huang and Thambiratnam (2001a) used the finite strip method, Guiterrez and Laura (1995) used dthe ifferential quadrature method, Zhao *et al.* (2002) used the discrete singular convolution method to solve the mentioned plate vibration problems. Because of its high accuracy, the Rayleight-Ritz method has been the most frequently used analytical method to appeal for vibration analysis of plates, as Narita and Hodgkinson (2005) did. Also Gorman (1991) and Bapat and Suryanarayan (1989) utilized the superposition method and the flexibility function approach as analytical techniques, respectively.

Several functions are used for the analysis of free vibration of point supported rectangular plates. These include vibrating beam functions (Kerstens, 1979), B-spline functions (Mizusawa and Kajita, 1987)] and orthogonal polynomial functions (Kim and Dickinson, 1987). On the other hand, Liew and Lam (1994) applied a set of orthogonal plate functions generated by using the Gram-Schmidt orthogonality relationship to elastic point supported rectangular plates. Lee and Lee (1997) used a new type of the admissible function. Kitipornchai *et al.* (1994) and Liew *et al.* (1994) applied the Lagrange multiplier method and the constrain function method to point supported Mindlin plates. Cheung and Zhou (1999, 2000) used the static beam function to composite plates and used the finite layer method to layered rectangular plates with point supports. Saadatpoure *et al.* (2000) studied vibration of plates having a general shape with internal point and line supports using the Galerkin method. Huang and Thambiratnam (2001b) applied a procedure incorporating the finite strip method together with spring systems for treating plates on elastic intermediate supports. Zhou (2002) used a set of static tapered beam functions which were the solutions of a tapered beam under a Taylor series of static loads developed as admissible functions for vibration analysis of point-supported rectangular plates with variable thickness in one or two directions. Again, Zhao *et al.* (2002) studied the problem of plate vibration under complex and irregular internal support conditions using the discrete singular convolution method. Kocatürk *et al.* (2004) used Lagrange equations to examine the steady state response to a sinusoidally varying force applied at the centre of a viscoelastically point-supported orthotropic elastic plate of rectangular shape with considered locations of added masses.

The Differential Quadrature Method (DQM) was proposed by Bellman and Casti (1971) in the early 1970's as an efficient numerical method to solve non-linear partial differential equations and applied to many areas of engineering problems. Especially, the Generalized Differential Quadrature Method (GDQM) has been used by various researches for efficient treatment of structural analysis problems. Analyses yielded good to excellent results for only a few discrete points due to the use of high order global basis functions in the computational domain. However, especially for real-world problems, DQM still lacks flexibility. Recently, Chen *et al.* (2000) extended the DQM to analysis of various structures and then it called the Quadrature Element Method (QEM). 49 degree of freedom (DOF) quadrature plate element was developed by Striz *et al.* (1994) to alleviate the lack of versatility and limitations of the existing high order series type approximation method. Different versions of the Differential Quadrature Method have been used for various applications. Hybrid approach was further developed by Han and Liew (1996) to solve the one-dimensional bending problem of the axisymmetric shear deformable circular plate, and by Liu and Liew (1998, 1999a,b) and Liu (2000) to solve two-dimensional bending and vibration problems of thick rectangular plates and polar plates having discontinuities. Wang and Gu (1997a,b) made an attempt to solve static problems of truss and beams and static and free vibration problems of thin plates. DQM was used by Liu and Liew (1999b) for the study of a two dimensional polar Reissner-Mindlin plate in the polar coordinate system by integrating the domain decomposition method (DDM). The Differential Quadrature Finite Difference Method (DQFDM) was proposed and applied by Chen (2004) for analysis of 2-D heat conduction in orthotropic media. Franciosi and Tomasiello (2004) applied a modified quadrature element method to perform static analysis of structures.

In this paper, the Quadrature Element Method is proposed and applied to analyze free vibration of point supported rectangular plates. Plates having different boundary conditions and various point topologies are studied. The results are compared with the studies using other approximating methods known in literature. First, interior and/or exterior point supported free plates and then, interior point supported plates having various boundary conditions are presented. Solutions are tested with the results of ABAQUS, a finite element program which has a wide spread use in the analysis of engineering problems.

2. Formulation of the quadrature plate element

The Hybrid Quadrature Element technique consists of a collocation method in conjunction with the Galerkin finite element technique to combines the high accuracy and rapid converging of DQM for efficient solution of differential equations with the generality of the finite element formulation (Chen *et al.*, 2000).

The quadrature plate element is closely related to the serendipity Lagragian element, but it has internal points and basis functions of high order (Chen *et al.*, 2000). Numerical procedures are extensively used in the element formulation to circumvent the problems caused by the use of high order basis functions. C_0 and C_1 inter-element compabilities are met exactly for the mid--surface, while the other C_2 or even C_3 compabilities are closely approximated at each boundary by the use of moderately high order basis functions. The 25 node rectangular element is given in Fig. 1. This plate element has also 49 degrees of freedom. These degrees of freedom, which belong to the plate element, are given in Table 1 (Chen *et al.*, 2000; Quan and Chang, 1989).

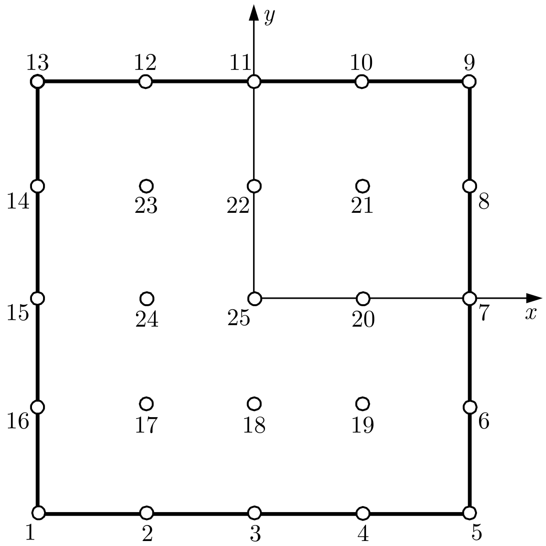


Fig. 1. Nodes of the Quadrature plate element

Table 1. Degrees of freedom for 25 node quadrature plate elements

Nodal number	DOF
1-5 9-13	$w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ \dfrac{\partial^2 w}{\partial x \partial y}$
2-3-4 10-11-12	$w,\ \dfrac{\partial w}{\partial y}$
6-7-8 14-15-16	$w,\ \dfrac{\partial w}{\partial x}$
17-18-19 20-21-22 23-24-25	w

The displacements of 25 nodes and 49 degrees of the freedom quadrature plate element are expressed in terms of polynomial type basis functions, i.e.

$$
\begin{aligned}
w(x,y) = & \sum_{i=1,5,9,13} \Big[N_{i1} w_i + N_{i2}\Big(\frac{\partial w}{\partial x}\Big)_i + N_{i3}\Big(\frac{\partial w}{\partial y}\Big)_i + N_{i4}\Big(\frac{\partial^2 w}{\partial x \partial y}\Big)_i\Big] \\
& + \sum_{i=2,3,4,10,11,12} \Big[N_{i1} w_i + N_{i2}\Big(\frac{\partial w}{\partial y}\Big)_i\Big] + \sum_{i=6,7,8,14,15,16} \Big[N_{i1} w_i + N_{i2}\Big(\frac{\partial w}{\partial x}\Big)_i\Big] \\
& + \sum_{\substack{i=17,18,19,20,\\ 21,22,23,24,25}} \Big[N_{i1} w_i\Big] = \mathbf{N}\mathbf{w}
\end{aligned}
\tag{2.1}
$$

where N_{ij} is the shape function which can be determined from the specified collocation points, and w_i, $(\partial w/\partial x)_i$, $(\partial w/\partial y)_i$, $(\partial^2 w/\partial x\partial y)_i$ are local DOFs associated with the node i.

The governing equation of the isotropic thin plate in small deflection free vibration is given by

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{\rho h \omega^2}{D} w \tag{2.2}$$

and Kirchhoff's plate theory, in which the bending strain of the element is given for an isotropic and homogeneous plate as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{2.3}$$

If Eq. (2.1) and Eq. (2.3) are combined, the strain-displacement relationship is stated by

$$\boldsymbol{\varepsilon} = -z \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \mathbf{N}\mathbf{w} = -z\mathbf{Q}\mathbf{w} \qquad \text{for} \qquad \mathbf{Q} = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \mathbf{N} \tag{2.4}$$

The stiffness matrix can be calculated for the area A

$$\mathbf{K} = \int_A \mathbf{Q}^{\mathrm{T}} \mathbf{D} \mathbf{Q} \, dA \tag{2.5}$$

where $\mathbf{D}$ is the rigidity matrix which can be calculated using constant thickness h, Poisson's ratio ν and the modulus of elasticity E

$$\mathbf{D} = \frac{Eh^2}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{2.6}$$

The consistent mass matrix can be calculated as

$$\mathbf{M} = \int_A \mathbf{N}^{\mathrm{T}} (\rho \mathbf{h}) \mathbf{N} \, dA \tag{2.7}$$

and the governing equation for plate free vibration can be written in the matrix form

$$(\mathbf{K}_s - \lambda^2 \mathbf{M}_s)\mathbf{w} = \mathbf{0} \tag{2.8}$$

where λ is the frequency parameter, and the subscribed s represents the whole discretized system.

3. Numerical application and discussions

Frequency parameters of free vibrations are described as $\lambda = \omega L^2 \sqrt{\rho h/D}$, where ω, L, ρ, h, D represent circular frequency, length of the plate, density, thickness and rigidity, respectively.

In order to obtain more accurate results, QEM solutions have been carried out by using 2×2 and 4×4 differential quadrate plate elements joined side by side along the x and y directions. When a larger number of plate elements are used more accurate results can be obtained, but the solution can be obtained with a larger linear system of equations. If there are simply supported boundary conditions on all edges of the plate considered then the quadrate plate element has only 25 DOFs. In other words, a set of 25×25 linear equations system has to be solved for one plate element. The size of the linear equations system is set to 400×400 for the same procedure needed be to solve with the same boundary conditions and the 4×4 plate element.

First, the number of plate elements that can be used for results having acceptable accuracy must be decided. Therefore, frequency parameters for three boundary conditions and four plate elements are obtained with QEM. Table 2 presents the frequency parameters λ of isotropic rectangular plates. It is interesting that acceptable accuracy results are obtained by QEM for all boundary conditions in the case of only one plate element.

Table 2. The first frequency parameters λ of isotropic square plates for some boundary conditions ($\lambda = \omega L^2\sqrt{\rho h/D}$)

	Exact	Number of use DQ plate elements			
	(Leissa, 1973)	1×1	2×2	3×3	4×4
S-S-S-S	19.73921	19.73921 $(7.0 \cdot 10^{-4})^*$	19.73921 $(2.2 \cdot 10^{-5})^*$	19.73921 $(4.3 \cdot 10^{-7})^*$	19.73921 $(1.6 \cdot 10^{-8})^*$
S-F-S-F	9.63138	9.63139	9.63138	9.63138	9.63138
S-C-S-S	23.64632	23.64700	23.64632	23.64632	23.64632

* Relative error in parenthesis has been evaluated using the analytical Leissa value ($2\pi^2$) [%]

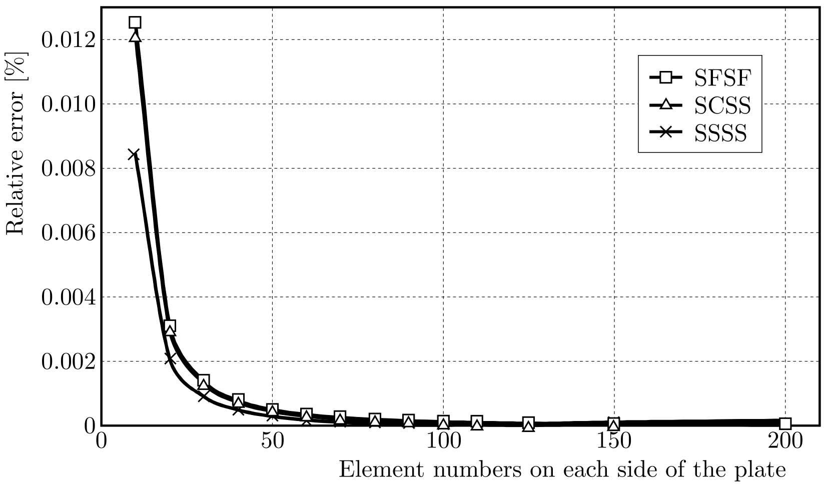


Fig. 2. Relative error determined by ABAQUS for different boundary conditions

Besides, the same boundary conditions given in Table 2 are solved using ABAQUS commercial finite codes. It is obvious that if more elements are used in computation, the error will be reduced. However, the required number of elements must be determined for acceptable accuracy. The variation of the relative error with selected degrees of freedom is given in Fig. 2 for different boundary conditions. Relative errors have been evaluated using the analytical results of Leissa (1973). This % error value of the relative difference is defined as (Analytical Leissa value-ABAQUS result)×100/(Analytical Leissa value). Naturally, the result changes when different boundary conditions are used. As in many literature sources the 4 node thin shell elements (S4R) are employed, the uniform mesh size and different element numbers on each side of the plate such as 10, 20, 50, 100, 200 and 400 scales are used to achieve convergent FEM solutions

(Rui *et al.*, 2015, 2016). In this study, the results have been given for all values from 100 SR4 shell elements on each side of the plate. For these elements, there are approximately 49,800 DOFs. As shown in Fig. 2, the biggest % relative error for SFSF boundary conditions to the selected number of elements is 0.01%.

In order to simplify the visualisation of types of supports which are used in tables and figures, symbols in Table 3 are to be used. The number of elements used in ABAQUS should be determined to obtain an acceptable solution for simply supported rectangular plates with point supports at the centre, as this type of problems is found in numerous literature items. Simply supported rectangular plates with a point support at centre are shown in Fig. 3. The results of QEM (2×2 and 4×4) are presented in Table 4 with other solutions for which different methods are applied. For the first five frequency parameters λ, all results are also in good agreement. Especially, the results of the finite strip element method used by Huang and Thambiratnam (2001) are strongly in agreement with QEM. If it is assumed that the first mode is 49.483 as it was taken from results of Huang's solution (Huang and Thambiratnam, 2001), Fig. 4 shows the change in the results from ABAQUS solution as a function of the number of elements on each side of the plate. It can be seen that the relative error according to Huang's results is approximately 0.03% for 100 elements on each side of the plate.

Table 3. Simplified support type symbols

Symbol	Support types
Null	Free
//////////////	Fixed
- - - - - - -	Simply
∘	Point

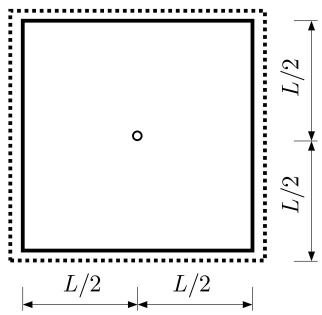


Fig. 3. Simply supported square plates with a point-support at center

As shown in Fig. 5, five boundary conditions and point support at the corner of the plate are considered. In Table 5, the results of Kim and Dickinson (1987) – orthogonal polynomial, Cheung and Zhou (2000) – static beam function, and Mizusawa and Kajita (1987) – finite spline, are presented. CFCF, CFSF, SFSF, CFFF, SFFF boundary conditions are considered and first five frequency parameters are presented. The natural frequencies are determined using both QEM and ABAQUS, and the obtained results are in good agreement with the analytical results reported in the literature.

For several cases, the results for plates with point supports are compared with other values given in the literature. As shown in Fig. 6, plates with different numbers of point supports at the interior and/or boundary are considered. All results obtained from ABAQUS and QEM solutions are presented in Table 6. Kato and Honma (1998), Kim and Dickinson (1987) used Rayleight-Ritz Method, Fan and Cheung (1984), Mizusawa and Kajita (1987) used Spline Finite Strip Element Method, Narita and Hodgkinson (2005) used Layerwise optimization method, Venkateswara *et*

Table 4. Frequency parameters λ of simply supported square plates with a point support at the center ($\lambda = \omega L^2\sqrt{\rho h/D}$)

Method	λ_1	λ_2	λ_3	λ_4	λ_5
Venkateswara *et al.* (1973)	–	–	52.62	–	–
Lee and Lee (1977)	–	–	53.088	–	–
Leissa (1969)	49.3	–	–	–	–
Saadatpour *et al.* (2000)]	49.348	–	–	–	–
Fan and Cheung (1984)	49.35	49.35	52.78	78.96	98.71
Kim and Dickinson (1987)]	49.348	49.348	53.170	78.959	98.696
Huang and Thambiratnam (2001b)	49.348	49.351	52.667	78.959	98.711
Present (ABAQUS)	49.362	49.362	52.643	78.975	98.784
Present (QEM, 2×2)	49.348	49.348	52.851	78.957	98.711
Present (QEM, 4×4)	49.348	49.348	52.677	78.957	98.696

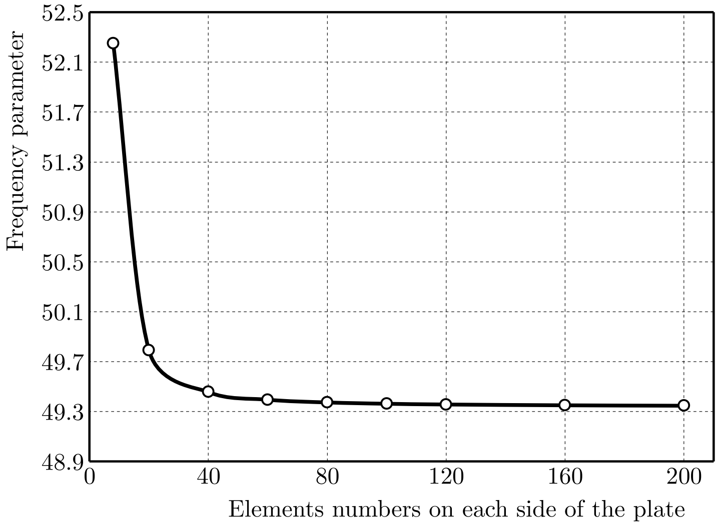


Fig. 4. First frequency parameters for simply supported square plate with a point support at center (ABAQUS solutions)

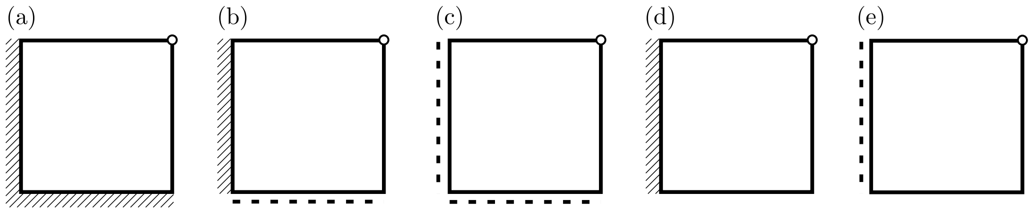


Fig. 5. Square plates with point supports at one corner for various boundary conditions

al. (1973) used Finite Element Method. Kocatürk *et al.* (2004) used the Lagrange Equation Method. The first five frequency parameters for eight different point supports situation are given in Table 6. It can be seen from Table 6, a very good agreement between QEM and those of Kato and Honma (1998), Kim and Dickinson (1987), Mizusawa and Kajita (1987), Narita and Hodgkinson (2005), Venkateswara *et al.* (1973), Kocatürk *et al.* (2004) are encountered.

Various point support topologies and four different types of boundary conditions are considered as shown in Table 7. The minimum distances are $L/4$ since four quadrature plate elements are used for solutions. Seven different situations are considered and the first five frequency parameters are calculated. SSSS, CCCC, SCSC and FCFS type of boundary conditions are selected.

Table 5. Frequency parameters λ of square plates with point supports at one corner for various boundary conditions ($\lambda = \omega L^2\sqrt{\rho h/D}$)

Fig.	Method	λ_1	λ_2	λ_3	λ_4	λ_5
5a	Cheung and Zhou (1999)	15.272	24.100	39.495	54.703	63.511
	Mizusawa and Kajita (1987)	15.12	23.70	39.37	53.53	62.54
	Kim and Dickinson (1987)	15.172	23.923	39.392	54.157	62.850
	Present (ABAQUS)	15.166	23.905	39.394	54.105	62.742
	Present (QEM, 2×2)	15.169	23.915	39.389	54.112	62.718
	Present (QEM, 4×4)	15.166	23.906	39.388	54.094	62.708
5b	Cheung and Zhou (1999)	12.021	21.348	35.140	47.916	58.903
	Mizusawa and Kajita (1987)	11.94	21.06	35.01	47.24	57.92
	Kim and Dickinson (1987)	11.940	21.175	35.015	47.398	58.144
	Present (ABAQUS)	11.939	21.167	35.018	47.399	58.096
	Present (QEM, 2×2)	11.939	21.172	35.014	47.393	58.076
	Present (QEM, 4×4)	11.939	21.167	35.014	47.388	58.069
5c	Cheung and Zhou (1999)	9.6801	17.496	30.713	44.178	51.873
	Mizusawa and Kajita (1987)	9.608	17.32	30.60	43.65	51.04
	Kim and Dickinson (1987)	9.6079	17.316	30.596	43.652	51.041
	Present (ABAQUS)	9.6079	17.317	30.598	43.663	51.058
	Present (QEM, 2×2)	9.6079	17.316	30.596	43.652	51.036
	Present (QEM, 4×4)	9.6079	17.316	30.596	43.652	51.035
5d	Cheung and Zhou (1999)	5.3351	16.054	22.000	29.536	43.894
	Mizusawa and Kajita (1987)	5.312	15.86	21.71	29.29	43.39
	Present (ABAQUS)	5.3261	15.912	21.813	29.403	43.499
	Present (QEM, 2×2)	5.3277	15.915	21.817	29.407	43.497
	Present (QEM, 4×4)	5.3268	15.912	21.812	29.403	43.494
5e	Cheung and Zhou (1999)	3.3395	12.033	17.419	25.886	38.982
	Mizusawa and Kajita (1987)	3.336	11.93	17.29	25.68	38.56
	Present (ABAQUS)	3.3357	11.927	17.293	25.681	38.561
	Present (QEM, 2×2)	3.3361	11.927	17.293	25.680	38.555
	Present (QEM, 4×4)	3.3361	11.927	17.293	25.679	38.555

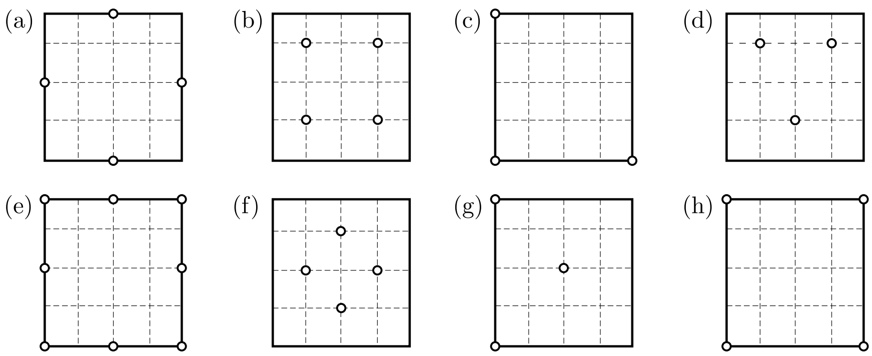


Fig. 6. Square plates with point supports

Besides, the results of point supported free plates are given in Table 8. The first five frequency parameters are presented for point supports on the interior and/or boundary of plates. The differences between the results of QEM and ABAQUS solutions are approximately 0.1%

Table 6. Comparison of frequency parameters λ of square plates with point supports ($\lambda = \omega L^2 \sqrt{\rho h/D}$)

Fig.	Method	λ_1	λ_2	λ_3	λ_4	λ_5
6a	Kato and Honma (1998)	13.51	18.03	19.05	19.05	27.26
	Kim and Dickinson (1987)	13.47	18.03	18.93	18.93	27.05
	Fan and Cheung (1984)	13.47	17.85	18.79	18.79	26.92
	Narita and Hodgkinson (2005)	13.47	18.14	19.02	19.02	–
	Present (ABAQUS)	13.468	17.835	18.780	18.780	26.910
	Present (QEM)	13.468	17.841	18.786	18.786	26.913
6b	Narita and Hodgkinson (2005)	19.60	23.40	33.17	33.17	–
	Present (ABAQUS)	19.598	23.380	32.580	32.580	34.985
	Present (QEM)	19.596	23.378	32.597	32.597	35.013
6c	Narita and Hodgkinson (2005)	3.299	9.894	15.77	19.60	–
	Present (ABAQUS)	3.298	9.893	15.769	19.598	26.618
	Present (QEM)	3.298	9.893	15.770	19.596	26.616
6d	Narita and Hodgkinson (2005)	9.512	14.78	21.34	29.09	–
	Present (ABAQUS)	9.486	14.659	21.309	28.841	33.586
	Present (QEM)	9.487	14.662	21.307	28.847	33.604
6e	Kato and Honma (1998)	18.03	35.62	35.62	38.68	61.06
	Kim and Dickinson (1987)	18.03	35.17	35.17	38.43	60.58
	Fan and Cheung (1984)	17.85	34.89	34.89	38.43	60.12
	Present (ABAQUS)	17.837	34.884	34.884	38.440	60.101
	Present (QEM)	17.843	34.882	34.882	38.432	60.086
6f	Narita and Hodgkinson (2005)	13.47	17.09	18.65	18.65	–
	Present (ABAQUS)	13.468	17.029	18.275	18.275	39.185
	Present (QEM)	13.468	17.030	18.284	18.284	39.215
6g	Narita and Hodgkinson (2005)	6.641	6.736	19.60	19.75	–
	Present (ABAQUS)	6.638	6.700	19.489	19.598	24.639
	Present (QEM)	6.639	6.701	19.495	19.596	24.639
6h	Narita and Hodgkinson (2005)	7.112	15.77	15.77	16.90	–
	Cheung and Zhou (1999)	7.136	15.800	15.805	19.710	38.710
	Mizusawa and Kajita (1987)	7.111	15.77	15.77	19.60	38.43
	Kocatürk *et al.* (2004)	7.1109	–	–	19.596	–
	Venkateswara *et al.* (1973)	7.1109	–	–	19.596	–
	Present (ABAQUS)	7.1112	15.769	15.769	19.598	38.440
	Present (QEM)	7.1109	15.770	15.770	19.596	38.432

4. Conclusions

The Quadrature Element Method is applied to analyze free vibration of point supported rectangular plates having different boundary conditions and various point topologies. The results are compared to other approximation methods. A very good agreement is observed with the data published in literature. A 25-node plate element is easier to process with commercial software. It is possible to apply the Quadrature Element Method to plates having more complex shapes and to obtain a better accuracy by means of joining plate elements side by side along the x and y directions.

Table 7. Frequency parameters λ of square plates with point supports for four boundary conditions ($\lambda = \omega L^2\sqrt{\rho h/D}$)

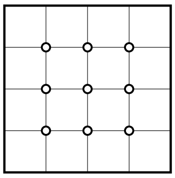
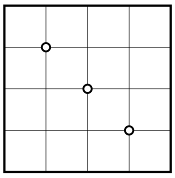
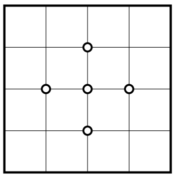
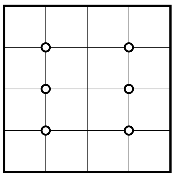
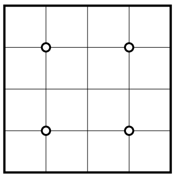
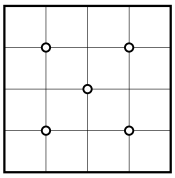
Support Position	Mod	S-S-S-S		C-C-C-C		C-S-C-S		F-C-F-S	
		QEM	ABAQUS	QEM	ABAQUS	QEM	ABAQUS	QEM	ABAQUS
	λ_1	167.78	168.08	187.97	188.10	170.35	170.64	48.538	48.515
	λ_2	167.78	168.08	187.97	188.10	184.95	184.97	50.128	50.086
	λ_3	182.71	182.74	207.96	207.92	185.99	186.13	82.215	82.172
	λ_4	182.71	182.74	215.98	215.71	205.14	205.04	82.772	82.712
	λ_5	197.39	197.65	242.16	242.60	206.70	206.96	133.17	133.19
	λ_1	49.348	49.362	73.394	73.437	60.807	60.829	26.227	26.222
	λ_2	62.106	62.071	86.985	86.931	73.233	73.194	33.799	33.787
	λ_3	91.269	91.232	105.57	105.55	100.02	99.999	61.801	61.796
	λ_4	98.696	98.784	131.58	131.76	115.97	116.083	66.573	66.574
	λ_5	128.30	128.38	151.28	151.16	141.12	140.983	77.381	77.385
	λ_1	78.957	78.975	108.22	108.27	94.586	94.625	42.012	41.987
	λ_2	91.269	91.228	121.28	121.25	104.68	104.66	42.899	42.859
	λ_3	91.269	91.228	121.28	121.24	110.16	110.10	58.187	58.196
	λ_4	101.69	101.61	139.20	139.12	120.37	120.30	61.114	61.109
	λ_5	167.78	168.08	204.49	204.88	170.35	170.64	99.245	99.301
	λ_1	67.760	67.759	74.089	74.075	71.703	71.701	38.804	38.798
	λ_2	91.269	91.232	105.57	105.55	104.68	104.66	48.538	48.515
	λ_3	131.52	131.51	162.74	162.81	162.17	162.25	77.124	77.123
	λ_4	167.78	168.08	187.97	188.10	170.33	170.64	80.303	80.260
	λ_5	167.78	168.08	207.05	207.49	193.81	193.78	82.215	82.172
	λ_1	52.677	52.644	55.185	55.150	53.966	53.931	38.203	38.193
	λ_2	91.269	91.232	105.57	105.55	92.350	92.298	41.679	41.647
	λ_3	91.269	91.232	105.57	105.55	104.68	104.66	52.440	52.410
	λ_4	98.696	98.784	131.58	131.76	110.61	110.70	77.139	77.137
	λ_5	146.83	146.80	180.45	180.55	168.11	168.16	79.983	79.957
	λ_1	91.269	91.232	105.57	105.55	92.350	92.298	39.993	39.972
	λ_2	91.269	91.232	105.57	105.55	103.43	103.33	41.679	41.647
	λ_3	98.696	98.784	116.08	115.81	104.68	104.66	75.963	75.960
	λ_4	104.81	104.58	131.58	131.76	120.48	120.44	79.983	79.957
	λ_5	167.78	168.07	207.05	207.49	170.35	170.64	96.537	96.564

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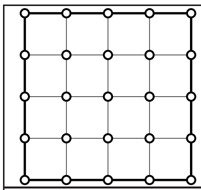
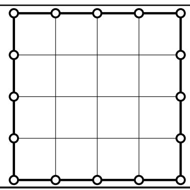
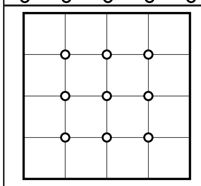
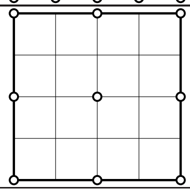
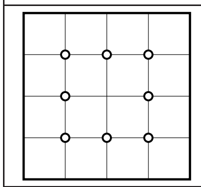
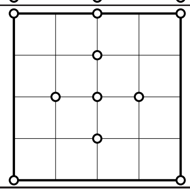
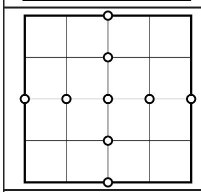
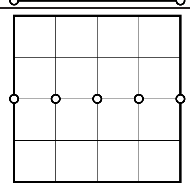
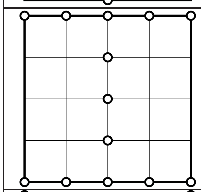
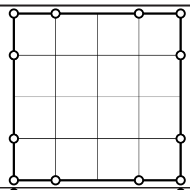
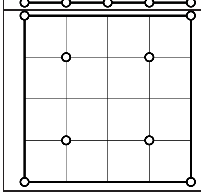
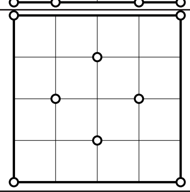
Table 8. Frequency parameters of free square plates with point supports ($\lambda = \omega L^2 \sqrt{\rho h/D}$)

Support position	Mod	QEM	ABAQUS	Support position	Mod	QEM	ABAQUS
	λ_1	145.65	145.75		λ_1	19.543	19.588
	λ_2	145.65	145.75		λ_2	48.235	48.233
	λ_3	146.74	146.84		λ_3	48.235	48.233
	λ_4	149.25	149.21		λ_4	74.958	74.921
	λ_5	158.85	158.81		λ_5	94.207	94.234
	λ_1	32.646	32.627		λ_1	34.882	34.884
	λ_2	32.646	32.627		λ_2	34.882	34.884
	λ_3	33.114	33.083		λ_3	38.432	38.440
	λ_4	35.013	34.985		λ_4	41.089	41.066
	λ_5	39.215	39.185		λ_5	68.499	68.484
	λ_1	31.518	31.476		λ_1	38.432	38.412
	λ_2	32.646	32.627		λ_2	39.215	39.185
	λ_3	32.646	32.627		λ_3	39.870	39.839
	λ_4	35.013	34.985		λ_4	39.870	39.839
	λ_5	39.215	39.185		λ_5	41.093	41.042
	λ_1	13.468	13.412		λ_1	13.468	13.468
	λ_2	20.987	20.982		λ_2	13.856	13.855
	λ_3	20.987	20.982		λ_3	20.987	20.984
	λ_4	26.646	26.636		λ_4	34.801	34.804
	λ_5	69.265	69.286		λ_5	39.941	39.936
	λ_1	16.111	16.110		λ_1	17.929	17.926
	λ_2	22.635	22.631		λ_2	41.882	41.867
	λ_3	46.224	46.238		λ_3	41.882	41.867
	λ_4	49.757	49.762		λ_4	60.130	60.127
	λ_5	74.639	74.661		λ_5	74.958	74.921
	λ_1	19.596	19.598		λ_1	36.964	36.952
	λ_2	34.907	34.901		λ_2	38.432	38.440
	λ_3	34.907	34.901		λ_3	39.215	39.185
	λ_4	44.148	44.152		λ_4	39.870	39.839
	λ_5	55.391	55.340		λ_5	39.870	39.839

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