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Utility functions invariant with respect to some classes of transformations*

by

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Abstract: Inspired by some relevant recent results of A. E. Abbas we determine utility functions invariant with respect to some classes of transformations.

Keywords: utility function, invariance, functional equation, additive function, exponential function

1. Introduction

Given a nonempty set X, a function $p : X \to [0,1]$ is said to be *a simple probability distribution* (or *a lottery*) on X provided the set $supp\ (p) := \{x \in X \mid p(x) > 0\}$ is finite and $\sum_{x\in supp\ (p)} p(x) = 1$. Elements of X are called *outcomes*. A family of all lotteries on X we will denote by $\Delta(X)$. Assume that $\succeq$ is a preference relation on $\Delta(X)$. The relation $\succeq$ represents the relative merits of any two lotteries for a decision maker. According to the classical result of von Neumann and Morgenstern, every preference relation $\succeq$ on $\Delta(X)$ satisfying some additional assumptions (completeness, transitivity, continuity, independence) can be represented by a utility function, that is, there exists a function $U : \Delta(X) \to \mathbb{R}$ such that, for every $p, q \in \Delta(X)$, we have

$$p \succeq q \Longleftrightarrow U(p) \geq U(q).$$

Moreover, every such function posseses *the Bernoulli utility function*, that is, a function $u : X \to \mathbb{R}$ such that

$$U(p) = \sum_{x\in supp\ (p)} p(x)u(x) \text{ for } p \in \Delta(X).$$

It is known that two utility functions U_1 and U_2 having the Bernoulli utility functions u_1 and u_2, respectively, represent the same preference relation over

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lotteries if and only if there exist $K \in (0,\infty)$ and $L \in \mathbb{R}$ such that $u_2(x) = Ku_1(x) + L$ for $x \in X$. In the sequel we will deal with the case of $X = \mathbb{R}$.

One of the fundamental problems in decision analysis under uncertainty is to determine the form of a utility function representing a decision maker's preference relation over the lotteries. There are several approaches to this problem. One of them is based on the notion of invariance. Given a non-degenerate interval I, a utility function $U : \Delta(\mathbb{R}) \to \mathbb{R}$ having the Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ is said to be *invariant with respect to a family of transformations*

$$\Gamma = \{\gamma_t : \mathbb{R} \to \mathbb{R} | t \in I\} \tag{1}$$

provided, for every $t \in I$, a utility function $U_t : \Delta(\mathbb{R}) \to \mathbb{R}$ given by

$$U_t(p) = \sum_{x \in supp\ (p)} p(x)u(\gamma_t(x)) \text{ for } p \in \Delta(\mathbb{R})$$

represents the same preference relation over $\Delta(\mathbb{R})$ as U. Since, for every $t \in I$, $u \circ \gamma_t$ is the Bernoulli utility function of U_t, a utility function U is invariant with respect to the family of transformations Γ of the form (1) if and only if there exist functions $K : I \to (0,\infty)$ and $L : I \to \mathbb{R}$ such that

$$u(\gamma_t(x)) = K(t)u(x) + L(t) \text{ for } x \in \mathbb{R},\ t \in I. \tag{2}$$

It is known (see Pfanzagl, 1959) that a utility function having a continuous Bernoulli utility function u is invariant with respect to the shift transformation by an arbitrary real number if and only if u is either a linear or an exponential function. Recently Abbas (2007), Abbas, Aczél and Chudziak (2009), Abbas (2010) and Chudziak (2010) have determined the forms of the utility functions invariant with respect to more general classes of transformations. In particular, in Abbas (2010), the utility functions invariant with respect to a family of transformations $\Gamma = \{\gamma_t : \mathbb{R} \to \mathbb{R} | t \in I\}$, where

$$\gamma_t(x) = v^{-1}(k(t)v(x) + l(t)) \text{ for } x \in \mathbb{R},\ t \in I \tag{3}$$

with some continuous and strictly monotone function $v : \mathbb{R} \to \mathbb{R}$ and functions $k : I \to (0,\infty)$ and $l : I \to \mathbb{R}$ such that

$$k(t)v(x) + l(t) \in v(\mathbb{R}) \text{ for } x \in \mathbb{R},\ t \in I, \tag{4}$$

have been considered. This family contains, as particular cases, several classes playing an important role in the utility theory. For more details we refer to Abbas (2010).

2. The main result

The aim of this paper is to give a complete description of the forms of utility functions invariant with respect to the family of transformations of the form (3). In order to avoid a trivial case, we will assume that the family is non-degenerate, that is - it does not consist just of the identity transformation. The following theorem is the main result of the paper.

THEOREM 1. *Assume that I is a non-degenerate interval and Γ is a non-degenerate family of transformations of the form* (3) *with some continuous and strictly monotone function $v : \mathbb{R} \to \mathbb{R}$ and functions $k : I \to (0,\infty)$ and $l : I \to \mathbb{R}$ satisfying* (4)*. A utility function $U : \Delta(\mathbb{R}) \to \mathbb{R}$ having a continuous Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ is invariant with respect to the family Γ if and only if one of the following cases holds:*

(*i*) *Γ is arbitrary and*

$$u(x) = av(x) + b \quad \text{for } x \in \mathbb{R} \tag{5}$$

with some $a, b \in \mathbb{R}$;

(*ii*) *Γ consists of a single transformation of the form*

$$\gamma(x) = v^{-1}(v(x) + l) \quad \text{for } x \in \mathbb{R} \tag{6}$$

with some $l \in \mathbb{R} \setminus \{0\}$ and either

$$u(x) = bv(x) + P\left(\frac{v(x)}{l}\right) \quad \text{for } x \in \mathbb{R} \tag{7}$$

or

$$u(x) = a^{v(x)} P\left(\frac{v(x)}{l}\right) + b \quad \text{for } x \in \mathbb{R}, \tag{8}$$

where $P : \mathbb{R} \to \mathbb{R}$ is a continuous 1*-periodic function, $a \in (0,\infty) \setminus \{1\}$ and $b \in \mathbb{R}$;*

(*iii*) *Γ consists of a single transformation of the form*

$$\gamma(x) = v^{-1}(kv(x) + l) \quad \text{for } x \in \mathbb{R} \tag{9}$$

with some $k \in (0,\infty) \setminus \{1\}$, $l \in \mathbb{R}$ and either

$$u(x) = \begin{cases} \left(\frac{l}{1-k} - v(x)\right)^s P_1\left(\log_k\left(\frac{l}{1-k} - v(x)\right)\right) + c & \text{whenever } v(x) < \frac{l}{1-k}, \\ c & \text{whenever } v(x) = \frac{l}{1-k}, \\ \left(v(x) - \frac{l}{1-k}\right)^s P_2\left(\log_k\left(v(x) - \frac{l}{1-k}\right)\right) + c & \text{whenever } v(x) > \frac{l}{1-k}, \end{cases} \tag{10}$$

where $P_1, P_2 : \mathbb{R} \to \mathbb{R}$ are continuous 1*-periodic functions, $s \in \mathbb{R} \setminus \{0\}$ whenever $\frac{l}{1-k} \notin v(\mathbb{R})$ and $s \in (0,\infty)$, otherwise; or $\frac{l}{1-k} \notin v(\mathbb{R})$ and*

$$u(x) = a \log_k \left| v(x) - \frac{l}{1-k} \right| + P\left(\log_k \left| v(x) - \frac{l}{1-k} \right|\right) \quad \text{for } x \in \mathbb{R}, \tag{11}$$

where $P : \mathbb{R} \to \mathbb{R}$ is a continuous 1*-periodic function and $a \in \mathbb{R}$;*

(iv)

$$\gamma_t(x) = v^{-1}(v(x) + l(t)) \text{ for } x \in \mathbb{R},\ t \in I \tag{12}$$

with some nonconstant continuous function $l : \mathbb{R} \to \mathbb{R}$ *and*

$$u(x) = ba^{v(x)} + c \text{ for } x \in \mathbb{R}, \tag{13}$$

where $a \in (0, \infty) \setminus \{1\}$, $b \in \mathbb{R} \setminus \{0\}$ *and* $c \in \mathbb{R}$*;*

(v)

$$\gamma_t(x) = v^{-1}(k(t)(v(x) - d) + d) \text{ for } x \in \mathbb{R},\ t \in I \tag{14}$$

with some $d \in \mathbb{R}$ *and nonconstant continuous function* $k : \mathbb{R} \to (0, \infty)$ *and either*

$$u(x) = \begin{cases} a(d - v(x))^s + c & \text{whenever } v(x) < d, \\ c & \text{whenever } v(x) = d, \\ b(v(x) - d)^s + c & \text{whenever } v(x) > d, \end{cases} \tag{15}$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$ *whenever* $v(\mathbb{R}) \subset (-\infty, d)$, $b \neq 0$ *whenever* $v(\mathbb{R}) \subset (d, \infty)$ *and* $|a|+|b| > 0$, *otherwise,* $s \in \mathbb{R}\setminus\{0\}$ *whenever* $d \notin v(\mathbb{R})$ *and* $s \in (0, \infty)$, *otherwise; or* $d \notin v(\mathbb{R})$ *and*

$$u(x) = a \ln|v(x) - d| + b \text{ for } x \in \mathbb{R}, \tag{16}$$

where $a \in \mathbb{R} \setminus \{0\}$ *and* $b \in \mathbb{R}$.

3. Proof of Theorem 1

Assume that $U : \Delta(\mathbb{R}) \to \mathbb{R}$ is a utility function having a continuous Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$. As we have already noted U is invariant with respect to a family of transformations Γ if and only if there exist functions $K : I \to (0, \infty)$ and $L : I \to \mathbb{R}$ such that (2) holds. Since the family Γ consists of the transformations of the form (3), condition (2) becomes

$$u(v^{-1}(k(t)v(x) + l(t))) = K(t)u(x) + L(t) \text{ for } x \in \mathbb{R},\ t \in I. \tag{17}$$

Straightforward calculations show that if one of the possibilities (i)-(v) holds, then (17) is satisfied with some functions $K : I \to (0, \infty)$ and $L : I \to \mathbb{R}$, whence U is invariant with respect to a family of transformations of the form (3).

Now, assume that (17) holds. Then, taking $f := u \circ v^{-1}$, we obtain

$$f(k(t)x + l(t)) = K(t)f(x) + L(t) \text{ for } x \in v(\mathbb{R}),\ t \in I. \tag{18}$$

If f is constant then (5) holds with $a = 0$. So, assume that f is nonconstant. In the sequel, we will use several times the following fact: if J is an interval unbounded above and a continuous function $p : J \to \mathbb{R}$ satisfies the condition

$$p(x + 1) = p(x) \text{ for } x \in J \tag{19}$$

then there exists a unique continuous 1-periodic extension of p, that is a continuous function $P:\mathbb{R}\to\mathbb{R}$ such that $P(x+1)=P(x)$ for $x\in\mathbb{R}$ and

$$p(x)=P(x) \text{ for } x\in J. \tag{20}$$

In fact, it is enough to define $P:\mathbb{R}\to\mathbb{R}$ as follows: $P(x)=p(x+n(x))$ for $x\in\mathbb{R}$, where $n(x):=\min\{n\in\mathbb{N}\cup\{0\}|x+n(x)\in J\}$.

Consider the following two cases:
1. k and l are constant;
2. k or l is nonconstant.

Case 1. In this case the family Γ consists of a single transformation of the form (9). Furthermore, as Γ is non-degenerate, we have $k\neq 1$ or $l\neq 0$. Note also that in this case (18) takes the form

$$f(kx+l)=K(t)f(x)+L(t) \text{ for } x\in v(\mathbb{R}),\ t\in I. \tag{21}$$

Since f is non–constant, taking $x_1,x_2\in v(\mathbb{R})$ with $f(x_1)\neq f(x_2)$, in view of (21), we get

$$K(t)=\frac{f(kx_1+l)-f(kx_2+l)}{f(x_1)-f(x_2)} \text{ for } t\in I.$$

Thus K is constant and, by (21), so is L. Therefore, (21) becomes

$$f(kx+l)=Kf(x)+L \text{ for } x\in v(\mathbb{R}). \tag{22}$$

Consider the following four subcases:
1.1. $k=K=1$,
1.2. $k=1$ and $K\neq 1$,
1.3. $k\neq 1$ and $K=1$,
1.4. $k\neq 1$ and $K\neq 1$.

Subcase 1.1. First note that (6) is valid. Moreover, making use of (4), we obtain that $x+l\in v(\mathbb{R})$ for $x\in v(\mathbb{R})$, whence $x+1\in\frac{1}{l}v(\mathbb{R})$ for $x\in\frac{1}{l}v(\mathbb{R})$. Therefore, $\frac{1}{l}v(\mathbb{R})$ is an interval unbounded above. Furthermore, in view of (22), a function $\tilde{p}:\frac{1}{l}v(\mathbb{R})\to\mathbb{R}$ given by

$$\tilde{p}(x)=e^{f(lx)-Lx} \text{ for } x\in\frac{1}{l}v(\mathbb{R})$$

is continuous and satisfies $\tilde{p}(x+1)=\tilde{p}(x)$ for $x\in\frac{1}{l}v(\mathbb{R})$. Thus, there exists a continuous 1-periodic function $\tilde{P}:\mathbb{R}\to\mathbb{R}$ such that $\tilde{p}(x)=\tilde{P}(x)$ for $x\in\frac{1}{l}v(\mathbb{R})$. Moreover

$$(u\circ v^{-1})(x)=f(x)=\frac{L}{l}x+ln\ \tilde{P}\left(\frac{x}{l}\right) \text{ for } x\in v(\mathbb{R}).$$

Thus, (7) holds with $b:=\frac{L}{l}$ and $P:=ln\ \tilde{P}$.

Subcase 1.2. Again we have (6). Furthermore, as previously, $\frac{1}{l}v(\mathbb{R})$ is an unbounded above interval and, in view of (22), a function $p : \frac{1}{l}v(\mathbb{R}) \to \mathbb{R}$ of the form

$$p(x) = K^{-x}\left(f(lx) - \frac{L}{1-K}\right) \quad \text{for} \quad x \in \frac{1}{l}v(\mathbb{R})$$

is continuous and satisfies (19) with $J := \frac{1}{l}v(\mathbb{R})$. Therefore, there exists a continuous 1-periodic function $P : \mathbb{R} \to \mathbb{R}$ such that (20) holds with $J = \frac{1}{l}v(\mathbb{R})$. Hence

$$(u \circ v^{-1})(x) = f(x) = K^{\frac{x}{l}}P\left(\frac{x}{l}\right) + \frac{L}{1-K} \quad \text{for} \quad x \in v(\mathbb{R}),$$

which means that (8) holds with $a := K^{\frac{1}{l}}$ and $b := \frac{L}{1-K}$.

Subcase 1.3. Suppose that $\frac{l}{1-k} \in v(\mathbb{R})$. Let $\tau : v(\mathbb{R}) \to \mathbb{R}$ be given by $\tau(x) = kx + l$ for $x \in v(\mathbb{R})$. Then, by (4), $\tau(x) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$ and so, by induction, $\tau^n(x) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$ and $n \in \mathbb{N}$. Furthermore, if $k < 1$, then

$$\lim_{n\to\infty} \tau^n(x) = \lim_{n\to\infty}\left(k^n x + l\sum_{i=0}^{n-1} k^i\right) = \frac{l}{1-k} \quad \text{for} \quad x \in v(\mathbb{R}). \tag{23}$$

If $k > 1$ then

$$\tau^{-1}(x) = \frac{x-l}{k} \in \left(x, \frac{l}{1-k}\right) \quad \text{whenever} \quad x \in v(\mathbb{R}) \cap \left(-\infty, \frac{l}{1-k}\right)$$

and

$$\tau^{-1}(x) = \frac{x-l}{k} \in \left(\frac{l}{1-k}, x\right) \quad \text{whenever} \quad x \in v(\mathbb{R}) \cap \left(\frac{l}{1-k}, \infty\right).$$

Thus, $\tau^{-n}(x) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$ and $n \in \mathbb{N}$. Moreover

$$\lim_{n\to\infty} \tau^{-n}(x) = \lim_{n\to\infty}\left(k^{-n} x - l\sum_{i=1}^{n} k^{-i}\right) = \frac{l}{1-k} \quad \text{for} \quad x \in v(\mathbb{R}). \tag{24}$$

Note also that, taking in (22) $x = \frac{l}{1-k}$, we get $L = 0$ and so

$$f(\tau(x)) = f(x) \quad \text{for} \quad x \in v(\mathbb{R}). \tag{25}$$

Therefore, as f is continuous, applying (23) if $k < 1$, and (24) if $k > 1$, we conclude that $f(x) = f\left(\frac{l}{1-k}\right)$ for $x \in v(\mathbb{R})$. Hence, f is constant, which yields a contradiction.

In this way we have proved that $\frac{l}{1-k} \notin v(\mathbb{R})$. Furthermore, since v is continuous, we have either $v(\mathbb{R}) - \frac{l}{1-k} \subset (-\infty, 0)$ or $v(\mathbb{R}) - \frac{l}{1-k} \subset (0, \infty)$. Let $\tilde{f} : \left(v(\mathbb{R}) - \frac{l}{1-k}\right) \to \mathbb{R}$ be given by

$$\tilde{f}(x) = e^{f(x + \frac{l}{1-k})} \quad \text{for} \quad x \in v(\mathbb{R}) - \frac{l}{1-k}. \tag{26}$$

Note that, according to (22), for every $x \in v(\mathbb{R}) - \frac{l}{1-k}$, we have

$$f\left(kx + \frac{l}{1-k}\right) = f\left(k\left(x + \frac{l}{1-k}\right) + l\right) = f\left(x + \frac{l}{1-k}\right) + L.$$

Thus, $\tilde{f}$ satisfies the following functional equation

$$\tilde{f}(kx) = e^L \tilde{f}(x) \text{ for } x \in v(\mathbb{R}) - \frac{l}{1-k}. \tag{27}$$

If $v(\mathbb{R}) - \frac{l}{1-k} \subset (-\infty, 0)$ then the set $\Lambda_1 := \{\log_k(-x) | x \in v(\mathbb{R}) - \frac{l}{1-k}\}$ is an interval. Moreover, Λ_1 is unbounded above. In fact, in view of (4), for every $x \in v(\mathbb{R}) - \frac{l}{1-k}$, we have

$$kx = k\left(x + \frac{l}{1-k}\right) + l - \frac{l}{1-k} \in v(\mathbb{R}) - \frac{l}{1-k},$$

which means that

$$1 + \log_k(-x) = \log_k(-kx) \in \Lambda_1.$$

Note also that, according to (27), a function $\tilde{p} : \Lambda_1 \to \mathbb{R}$ given by $\tilde{p}(x) = \tilde{f}(-k^x)e^{-Lx}$ for $x \in \Lambda_1$, is continuous and satisfies $\tilde{p}(x+1) = \tilde{p}(x)$ for $x \in \Lambda_1$. Thus, there exists a continuous 1-periodic function $\tilde{P} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{p}(x) = \tilde{P}(x)$ for $x \in \Lambda_1$. Furthermore, we have

$$\tilde{f}(x) = e^{L \log_k(-x)} \tilde{P}(\log_k(-x)) \text{ for } x \in v(\mathbb{R}) - \frac{l}{1-k},$$

which, in view of (26), gives

$$(u \circ v^{-1})(x) = f(x) = \ln \tilde{f}\left(x - \frac{l}{1-k}\right)$$

$$= L \log_k\left(\frac{l}{1-k} - x\right) + \ln \tilde{P}\left(\log_k\left(\frac{l}{1-k} - x\right)\right) \text{ for } x \in v(\mathbb{R}).$$

If $v(\mathbb{R}) - \frac{l}{1-k} \subset (0, \infty)$ then, taking $\Lambda_2 := \{\log_k x | x \in v(\mathbb{R}) - \frac{l}{1-k}\}$ and $\tilde{p} : \Lambda_2 \to \mathbb{R}$ of the form $\tilde{p}(x) = \tilde{f}(k^x)e^{-Lx}$ for $x \in \Lambda_2$, in a similar way we obtain that there exists a continuous 1-periodic function $\tilde{P} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{p}(x) = \tilde{P}(x)$ for $x \in \Lambda_2$. Hence

$$(u \circ v^{-1})(x) = f(x) = L \log_k\left(x - \frac{l}{1-k}\right) + \ln \tilde{P}\left(\log_k\left(x - \frac{l}{1-k}\right)\right) \text{ for } x \in v(\mathbb{R}).$$

Therefore, (11) holds with $a := L$ and $P := \ln \tilde{P}$.

Subcase 1.4. Let $\tilde{f} : \left(v(\mathbb{R}) - \frac{l}{1-k}\right) \to \mathbb{R}$ be given by

$$\tilde{f}(x) = f\left(x + \frac{l}{1-k}\right) - \frac{L}{1-K} \text{ for } x \in v(\mathbb{R}) - \frac{l}{1-k}. \tag{28}$$

Then, making use of (22), we get

$$\tilde{f}(kx) = K\tilde{f}(x) \ \text{ for } \ x \in v(\mathbb{R}) - \frac{l}{1-k}. \tag{29}$$

So, arguing as in the case of (27), we conclude that

$$\tilde{f}(x) = \begin{cases} K^{\log_k(-x)} P_1(\log_k(-x)) & \text{for} \quad x \in \left(v(\mathbb{R}) - \frac{l}{1-k}\right) \cap (-\infty, 0), \\ K^{\log_k x} P_2(\log_k x) & \text{for} \quad x \in \left(v(\mathbb{R}) - \frac{l}{1-k}\right) \cap (0, \infty), \end{cases}$$

where $P_1, P_2 : \mathbb{R} \to \mathbb{R}$ are continuous 1-periodic functions. Therefore, since $K^{\log_k x} = x^{\log_k K}$ for $x \in (0, \infty)$, in view of (28), we obtain

$$f(x) = \begin{cases} \left(\frac{l}{1-k} - x\right)^{\log_k K} P_1\left(\log_k\left(\frac{l}{1-k} - x\right)\right) + \frac{L}{1-K} \text{ for } x \in v(\mathbb{R}) \cap \left(-\infty, \frac{l}{1-k}\right), \\ \left(x - \frac{l}{1-k}\right)^{\log_k K} P_2\left(\log_k\left(x - \frac{l}{1-k}\right)\right) + \frac{L}{1-K} \text{ for } x \in v(\mathbb{R}) \cap \left(\frac{l}{1-k}, \infty, \right). \end{cases}$$

Furthermore, if $\frac{l}{1-k} \in v(\mathbb{R})$, then $0 \in v(\mathbb{R}) - \frac{l}{1-k}$ and so, in view of (29), we get $\tilde{f}(0) = K\tilde{f}(0)$. Since $K \neq 1$, this implies that $\tilde{f}(0) = 0$, whence, by (28), we obtain $f(\frac{l}{1-k}) = \frac{L}{1-K}$. Note also that from the continuity of f it follows that $\log_k K > 0$ whenever $\frac{l}{1-k} \in v(\mathbb{R})$. Consequently, as $f = u \circ v^{-1}$, we get (10) with $s := \log_k K$ and $c := \frac{L}{1-K}$.

Case 2. Similarly as in the previous case, we distinguish four subcases:
2.1. k and K are identically 1,
2.2. k is identically 1 and K is not identically 1,
2.3. k is not identically 1 and $K(t) = 1$ for every $t \in I$ with $k(t) \neq 1$,
2.4. there is a $t_0 \in I$ such that $k(t_0) \neq 1$ and $K(t_0) \neq 1$.

Subcase 2.1. In this case l is nonconstant and (18) becomes

$$f(x + l(t)) = f(x) + L(t) \ \text{ for } \ x \in v(\mathbb{R}), \ t \in I. \tag{30}$$

Thus, taking a $t_0 \in I$ such that $l(t_0) \neq 0$, in view of (4), we obtain that $x + 1 \in \frac{1}{l(t_0)} v(\mathbb{R})$ for $x \in \frac{1}{l(t_0)} v(\mathbb{R})$. Hence, $\frac{1}{l(t_0)} v(\mathbb{R})$ is an unbounded above interval. Furthermore, by applying (30) with $t = t_0$, we get that a function $p : \frac{1}{l(t_0)} v(\mathbb{R}) \to \mathbb{R}$ given by

$$p(x) = f(l(t_0)x) - L(t_0)x \ \text{ for } \ x \in \frac{1}{l(t_0)} v(\mathbb{R}), \tag{31}$$

is continuous and satisfies (19) with $J := \frac{1}{l(t_0)} v(\mathbb{R})$. Thus, there exists a continuous 1-periodic function $P : \mathbb{R} \to \mathbb{R}$ such that (20) holds with $J := \frac{1}{l(t_0)} v(\mathbb{R})$. Moreover

$$f(x) = \frac{L(t_0)}{l(t_0)} x + P\left(\frac{x}{l(t_0)}\right) \ \text{ for } \ x \in v(\mathbb{R}). \tag{32}$$

Inserting into (30) f of the form (32), we get

$$P\left(\frac{x+l(t)}{l(t_0)}\right)=P\left(\frac{x}{l(t_0)}\right)+L(t)-\frac{L(t_0)}{l(t_0)}l(t)\ \text{ for }\ x\in v(\mathbb{R}),t\in I. \tag{33}$$

Let $h: l(I)\to\mathbb{R}$ be given by

$$h(z)=P\left(\frac{x_0+z}{l(t_0)}\right)+\frac{L(t_0)}{l(t_0)}z-P\left(\frac{x_0}{l(t_0)}\right)\ \text{ for }\ z\in l(I), \tag{34}$$

where $x_0\in v(\mathbb{R})$ is fixed. Then, taking in (33) $x=x_0$, we get $L(t)=h(l(t))$ for $t\in I$. Hence, in view of (33), for every $x\in\frac{1}{l(t_0)}v(\mathbb{R})$ and $y\in\frac{1}{l(t_0)}l(I)$, we have

$$P(x+y)=P(x)+g(y), \tag{35}$$

where $g(y):=h(l(t_0)y)-L(t_0)y$ for $y\in\frac{1}{l(t_0)}l(I)$. In particular, (35) holds for every $(x,y)\in\frac{1}{l(t_0)}v(\mathbb{R})\times int\ \frac{1}{l(t_0)}l(I)$ and this set is a nonempty, open and connected subset of $\mathbb{R}^2$ because v is continuous and strictly monotone and l is continuous and nonconstant. Therefore, as P is continuous, from Kuczma (1985) (p. 311) and Sobek (2010) (Corollary 1) we derive that there are $A,b\in\mathbb{R}$ such that $P(x)=Ax+b$ for $x\in\frac{1}{l(t_0)}v(\mathbb{R})$. Hence, as P is 1-periodic, we get $A=0$ and so, taking into account (32), we obtain

$$(u\circ v^{-1})(x)=f(x)=\frac{L(t_0)}{l(t_0)}x+b\ \text{ for }\ x\in v(\mathbb{R}).$$

Thus (5) holds with $a:=\frac{L(t_0)}{l(t_0)}$.

Subcase 2.2. In this case (12) holds and (18) takes the form

$$f(x+l(t))=K(t)f(x)+L(t)\ \text{ for }\ x\in v(\mathbb{R}),\ t\in I. \tag{36}$$

Fix a $t_0\in I$ with $K(t_0)\neq 1$. Since f is nonconstant, taking in (36) $t=t_0$, we get $l(t_0)\neq 0$. Moreover, a straightforward calculation shows that $\frac{1}{l(t_0)}v(\mathbb{R})$ is an interval unbounded above and a function $p:\frac{1}{l(t_0)}v(\mathbb{R})\to\mathbb{R}$ given by

$$p(x)=K(t_0)^{-x}\left(f(l(t_0)x)-\frac{L(t_0)}{1-K(t_0)}\right)\ \text{ for }\ x\in\frac{1}{l(t_0)}v(\mathbb{R})$$

is continuous and satisfies (19) with $J:=\frac{1}{l(t_0)}v(\mathbb{R})$. Thus there exists a continuous 1-periodic function $P:\mathbb{R}\to\mathbb{R}$ such that (20) holds with $J:=\frac{1}{l(t_0)}v(\mathbb{R})$. Consequently

$$f(x)=A^{\frac{x}{l(t_0)}}P\left(\frac{x}{l(t_0)}\right)+c\ \text{ for }\ x\in v(\mathbb{R}), \tag{37}$$

where $A:=K(t_0)$ and $c:=\frac{L(t_0)}{1-K(t_0)}$. Putting into (36) f of the form (37), we obtain

$$A^{\frac{x+l(t)}{l(t_0)}}P\left(\frac{x+l(t)}{l(t_0)}\right)+c=K(t)A^{\frac{x}{l(t_0)}}P\left(\frac{x}{l(t_0)}\right)+cK(t)+L(t)\ \text{ for }\ x\in v(\mathbb{R}),\ t\in I.$$

Since, by (4), $x+l(t_0) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$, we have by iteration $x+nl(t_0) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$, $n \in \mathbb{N}$. Thus, setting in the last equality $x + nl(t_0)$ in a place of x, for every $x \in v(\mathbb{R})$, $t \in I$ and $n \in \mathbb{N}$, we get

$$A^{\frac{x+nl(t_0)+l(t)}{l(t_0)}} P\left(\frac{x+nl(t_0)+l(t)}{l(t_0)}\right)+c = K(t)A^{\frac{x+nl(t_0)}{l(t_0)}} P\left(\frac{x+nl(t_0)}{l(t_0)}\right)+cK(t)+L(t),$$

whence, as P is 1-periodic, we obtain

$$A^n A^{\frac{x+l(t)}{l(t_0)}} P\left(\frac{x+l(t)}{l(t_0)}\right) + c = K(t)A^n A^{\frac{x}{l(t_0)}} P\left(\frac{x}{l(t_0)}\right) + cK(t) + L(t).$$

Therefore, taking into account (37), for every $x \in v(\mathbb{R})$, $t \in I$ and $n \in \mathbb{N}$, we get

$$A^n(f(x+l(t)) - c - K(t)(f(x) - c)) = c(K(t) - 1) + L(t).$$

Since $A \neq 1$, this means that

$$f(x+l(t)) - c = K(t)(f(x) - c) \text{ for } x \in v(\mathbb{R}),\ t \in I. \tag{38}$$

Fix an $x_0 \in v(\mathbb{R})$ with $f(x_0) \neq c$ and define a function $h : l(I) \to \mathbb{R}$ in the following way $h(z) = \frac{f(x_0+z)-c}{f(x_0)-c}$ for $z \in l(I)$. Then, taking in (38) $x = x_0$, we get $K(t) = h(l(t))$ for $t \in I$. Hence, from (38) we derive that

$$f(x+y) - c = (f(x) - c)h(y) \text{ for } x \in v(\mathbb{R}),\ y \in l(I).$$

The last equality holds, in particular, for every $(x, y) \in v(\mathbb{R}) \times int\, l(I)$ and this set is a nonempty, open and connected subset of $\mathbb{R}^2$. Thus, as f is continuous, from Kuczma (1985) (p. 311) and Sobek (2010) (Corollary 2) it follows that either $f(x+y) - c = 0$ for $(x, y) \in v(\mathbb{R}) \times int\, l(I)$, or there exist $\alpha, b \in \mathbb{R} \setminus \{0\}$ such that

$$f(x+y) - c = be^{\alpha x} \text{ for } x \in v(\mathbb{R}). \tag{39}$$

Note, however, that since f is nonconstant and $K(t) > 0$ for $t \in I$, the first possibility is excluded by (38). Therefore (39) holds and so, as $f = u \circ v^{-1}$, we get (13) with $a := e^{\alpha}$.

Subcase 2.3. Fix a $t_0 \in I$ with $k(t_0) \neq 1$. Suppose that $k(t_0) < 1$. Since k is continuous, there is a maximal open interval $I_0 \subset I$ containing t_0 such that $k(t) < 1$ for $t \in I_0$. Note that

$$K(t) = 1 \text{ for } t \in I_0. \tag{40}$$

Let $D : I_0 \to \mathbb{R}$ be given by

$$D(t) = \frac{l(t)}{1-k(t)} \text{ for } t \in I_0. \tag{41}$$

For every $t \in I_0$, let $\tau_t : v(\mathbb{R}) \to \mathbb{R}$ be of the form $\tau_t(x) = k(t)x + l(t)$ for $x \in v(\mathbb{R})$. According to (4), for every $x \in v(\mathbb{R})$ and $t \in I_0$, we have $\tau_t(x) \in v(\mathbb{R})$, so $\tau_t^n(x) \in v(\mathbb{R})$ for $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} \tau_t^n(x) = D(t)$ for $x \in v(\mathbb{R})$. Therefore, for every $t \in I_0$, $D(t)$ is an accumulation point of the interval $v(\mathbb{R})$. Furthermore, arguing as in the subcase 1.3, we obtain that $D(t) \notin v(\mathbb{R})$ for $t \in I_0$. Since D is a continuous function defined on the interval I_0, this means that D is constant, say $D = d$ with some $d \in \mathbb{R}$. Hence $d \notin v(\mathbb{R})$ and, in view of (41), we get

$$l(t) = d(1 - k(t)) \text{ for } t \in I_0. \tag{42}$$

Note also that k is nonconstant. Otherwise we would have $I_0 = I$ and so, in view of (42), we would obtain that l is also constant, which is excluded in this case. Since I_0 is maximal and k is nonconstant, the set $k(I_0)$ has a nonempty interior. Moreover, as $d \notin v(\mathbb{R})$, we have either $v(\mathbb{R}) \subset (-\infty, d)$ or $v(\mathbb{R}) \subset (d, \infty)$. Since the proof in both cases is similar, assume, for instance, that the second possibility holds. Then, from (18) and (42) we derive that

$$f(k(t)(x - d) + d) = f(x) + L(t) \text{ for } x \in v(\mathbb{R}),\ t \in I_0. \tag{43}$$

Thus, taking $\tilde{f} : (v(\mathbb{R}) - d) \to \mathbb{R}$ of the form

$$\tilde{f}(x) = f(x + d) \text{ for } x \in v(\mathbb{R}) - d, \tag{44}$$

we obtain

$$\tilde{f}(k(t)x) = \tilde{f}(x) + L(t) \text{ for } x \in v(\mathbb{R}) - d,\ t \in I_0.$$

Hence

$$\tilde{f}(k(t)x) = \tilde{f}(x) + H(k(t)) \text{ for } x \in v(\mathbb{R}) - d,\ t \in I_0,$$

where $H : k(I_0) \to \mathbb{R}$ is given by $H(z) = \tilde{f}(zx_0) - \tilde{f}(x_0)$ for $z \in k(I_0)$, with a fixed $x_0 \in v(\mathbb{R}) - d$. Consequently, for every $x \in v(\mathbb{R}) - d$ and $y \in k(I_0)$, we have

$$\tilde{f}(xy) = \tilde{f}(x) + H(y).$$

In particular, the last equality holds for every $(x, y) \in (v(\mathbb{R}) - d) \times int\ k(I_0)$ and this set is a nonempty, open and connected subset of $(0, \infty)^2$. Moreover, as f is continuous and nonconstant, so is $\tilde{f}$. Thus from Kuczma (1985) (p. 311) and Sobek (2010) (Corollary 1) we derive that there exist $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$\tilde{f}(x) = a \ln x + b \text{ for } x \in v(\mathbb{R}) - d.$$

Therefore, in view of (44), we get

$$f(x) = a \ln(x - d) + b \text{ for } x \in v(\mathbb{R}), \tag{45}$$

which implies (16). Furthermore, by inserting into (18) f of the form (45), we obtain

$$a \ln(k(t)x + l(t) - d) + b = aK(t) \ln(x - d) + bK(t) + L(t) \text{ for } x \in v(\mathbb{R}),\ t \in I.$$

Differentiating the last equality with respect to x, after the standard computations, we obtain

$$l(t) - d(1-k(t)) = \frac{k(t)}{K(t)}(1-K(t))(x-d) \ \text{ for } \ x \in v(\mathbb{R}),\ t \in I.$$

Hence

$$l(t) = d(1-k(t)) \ \text{ for } \ t \in I, \tag{46}$$

which implies (14).

Next suppose that $k(t_0) > 1$. Then

$$\tau_{t_0}^{-1}(x) = \frac{x - l(t_0)}{k(t_0)} \in \left(x, \frac{l(t_0)}{1-k(t_0)}\right) \ \text{ whenever } \ x \in v(\mathbb{R}) \cap \left(-\infty, \frac{l(t_0)}{1-k(t_0)}\right)$$

and

$$\tau_{t_0}^{-1}(x) = \frac{x - l(t_0)}{k(t_0)} \in \left(\frac{l(t_0)}{1-k(t_0)}, x\right) \ \text{ whenever } \ x \in v(\mathbb{R}) \cap \left(\frac{l(t_0)}{1-k(t_0)}, \infty\right).$$

Hence, arguing as in the subcase 1.3, we obtain that $d := \frac{l(t_0)}{1-k(t_0)} \notin v(\mathbb{R})$ and a function $\tilde{f} : (v(\mathbb{R}) - d) \to \mathbb{R}$ given by

$$\tilde{f}(x) = e^{f(x+d)} \ \text{ for } \ x \in v(\mathbb{R}) - d, \tag{47}$$

satisfies equation

$$\tilde{f}(k(t_0)x) = e^{L(t_0)}\tilde{f}(x) \ \text{ for } \ x \in v(\mathbb{R}) - d. \tag{48}$$

Since $d \notin v(\mathbb{R})$ we have either $v(\mathbb{R}) - d \subset (-\infty, 0)$ or $v(\mathbb{R}) - d \subset (0, \infty)$. Assume, for instance, that the second possibility holds. Then the set $\Lambda := \{\log_{k(t_0)} x | x \in v(\mathbb{R}) - d\}$ is an interval unbounded above and, in view of (48), a function $\tilde{p} : \Lambda \to \mathbb{R}$ of the form $\tilde{p}(x) = \tilde{f}(k(t_0)^x)e^{-L(t_0)x}$ for $x \in \Lambda$, is continuous and satisfies $\tilde{p}(x+1) = \tilde{p}(x)$ for $x \in \Lambda$. Thus there exists a continuous 1-periodic function $\tilde{P} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{p}(x) = \tilde{P}(x)$ for $x \in \Lambda$. Moreover

$$\tilde{f}(x) = e^{L(t_0)\log_{k(t_0)} x}\tilde{P}(\log_{k(t_0)} x) \ \text{ for } \ x \in v(\mathbb{R}) - d.$$

Hence, in view of (47), we get

$$f(x) = L(t_0)\log_{k(t_0)}(x-d) + P(\log_{k(t_0)}(x-d)) \ \text{ for } \ x \in v(\mathbb{R}), \tag{49}$$

where $P := \ln\ \tilde{P}$. Putting into (18) f of the form (49), we obtain

$$L(t_0)\log_{k(t_0)}(k(t)x + l(t) - d) + P(\log_{k(t_0)}(k(t)x + l(t) - d))$$
$$= K(t)[L(t_0)\log_{k(t_0)}(x-d) + P(\log_{k(t_0)}(x-d))] + L(t) \ \text{ for } \ x \in v(\mathbb{R}), t \in I.$$

Note also that, according to (4), $k(t_0)(x-d)+d = k(t_0)x + l(t_0) \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$, so by iteration, we get $k(t_0)^n(x-d)+d \in v(\mathbb{R})$ for $x \in v(\mathbb{R})$, $n \in \mathbb{N}$. Therefore, setting in the last equality $k(t_0)^n(x-d)+d$ in a place of x, for every $x \in v(\mathbb{R})$, $t \in I$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} &L(t_0)\log_{k(t_0)}(k(t)k(t_0)^n(x-d)+l(t)-d(1-k(t))) \\ &+P(\log_{k(t_0)}(k(t)k(t_0)^n(x-d)+l(t)-d(1-k(t)))) \\ &= K(t)[L(t_0)\log_{k(t_0)}(k(t_0)^n(x-d)) + P(\log_{k(t_0)}(k(t_0)^n(x-d)))] + L(t) \end{aligned}$$

whence, as P is 1-periodic, we get

$$\begin{aligned} &L(t_0)n + L(t_0)\log_{k(t_0)}\left(k(t)(x-d)+\frac{l(t)-d(1-k(t))}{k(t_0)^n}\right) \\ &+P\left(\log_{k(t_0)}\left(k(t)(x-d)+\frac{l(t)-d(1-k(t))}{k(t_0)^n}\right)\right) \\ &= K(t)[L(t_0)n + L(t_0)\log_{k(t_0)}(x-d) + P(\log_{k(t_0)}(x-d))] + L(t). \end{aligned}$$

Thus, letting $n \to \infty$ and using the continuity of P, for every $x \in v(\mathbb{R})$ and $t \in I$, we obtain

$$\begin{aligned} &L(t_0)\log_{k(t_0)}(k(t)(x-d)) + P(\log_{k(t_0)}(k(t)(x-d))) \\ &-K(t)[L(t_0)\log_{k(t_0)}(x-d) + P(\log_{k(t_0)}(x-d))] - L(t) = \lim_{n\to\infty} L(t_0)(K(t)-1)n. \end{aligned}$$

Hence either K is identically 1 and

$$\begin{aligned} &L(t_0)\log_{k(t_0)}(k(t)(x-d)) + P(\log_{k(t_0)}(k(t)(x-d))) \\ &= L(t_0)\log_{k(t_0)}(x-d) + P(\log_{k(t_0)}(x-d)) + L(t) \quad \text{for } x \in v(\mathbb{R}), t \in I \end{aligned}$$

or K is not identically 1, $L(t_0) = 0$ and

$$P(\log_{k(t_0)}(k(t)(x-d))) = K(t)P(\log_{k(t_0)}(x-d)) + L(t) \quad \text{for } x \in v(\mathbb{R}), t \in I. \tag{50}$$

In the first case, making use of (49), we get

$$f(k(t)(x-d)+d) = f(x) + L(t) \quad \text{for } x \in v(\mathbb{R}),\ t \in I.$$

Therefore, arguing as in the case of (43), we obtain (45) and (46). Thus (14) and (16) hold.

Next, assume that K is not identically 1, $L(t_0) = 0$ and (50) holds. Then, in view of (49), we get

$$f(x) = P(\log_{k(t_0)}(x-d)) \quad \text{for } x \in v(\mathbb{R}), \tag{51}$$

so according to (50), we have

$$f(k(t)(x-d)+d) = K(t)f(x) + L(t) \quad \text{for } x \in v(\mathbb{R}),\ t \in I. \tag{52}$$

Hence

$$\tilde{f}(k(t)x) = K(t)\tilde{f}(x) + L(t) \ \text{ for } \ x \in v(\mathbb{R}) - d, \ t \in I, \tag{53}$$

where $\tilde{f} : (v(\mathbb{R}) - d) \to \mathbb{R}$ is given by (44). Since f is nonconstant, so if $\tilde{f}$. Thus, taking $x_1, x_2 \in v(\mathbb{R}) - d$ with $\tilde{f}(x_1) \neq \tilde{f}(x_2)$, from (53) we deduce that

$$K(t) = G(k(t)) \ \text{ for } \ t \in I, \tag{54}$$

where $G : k(I) \to \mathbb{R}$ is of the form

$$G(z) = \frac{\tilde{f}(zx_1) - \tilde{f}(zx_2)}{\tilde{f}(x_1) - \tilde{f}(x_2)} \ \text{ for } \ z \in k(I). \tag{55}$$

Consequently, making use of (53) and (54), we obtain that $L(t) = H(k(t))$ for $t \in I$, where $H : k(I) \to \mathbb{R}$ is given by

$$H(z) = \tilde{f}(zx_1) - G(z)\tilde{f}(x_1) \ \text{ for } \ z \in k(I). \tag{56}$$

Hence, in view of (53) and (54), for every $x \in v(\mathbb{R}) - d$ and $y \in k(I)$, we get

$$\tilde{f}(xy) = \tilde{f}(x)G(y) + H(y). \tag{57}$$

Thus, (57) holds for every $(x, y) \in (v(\mathbb{R}) - d) \times int\ k(I)$ and this set is a nonempty, open and connected subset of $(0, \infty)^2$. Furthermore, as K is not identically 1, taking into account (40), we obtain that K is nonconstant. Hence, in view of (54), G is nonconstant as well. Therefore, since $\tilde{f}$ is continuous and nonconstant, from Kuczma (1985) (p. 311) and Sobek (2010) (Corollary 3) we derive that there exist $a, s \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{R}$ such that

$$\tilde{f}(x) = ax^s + c \ \text{ for } \ x \in v(\mathbb{R}) - d.$$

Thus, making use of (44), we get

$$f(x) = a(x - d)^s + c \ \text{ for } \ x \in v(\mathbb{R}). \tag{58}$$

So, in view of (51), we obtain

$$\begin{aligned} ak(t_0)^s(x - d)^s + c &= f(k(t_0)(x - d) + d) = P(\log_{k(t_0)}(k(t_0)(x - d))) \\ &= P(\log_{k(t_0)}(x - d)) = f(x) = a(x - d)^s + c \ \text{ for } \ x \in v(\mathbb{R}). \end{aligned}$$

Hence $k(t_0) = 1$, which yields a contradiction.

Subcase 2.4. Suppose that $k(t_0) < 1$. Then, arguing as in the previous subcase, we obtain that k is nonconstant and (42) holds with some $d \in \mathbb{R} \setminus v(\mathbb{R})$, where $I_0 \subset I$ is a maximal open interval containing t_0 such that $k(t) < 1$ for $t \in I_0$. Therefore, assuming that $v(\mathbb{R}) \subset (d, \infty)$ and making use of (18) and (42), we get

$$\tilde{f}(xy) = \tilde{f}(x)G(y) + H(y) \ \text{ for } \ x \in v(\mathbb{R}) - d, \ y \in k(I_0),$$

where $\tilde{f}$, G and H are given by (44), (55) and (56), respectively (with I_0 instead of I). So, similarly as in the case of (57), we get (58) with some $a, s \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{R}$. If $s = 1$ then from (58) follows (5) with $b := c - ad$. If $s \neq 1$, then by inserting into (18) f of the form (58), we get

$$a(k(t)x + l(t) - d)^s + c = K(t)[a(x-d)^s + c] + L(t) \quad \text{for} \quad x \in v(\mathbb{R}),\ t \in I.$$

Hence, differentiating the last equality with respect to x, after straightforward computation, we obtain

$$l(t) - d(1-k(t)) = \left(\left(\frac{K(t)}{k(t)} \right)^{\frac{1}{s-1}} - k(t) \right) (x-d) \quad \text{for} \quad x \in v(\mathbb{R}), t \in I.$$

Thus, (46) holds, which implies (14). Furthermore, from (58) we derive (15) (with $b := a$).

Now, suppose that $k(t_0) > 1$. Setting in (18) $t = t_0$ and arguing as in the subcase 1.4, we obtain

$$f(x) = \begin{cases} (d-x)^s P_1(\log_{k(t_0)}(d-x)) + c & \text{for} \quad x \in v(\mathbb{R}) \cap (-\infty, d), \\ c & \text{for} \quad x \in v(\mathbb{R}) \cap \{d\}, \\ (x-d)^s P_2(\log_{k(t_0)}(x-d)) + c & \text{for} \quad x \in v(\mathbb{R}) \cap (d, \infty), \end{cases} \tag{59}$$

where $d := \frac{l(t_0)}{1-k(t_0)}$, $c := \frac{L(t_0)}{1-K(t_0)}$, $P_1, P_2 : \mathbb{R} \to \mathbb{R}$ are continuous 1-periodic functions, $s \in \mathbb{R} \setminus \{0\}$ whenever $d \notin v(\mathbb{R})$ and $s \in (0, \infty)$, otherwise.

Suppose that $v(\mathbb{R}) \cap (d, \infty) \neq \emptyset$. Let $x \in v(\mathbb{R}) \cap (d, \infty)$ and $t \in I$. Then, similarly as in the previous subcase, we get that $k(t_0)^n (x-d) + d \in v(\mathbb{R}) \cap (d, \infty)$ for $n \in \mathbb{N}$. Furthermore, as $k(t_0) > 1$, for sufficiently large $n \in \mathbb{N}$, we have

$$k(t)(k(t_0)^n(x-d) + d) + l(t) \in v(\mathbb{R}) \cap (d, \infty).$$

Thus, taking into account (18) and (59), for sufficiently large $n \in \mathbb{N}$, we obtain

$$[k(t)k(t_0)^n(x-d) + l(t) - d(1-k(t))]^s P_2(\log_{k(t_0)}(k(t)k(t_0)^n(x-d) + l(t) - d(1-k(t))))$$
$$= K(t)(k(t_0)^n(x-d))^s P_2(\log_{k(t_0)}(k(t_0)^n(x-d))) + c(K(t)-1) + L(t).$$

Hence, as P_2 is 1-periodic, we get

$$[k(t)k(t_0)^n(x-d) + l(t) - d(1-k(t))]^s P_2\left(\log_{k(t_0)} \left(k(t)(x-d) + \frac{l(t) - d(1-k(t))}{k(t_0)^n} \right) \right)$$
$$= K(t)(k(t_0)^n(x-d))^s P_2(\log_{k(t_0)}(x-d)) + c(K(t)-1) + L(t).$$

Dividing both sides of the last equality by $k(t_0)^{ns}$, letting $n \to \infty$ and using the continuity of P_2, we obtain

$$(k(t)(x-d))^s P_2(\log_{k(t_0)}(k(t)(x-d))) = K(t)(x-d)^s P_2(\log_{k(t_0)}(x-d)).$$

Hence, in view of (59), we get $f(k(t)(x-d))+d)-c=K(t)(f(x)-c)$. In this way we have proved that

$$f(k(t)(x-d))+d)-c=K(t)(f(x)-c) \text{ for } x\in v(\mathbb{R})\cap(d,\infty), t\in I. \quad (60)$$

Thus

$$\tilde{f}(k(t)x)=K(t)\tilde{f}(x) \text{ for } x\in(v(\mathbb{R})-d)\cap(0,\infty), t\in I, \quad (61)$$

where $\tilde{f}:(v(\mathbb{R})-d)\cap(0,\infty)\to\mathbb{R}$ is given by

$$\tilde{f}(x)=f(x+d)-c \text{ for } x\in(v(\mathbb{R})-d)\cap(0,\infty). \quad (62)$$

If $\tilde{f}(x_0)\neq 0$ for some $x_0\in(v(\mathbb{R})-d)\cap(0,\infty)$ then, taking a function $G: k(I)\to\mathbb{R}$ of the form $G(z)=\frac{\tilde{f}(zx_0)}{\tilde{f}(x_0)}$ for $z\in k(I)$, from (61) we deduce that $K(t)=G(k(t))$ for $t\in I$. Thus, $G(y)>0$ for $y\in k(I)$ and, in view of (61), for every $x\in(v(\mathbb{R})-d)$ and $y\in k(I)$, we get

$$\tilde{f}(xy)=\tilde{f}(x)G(y).$$

The last equality holds, in particular, for every $(x,y)\in(v(\mathbb{R})-d)\times int\ k(I)$ and, as k is nonconstant and continuous, this set is a nonempty, open and connected subset of $(0,\infty)^2$. Moreover $\tilde{f}(x_0y)=\tilde{f}(x_0)G(y)\neq 0$ for $y\in k(I)$ and $\tilde{f}$ is continuous. Thus, taking into account Kuczma (1985) (p. 311) and Sobek (2010) (Corollary 2), we conclude that there exist $\beta\in\mathbb{R}$ and $s\in\mathbb{R}\setminus\{0\}$ such that

$$\tilde{f}(x)=\beta x^s \text{ for } x\in(v(\mathbb{R})-d)\cap(0,\infty).$$

Hence, in view of (62), we have

$$f(x)=\beta(x-d)^s+c \text{ for } x\in v(\mathbb{R})\cap(d,\infty). \quad (63)$$

Note also that if $\tilde{f}(x)=0$ for $x\in(v(\mathbb{R})-d)\cap(0,\infty)$ then (63) holds with $\beta=0$.

In a similar way we obtain that if $v(\mathbb{R})\cap(-\infty,d)\neq\emptyset$ then

$$f(x)=\alpha(d-x)^{s_1}+c \text{ for } x\in v(\mathbb{R})\cap(-\infty,d), \quad (64)$$

with some $s_1\in\mathbb{R}\setminus\{0\}$ and $\alpha\in\mathbb{R}$.

Assume that $d\notin v(\mathbb{R})$. If $v(\mathbb{R})\subset(-\infty,d)$ then $\alpha\neq 0$, so arguing as in the case $k(t_0)<1$, in view of (64), we obtain (5) (with $a:=-\alpha$ and $b:=c+\alpha d$) if $s_1=1$; and (14)-(15) (with $a:=\alpha$ and $s:=s_1$) if $s_1\neq 1$. Similarly, if $v(\mathbb{R})\subset(d,\infty)$, then, taking into account (63), we get $\beta\neq 0$ and so we obtain (5) (with $a:=\beta$ and $b:=c-\beta d$) if $s=1$; and (14)-(15) (with $b:=\beta$) if $s\neq 1$.

Assume that $d\in v(\mathbb{R})$. Since f is nonconstant and continuous this implies that $|\alpha|+|\beta|>0$; $s_1>0$ whenever $\alpha\neq 0$, $s>0$ whenever $\beta\neq 0$ and

$$f(d)=c. \quad (65)$$

If $s = s_1 = 1$ and $\beta = -\alpha$ then (5) holds with $a := \beta$ and $b := c - \beta d$. Assume that $\{s, s_1\} \neq \{1\}$ or $\beta \neq -\alpha$. Then, in view of (63) and (64), we have

$$f'(x) \neq 0 \ \text{ for } \ x \in v(\mathbb{R}) \setminus \{d\}.$$

Moreover, $f'(d)$ either does not exist or it is equal to 0. On the other hand, from (18) it follows that, for every $x \in v(\mathbb{R})$ and $t \in I$, f is differentiable at x if and only if f is differentiable at $k(t)x + l(t)$; and $f'(x) = 0$ if and only if $f'(k(t)x + l(t)) = 0$. Therefore $k(t)d + l(t) = d$ for $t \in I$. Hence (46) holds, which implies (14).

Now, if $\alpha = 0$ or $\beta = 0$, then we have (15) (with $b := \beta$ in the first case and $a := \alpha$, $s := s_1$ in the second case). If $\alpha \neq 0$ and $\beta \neq 0$, then, making use of (18), (46), (63) and (64), we obtain

$$\beta(k(t)^s - K(t))(x-d)^s = L(t) - c(1 - K(t)) \ \text{ for } \ x \in v(\mathbb{R}) \cap (d, \infty), t \in I$$

and

$$\alpha(k(t)^{s_1} - K(t))(x-d)^{s_1} = L(t) - c(1 - K(t)) \ \text{ for } \ x \in v(\mathbb{R}) \cap (-\infty, d), t \in I.$$

Hence, $K(t) = k(t)^s = k(t)^{s_1}$ for $t \in I$. Since k is nonconstant, this implies that $s = s_1$. Therefore, (63)-(65) imply (15) with $a := \alpha$ and $b := \beta$.

References

ABBAS, A. E. (2007) Invariant utility functions and certain equivalent transformations. *Decision Analysis* **4**, 17–31.

ABBAS, A. E. (2010) Invariant multiattribute utility functions. *Theory and Decision* **68**, 69–99.

ABBAS, A. E., ACZÉL, J. and CHUDZIAK, J. (2009) Invariance of multiattribute utility functions under shift transformations. *Result. Math.* **54**, 1–13.

CHUDZIAK, J. (2010) On a class of multiattribute utility functions invariant under shift transformations. *Acta Phys. Polon. A.* **117**, 673–675.

KUCZMA, M. (1985) *An Introduction to the Theory of Functional Equations and Inequalities.* Państwowe Wydawnictwo Naukowe, Uniwersytet Śląski, Warszawa-Kraków-Katowice.

PFANZAGL, J. (1959) A general theory of measurement. Applications to utility, *Naval Res. Logist. Quart.* **6**, 283–294.

SOBEK, B. (2010) Pexider equation on a restricted domain. *Demonstratio Math.* **43**, 81–88.