Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XX (2015)
39–47
DOI http://dx.doi.org/10.16926/m.2015.20.05

SOME DUAL LOGIC WITHOUT TAUTOLOGIES

ANETTA GÓRNICKA, ARKADIUSZ BRYLL

Abstract

On this paper we consider a logic dual to the logic C_{R_A} and prove that it does not contain tautologies.

1. Introduction

We consider a logic with the strongly adequate dual matrix and the empty set of tautologies. In a logic without tautologies there are no axioms, thus all theorems are proved on the base of some premisses with the use of fixed inference rules. In our approach a logic will be identified with a structural consequence operation [1].

Let S be a set of formulas of a propositional language.

A function C mapping 2^S into 2^S is a consequence if it fulfills the following conditions:

(1) $$X \subseteq C(X),$$

(2) $$X \subseteq Y \Rightarrow C(X) \subseteq C(Y),$$

(3) $$C(C(X)) \subseteq C(X),$$

for $X, Y \in 2^S$. Then,

(4) $$C(X \cup C(Y)) = C(X \cup Y).$$

The consequence dual to a consequence C, denoted by dC, is defined as follows [11]:

Definition 1.
$\alpha \in dC(X) \Leftrightarrow \exists_Y (Y \subseteq X \wedge card(Y) < \aleph_0 \wedge \bigcap\{C(\{\beta\})\colon \beta \in Y \subseteq C(\{\alpha\})\}$,
for any $\alpha \in S$ and any $X \subseteq S$.

[1]Terms of a logic and a consequence will be used interchangeably

The dual consequence dC has, among others, the following properties [2]:

Lemma 1.

For $\alpha, \beta \in S$:

a. $\beta \in dC(\{\alpha\}) \Leftrightarrow \alpha \in C(\{\beta\})$,

b. $dC(\emptyset) = \{\gamma \colon C(\{\gamma\}) = S\}$.

Therefore,

Corollary 1. $\gamma \notin dC(\emptyset) \Leftrightarrow C(\{\gamma\}) \neq S$.

With every propositional language $\mathrm{J} = (S, \mathbb{F})$, where $\mathbb{F}$ is a set of logical operators, we can associate an algebra $\mathrm{A} = (U, \mathbf{f})$ similar to J. By distinguishing in A a subset V ($\emptyset \neq V \subset U$), which we call the set of distinguished values, we obtain a logical matrix corresponding to the language J:

$$\mathfrak{M} = (U, V, \mathbf{f}).$$

The dual matrix to the matrix $\mathfrak{M}$ is defined by

$$\mathfrak{M}^d = (U, U - V, \mathbf{f}).$$

Matrices $\mathfrak{M}$ i $\mathfrak{M}^d$ differ only with respect to the set of distinguished values.

A consequence can be given by means of a set of rules $\mathcal{R}$ (rule consequence $C_{\mathcal{R}}$) or by means of a logical matrix $\mathfrak{M}$ (matrix consequence $C_{\mathfrak{M}}$). Here are the definitions:

Definition 2. $\alpha \in C_{\mathcal{R}}(X) \Leftrightarrow$ *(there exists a proof of α based on X and $\mathcal{R}$), $(X \cup \{\alpha\} \subseteq S)$.*

Definition 3. $\alpha \in C_{\mathfrak{M}}(X) \Leftrightarrow \forall_{h \in Hom}[h(X) \subseteq V \Rightarrow h(\alpha) \in V]$,
where Hom denotes the set of all homomorphisms of J *into* A.

Both $C_{\mathcal{R}}$ and $C_{\mathfrak{M}}$ fulfill the conditions of a consequence. Moreover, $C_{\mathcal{R}}$ is a finitistic consequence.

Let $E(\mathfrak{M})$ denote the content of a matrix $\mathfrak{M}$, i.e. the set of all its tautologies:

$$E(\mathfrak{M}) = \{\alpha \in S \colon \forall_{h \in Hom}(h(\alpha) \in V)\}. \tag{5}$$

By the definition

$$E(\mathfrak{M}) = C_{\mathfrak{M}}(\emptyset). \tag{6}$$

If $C_{\mathfrak{M}}(\emptyset) = \emptyset$, then $\mathfrak{M}$ does not contain any tautologies.
A logical matrix $\mathfrak{M}$ is said to be strongly adequate for a logic C if $C(X) = C_{\mathfrak{M}}(X)$ for every $X \subseteq S$.

Let T be a binary functor on S. We generalize T in the following way:

Definition 4.

a. $\mathrm{T}(\alpha) = \alpha$,
b. $\mathrm{T}(\alpha, \beta) = \mathrm{T}\alpha\beta$,
c. $\mathrm{T}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = \mathrm{T}(\mathrm{T}(\alpha_1, \ldots, \alpha_n), \alpha_{n+1})$.

It is easy to prove by means of induction the following facts.

Lemma 2.

a. *If a consequence C has, with regard to a functor* T*, the property*
$$C(\{\mathrm{T}\alpha\beta\}) = C(\{\alpha, \beta\}),$$
then
$$C(\{\mathrm{T}(\alpha_1, \ldots, \alpha_n)\}) = C\{\alpha_1, \ldots, \alpha_n\}).$$
b. *If a consequence C has, with regard to a functor* T*, the property*
$$C(\{\mathrm{T}\alpha\beta\}) = C(\{\alpha\}) \cap C(\{\beta\}),$$
then
$$C(\{\mathrm{T}(\alpha_1, \ldots, \alpha_n)\}) = \bigcap_{i=1}^{n} C(\{\alpha_i\}).$$

In our further considerations we apply the well known Lindenbaum Theorem [10]:

Theorem 1. *For every $\alpha \in S$ and $X \subseteq S$*
$$\alpha \notin C(X) \Rightarrow \exists_{Y \subseteq S}(C(Y) = Y \wedge \alpha \notin Y \wedge \forall_{\beta \notin Y}(\alpha \in C(Y \cup \{\beta\}))).$$

The set Y fulfilling the above condition is called a relatively maximal supersystem of X with regard to α. For every $\alpha \notin C(X)$ there can exist many different relatively maximal supersystems of X. The set of all such supersystems shall be denoted by L_X^α.

2. Main results

In [7] the logic C_{R_A} with the functor A of the alternative is considered. This logic is based on the set $R_A = \{r_1, r_2, r_3, r_4\}$ of inference rules, where
$$r_1 : \frac{\alpha}{A\alpha\beta}, \quad r_2 : \frac{A\alpha\alpha}{\alpha}, \quad r_3 : \frac{A\alpha\beta}{A\beta\alpha}, \quad r_4 : \frac{AA\alpha\beta\gamma}{A\alpha A\beta\gamma}.$$
The matrix
$$\mathfrak{M}_a = (\{0, 1\}, \{1\}, \{a\}),$$

where $a(x,y) = max(x,y)$, $x,y \in \{0,1\}$, is strongly adequate for the logic C_{R_A}.

It is proven in [7] that

Theorem 2. *$C_{R_A}(X) = C_{\mathfrak{M}_a}(X)$ for every $X \subseteq S$.*

The logic C_{R_A} does not contain tautologies since

$$C_{R_A}(\emptyset) = C_{\mathfrak{M}_a}(\emptyset) = E(\mathfrak{M}_a) = \emptyset \tag{7}$$

The equality $E(\mathfrak{M}_a) = \emptyset$ results from the fact that for any formula α the homomorphism h_0 assigning to all propositional variables the value 0 fulfills the condition $h(\alpha) = 0$. Therefore, it is not true that $\forall_{h \in Hom} h(\alpha) = 1$.

In [7] it is also proven that

Lemma 3. *For every $X \subseteq S$ and all $\alpha, \beta \in S$*

$$C_{R_A}(X \cup \{\alpha\}) \cap C_{R_A}(X \cup \{\beta\}) \subseteq C_{R_A}(X \cup \{A\alpha\beta\}).$$

Applying the rules r_1 i r_3 one can show that

Lemma 4. *For every $X \subseteq S$ and all $\alpha, \beta \in S$*

$$C_{R_A}(X \cup \{A\alpha\beta\}) \subseteq C_{R_A}(X \cup \{\alpha\}) \cap C_{R_A}(X \cup \{\beta\}).$$

Proof. By means of r_1 we obtain $A\alpha\beta \in C_{R_A}(\alpha)$ and $A\beta\alpha \in C_{R_A}(\beta)$. Applying r_3 we get $A\alpha\beta \in C_{R_A}(\{A\beta\alpha\})$. By monotonicity of C_{R_A} we have $C_{R_A}(\{A\alpha\beta\}) \subseteq C_{R_A}(\{\beta\})$. Then $A\alpha\beta \in C_{R_A}(\{\beta\}$. By the property (4) of a consequence, $C_{R_A}(X \cup \{A\alpha\beta\}) \subseteq C_{R_A}(X \cup C_{R_A}(\{\alpha\})) = C_{R_A}(X \cup \{\alpha\})$. Similarly $C_{R_A}(X \cup \{A\alpha\beta\}) \subseteq C_{R_A}(X \cup C_{R_A}(\{\beta\})) = C_{R_A}(X \cup \{\beta\})$. Thus $C_{R_A}(X \cup \{A\alpha\beta\}) \subseteq C_{R_A}(X \cup \{\alpha\}) \cap C_{R_A}(X \cup \{\beta\})$. □

From the above Lemmas we conclude that

Theorem 3. *For every $X \subseteq S$ and all $\alpha, \beta \in S$*

$$C_{R_A}(X \cup \{A\alpha\beta\}) = C_{R_A}(X \cup \{\alpha\}) \cap C_{R_A}(X \cup \{\beta\}).$$

Therefore,

Corollary 2. *For all $\alpha, \beta \in S$*

$$C_{R_A}(\{A\alpha\beta\}) = C_{R_A}(\{\alpha\}) \cap C_{R_A}(\{\beta\}).$$

Let us consider the consequence $C_{R_A^d}$ based on the following set of inference rules:

$$R_{A^d} = \{r_1^d, r_2^d\}, \quad \text{where } r_1^d \colon \frac{A\alpha\beta}{\alpha, \beta}, r_2^d \colon \frac{\alpha, \beta}{A\alpha\beta}.$$

The above rules express the classical property of the alternative: the alternative is false if and only if both its components are false. It is clear that the consequence $C_{R_A^d}$ has the following property:

Lemma 5. *For every $X \subseteq S$ and for all $\alpha, \beta \in S$*

$$C_{R_A^d}(X \cup \{A\alpha\beta\}) = C_{R_A^d}(X \cup \{\alpha, \beta\}).$$

Therefore,

Corollary 3. *For all $\alpha, \beta \in S$*

$$C_{R_A^d}(\{A\alpha\beta\}) = C_{R_A^d}(\{\alpha, \beta\}).$$

The matrix consequence $C_{\mathfrak{M}_a^d}$ with a matrice $\mathfrak{M}_a^d = (\{0,1\}, \{0\}, \{a\}\}$ dual with respect to the matrix $\mathfrak{M}_a$ is defined as follows:

Definition 5. *For every $\alpha \in S$*

$$\alpha \in C_{\mathfrak{M}_a^d}(X) \Leftrightarrow \forall_{h \in Hom}(h(X) \subseteq \{0\} \Rightarrow h(\alpha) = 0),$$

where Hom is the set of all homomorphisms, i.e., functions $h\colon S \longrightarrow \{0,1\}$ such that

$$h(A\alpha\beta) = a(h(\alpha, \beta)).$$

The set of tautologies of the matrix $\mathfrak{M}_a^d$ is empty, i.e.

$$E(\mathfrak{M}_a^d) = C_{\mathfrak{M}_a^d}(\emptyset) = \emptyset. \tag{8}$$

It results from the fact that for the homomorphism h assigning to all propositional variables the value 1 we have $h(\alpha) = 1$ for any α.

Let us notice that the set $C_{\mathfrak{M}_a^d}(X)$ is closed with regard to the rules from R_A^d, i.e.

$$C_{R_A^d}(C_{\mathfrak{M}_a^d}(X)) \subseteq C_{\mathfrak{M}_a^d}(X), \quad X \subseteq S. \tag{9}$$

We show that the matrix $\mathfrak{M}_a^d$ is strongly adequate for the logic $C_{R_A^d}$:

Theorem 4. $C_{R_A^d}(X) = C_{\mathfrak{M}_a^d}(X)$, *for every $X \subseteq S$.*

Proof. The inclusion $C_{R_A^d}(X) \subseteq C_{\mathfrak{M}_a^d}(X)$ results from (9) since $C_{R_A^d}(X) \subseteq C_{R_A^d}(C_{\mathfrak{M}_a^d}(X)) \subseteq C_{\mathfrak{M}_a^d}(X)$.
To prove $C_{\mathfrak{M}_a^d}(X) \subseteq C_{R_A^d}(X)$ let us assume that $\alpha \notin C_{R_A^d}(X)$. By Theorem 1, there exists a set $Y_0 \in L_X^\alpha$ such that

$$X \subseteq Y_0, \tag{10}$$

$$C_{R_A^d}(Y_0) = Y_0, \tag{11}$$

$$\alpha \notin Y_0, \tag{12}$$

$$\forall_{\beta \notin Y_0}(\alpha \in C_{R_A^d}(Y_0 \cup \{\beta\})). \tag{13}$$

According to Lemma 5 and property (11), we have

$$A\beta\gamma \in Y_0 \Leftrightarrow (\beta \in Y_0 \wedge \gamma \in Y_0). \tag{14}$$

Indeed, for any $\beta, \gamma \in S$ we get:
$A\beta\gamma \in Y_0 \Rightarrow Y_0 \cup \{A\beta\gamma\} = Y_0 \Rightarrow C_{R_A^d}(Y_0 \cup \{A\beta\gamma\}) = C_{R_A^d}(Y_0) = Y_0 = C_{R_A^d}(Y_0 \cup \{\beta, \gamma\})$. As $\beta, \gamma \in C_{R_A^d}(Y_0 \cup \{\beta, \gamma\}) = Y_0$, so $\beta, \gamma \in Y_0$. Therefore, $\beta, \gamma \in Y_0 \Rightarrow Y_0 \cup \{\beta, \gamma\} = Y_0 \Rightarrow C_{R_A^d}(Y_0 \cup \{\beta, \gamma\}) = C_{R_A^d}(Y_0) = Y_0 = C_{R_A^d}(Y_0 \cup \{A\beta\gamma\})$. Since $A\beta\gamma \in C_{R_A^d}(Y_0 \cup \{A\beta\gamma\}) = Y_0$, then $A\beta\gamma \in Y_0$.
We can consider the following homomorphism $h_{Y_0} : S \longrightarrow \{0, 1\}$ based on the set Y_0:

$$h_{Y_0}(\alpha) = \begin{cases} 0, & \text{gdy } \alpha \in Y_0 \\ 1, & \text{gdy } \alpha \notin Y_0. \end{cases} \tag{15}$$

We show that h_{Y_0} is a homomorphism. By (14) i (15) we get:
$h_{Y_0}(A\beta\gamma) = 0 \Leftrightarrow A\beta\gamma \in Y_0 \Leftrightarrow \beta, \gamma \in Y_0 \Leftrightarrow h_{Y_0}(\beta) = 0 \wedge h_{Y_0}(\gamma) = 0 \Leftrightarrow a(h_{Y_0}(\beta), h_{Y_0}(\gamma)) = 0$;
$h_{Y_0}(A\beta\gamma) = 1 \Leftrightarrow A\beta\gamma \notin Y_0 \Leftrightarrow \beta \notin Y_0 \vee \gamma \notin Y_0 \Leftrightarrow h_{Y_0}(\beta) = 1 \vee h_{Y_0}(\gamma) = 1 \Leftrightarrow a(h_{Y_0}(\beta), h_{Y_0}(\gamma)) = 1$.
Thus,

$$h_{Y_0}(A\beta\gamma) = a(h_{Y_0}(\beta), h_{Y_0}(\gamma)). \tag{16}$$

According to (10), $h_{Y_0}(X) \subseteq h_{Y_0}(Y_0)$ for any $X \subseteq S$. As $h_{Y_0}(Y_0) = \{h_{Y_0}(\delta) : \delta \in Y_0\} = \{0\}$, then $h_{Y_0}(X) \subseteq \{0\}$.
By (12) we have $h_{Y_0}(\alpha) = 1$. Then, $\exists_{h \in Hom}(h(X) \subseteq \{0\} \wedge h(\alpha) = 1)$ and, by Definition 5, we obtain $\alpha \notin C_{\mathfrak{M}_a^d}(X)$. Therefore, $C_{\mathfrak{M}_a^d}(X) \subseteq C_{R_A^d}(X)$ and having $C_{R_A^d}(X) \subseteq C_{\mathfrak{M}_a^d}(X)$ we get $C_{R_A^d} = C_{\mathfrak{M}_a^d}$. □

3. Final remarks

We show that the logic $C_{R_A^d}$ is dual to the logic C_{R_A} (in the sense of Definition 1).
First, we prove by means of induction (with respect on the complexity of formulas) that

Lemma 6. $dC_{R_A}(\emptyset) = \emptyset$.

Proof. If α is a variable, then $C_{R_A}(\{\alpha\}) \neq S$, because we cannot get (by means of the rules from R_A) any formula from S starting from a single propositional variable.
Assume inductively that $C_{R_A}(\{\alpha_1\}) \neq S$ i $C_{R_A}(\{\alpha_2\}) \neq S$ holds for formulas α_1, α_2. Then, by Corollary 2, regarding the formula $A\alpha_1\alpha_2$ we get

$C_{R_A}(\{A\alpha_1\alpha_2\}) = C_{R_A}(\{\alpha_1\}) \cap C_{R_A}(\{\alpha_2\}) \neq S$. Then, according to Corollary 1, we have $C_{R_A}(\{\gamma\}) \neq S$ for any $\gamma \in S$ and then $\gamma \notin dC_{R_A^d}(\emptyset)$, so $dC_{R_A}(\emptyset) = \emptyset$. □

From Theorem 4 and property (8), we have $C_{R_A^d}(\emptyset) = \emptyset$, then, by Lemma 6:

Lemma 7. $dC_{R_A}(\emptyset) = C_{R_A^d}(\emptyset)$.

Now, we prove

Lemma 8. *If $\alpha \in C_{R_A^d}(X)$, then $\alpha \in dC_{R_A}(X)$ for any $X \subseteq S$.*

Proof. Let $\alpha \in C_{R_A^d}(X)$. Since $C_{R_A^d}$ is a finitary consequence, then there exists a finite subset Y_0 of the set$X$ such that $\alpha \in C_{R_A^d}(Y_0)$. Let us notice that $Y_1 \neq \emptyset$. Indeed, if $Y_1 = \emptyset$, then $\alpha \in C_{R_A^d}(\emptyset)$ and as $C_{R_A^d}(\emptyset) = \emptyset$, we get $\alpha \in \emptyset$, which leads to a contradiction. Thus, let us assume $Y_0 = \{\beta_1, \ldots, \beta_n\}$. By Corollary 3 and Lemma 2a., we obtain

$$\alpha \in C_{R_A^d}(\{\beta_1, \ldots, \beta_n\}) = C_{R_A^d}(\{A(\beta_1, \ldots, \beta_n)\}).$$

Then, by Theorem 4, $\alpha \in C_{\mathfrak{M}_a^d}(\{A(\beta_1, \ldots, \beta_n)\})$, so

$$\forall_{h \in Hom}(h(A(\beta_1, \ldots, \beta_n)) = 0 \Rightarrow h(\alpha) = 0).$$

We get $\forall_{h \in Hom}(h(\alpha) = 1 \Rightarrow h(A(\beta_1, \ldots, \beta_n) = 1)$, then $A(\beta_1, \ldots, \beta_n) \in C_{\mathfrak{M}_a}(\{\alpha\}) = C_{R_A}(\{\alpha\})$, hence, by Corollary 2 and Lemma 2b we conclude that

$$\bigcap\{C_{R_A}(\{\beta\}) : \beta \in Y_0\} \subseteq C_{R_A}(\{\alpha\}).$$

Therefore,

$$\exists_Y (Y \subseteq X \wedge card(Y) < \aleph_0 \wedge \bigcap\{C_{R_A}(\{\beta\})\colon \beta \in Y \subseteq C_{R_A}(\{\alpha\})).$$

According to Definition 1, we get $\alpha \in dC_{R_A}(X)$.

□

Lemma 9. *If $X \neq \emptyset$, then $dC_{R_A}(X) \subseteq C_{R_A^d}(X))$ for every $X \subseteq S$.*

Proof. Let $X \neq \emptyset$ and let us suppose $\alpha \in dC_{R_A}(X)$. By Definition 1, there exists a set Y_0 such that

$$Y \subseteq X \wedge card(Y) < \aleph_0 \wedge \bigcap\{C_{R_A}(\{\beta\})\colon \beta \in Y \subseteq C_{R_A}(\{\alpha\})\}.$$

Let us consider two cases: $Y_0 = \emptyset$ or $Y_0 \neq \emptyset$.

Let $Y_0 = \emptyset$, then $\bigcap\{C_{R_A}(\{\beta\})\colon \beta \in \emptyset\} = S$. Therefore, $C_{R_A}(\alpha) = S$ and $\forall_{\gamma \in S}(\gamma \in C_{R_A}(\{\alpha\})$. According to our assumption $X \neq \emptyset$, there is $\gamma_1 \in X$, hence $\gamma_1 \in S$. Therefore $\gamma_1 \in C_{R_A}(\{\alpha\}) = C_{\mathfrak{M}_a}(\{\alpha\})$. By Definition of the matrix consequence we have $\forall_{h \in Hom}(h(\alpha) = 1 \Rightarrow h(\gamma_1) = 1)$, so

$\forall_{h\in Hom}(h(\gamma_1) = 0 \Rightarrow h(\alpha) = 0)$, hence $\alpha \in C_{\mathfrak{M}_a^d}(\{\gamma_1\}) = C_{R_A^d}(\{\gamma_1\})$. However, $C_{R_A^d}(\{\gamma_1\}) \subseteq C_{R_A^d}(X)$, then $\alpha \in C_{R_A^d}(X)$.

If $Y_0 = \{\beta_1, \ldots, \beta_n\}$, then, by Corollary 2 and Lemma 2b, we obtain that $\bigcap\{C_{R_A}(\{\beta\})\colon \beta \in Y_1\} = C_{R_A}(\{A(\beta_1, \ldots, \beta_n)\}$, so $C_{R_A}(\{A(\beta_1, \ldots, \beta_n)\}) \subseteq C_{R_A}(\{\alpha\})$, and hence $A(\beta_1, \ldots, \beta_n) \in C_{R_A}(\{\alpha\}) = C_{\mathfrak{M}_a}(\{\alpha\})$. Therefore, $\forall_{h\in Hom}(h(\alpha) = 1 \Rightarrow h(A(\beta_1, \ldots, \beta_n) = 1)$, so $\forall_{h\in Hom}(h(A(\beta_1, \ldots, \beta_n) = 0 \Rightarrow h(\alpha) = 0)$. Then, $\alpha \in C_{\mathfrak{M}_a^d}(\{A(\beta_1, \ldots, \beta_n)\}) = C_{R_A^d}(\{A(\beta_1, \ldots, \beta_n)\})$, hence, according to Corollary 3, $\alpha \in C_{R_A^d}(Y_0)$. Since $Y_0 \subseteq X$, we get $\alpha \in C_{R_A^d}(Y_0) \subseteq C_{R_A^d}(X)$, hence $\alpha \in C_{R_A^d}(X)$.

Then, we have proved that in both cases $dC_{R_A}(X) \subseteq C_{R_A^d}(X)$ for every $X \neq \emptyset$. □

According to Lemmas 8, 9 i 10 we obtain

Theorem 5.

$$C_{R_A^d} = dC_{R_A}.$$

It means that the logic $C_{R_A^d}$ is dual with respect to the logic C_{R_A}. It does not contain tautologies, neither. According to Theorem 4 we can conclude that the logic $C_{R_A^d}$ is de facto a conjuctional logic expressed by means of the operator A. To notice this fact it is enough to look closely at the rules r_1^d and r_2^d from R_a^d.

References

[1] G. Asser *Einfürung in die mathematische Logik*, Teil I. Leipzig, 1959.

[2] G. Bryll *Metody odrzucania wyrażeń*, Akademicka Oficyna Wydawnicza PLJ, Warszawa, 1996.

[3] G. Bryll, A. Chotomska, L. Jaworski, *A formalization of logik without tautologies*, Prace Naukowe WSP w Częstochowie, Matematyka VIII (2000–2001), 9–12.

[4] A. Górnicka, *Axiomatization of the sentential logic dual to Sobociński's n — valued logic*, Bulletin of the Section of Logic, vol. 40:1/2 (2011), pp. 47–54.

[5] A. Górnicka, *Badania nad pewnymi logikami dualnymi*, Rozprawa doktorska (maszynopis), Wrocław, 2001.

[6] A. Górnicka, *Dual consequence operations associated with a certain class of logical matrices*, Bulletin of the Section of Logic, vol.29/4, (2000), 1–8.

[7] L. Jaworski, *Z badań nad logikami bez tautologii*, Rozprawa doktorska (maszynopis), Opole, 2005–2007.

[8] J. Łoś, *An algebraic proof of completeness for the two - valued propositional calculus*, Colloquium Mathematicum, no.2, (1951), 263-240.

[9] G. Malinowski, M. Spasowski, *Dual counterparts of Łukasiewicz's sentential calculi*, Studia Logica, vol.33, no.2 (1974), 153–162.

[10] W.A. Pogorzelski, *The classical propositional calculus* (in Polish), PWN Warszawa, 1978.

[11] R. Wójcicki, *Dual counterparts of consequence operations*, Bulletin of the Section of Logic, vol.2, no.1 (1973), 201–214.

Received: August 2015

Anetta Górnicka
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
42-200 Częstochowa, Al. Armii Krajowej 13/15, Poland
E-mail address: a.gornicka@ajd.czest.pl

Arkadiusz Bryll
Częstochowa University of Technology,
The Faculty of Menagement,
42-201 Częstochowa, ul. Dąbrowskiego 69, room. nr 2,Poland
E-mail address: tex@op.pl