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Shock models under policy N

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Abstract

We present the life distribution of a device subject to shocks governed by phase-type distributions. The probability of failures after shock follows discrete phase-type distribution. Lifetimes between shocks are affected by the number of cumulated shocks and they follow continuous phase-type distributions. The device can support a maximum of N shocks. We calculate the distribution of the lifetime of the device and illustrate the calculations by means of a numerical application. Computational aspects are introduced. This model extends other previously considered in the literature.

1. Introduction

The classical shocks model of Esary et al. [2] study the lifetime of a device subject to shocks that arrive randomly following a Poisson process $\{N(t),t\geq 0\}$. The device has a probability $\bar{P}_k$ of survival to k shocks. Then, the survival function of the model H(t) is given by:

$$\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t)=k\}\bar{P}_k \qquad (1)$$

The authors study this model under a non-parametric methodology, considering reliability classes. Neuts et al. [5], introduced phase-type distributions and calculated explicitly the lifetime distribution of the device. Manoharan et al. [3] considered a finite mixture of homogeneous Poisson process as arrival process. In these previous papers, the number of shocks that arrive to the device is unlimited. In [6] PH-distributions are used to study a model submitted to a limited number of failures.

We present a model limiting the number of shocks that the device can support. The probability of failure due to the shocks follows a discrete phase-type distribution. The interarrival times between shocks depend on the number of cumulated shocks.

The process that governs the system is a Markov one with vectorial state space. We calculate the lifetime distribution of the device and present a numerical example illustrating the calculations.

In Section 2 the shock model is presented. In Section 3 the Markov model that governs the system is constructed, and the lifetime distribution of the device determined. In Section 4 a numerical application is performed.

Given that the phase-type distributions play a fundamental role throughout the paper, we define them in the discrete and continuous cases. More details about these distributions can be seen in Neuts [4].

2. Definitions

Definition 1 The distribution H(·) on [o,∞[is a phase-type distribution (PH-distribution) with representation (α,T) if it is the distribution of the time untill the absorption in a Markov process on the states {1,...,m,m+1} generator

$$\begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}$$

and initial row probability vector α of order *m*. We assume that the states {1,...,m} are all transient. Throughout this paper *e* denotes a column vector with all components equal to one the dimension of which is determined by the context. The matrix T of order *m* is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $-Te=T^0$. The distribution of H(·) is given by

$$H(x)=1-\alpha\exp(Tx)e,\ x\geq 0$$

It will be denoted that H(·) follows PH(α,T) distribution.

Definition 2 A density $\{p_k\}$ of the nonnegative integers is of phase type if and only if there exists a finite Markov chain with transition probability matrix P of order n+1 of the form

$$\begin{pmatrix} S & S^0 \\ 0 & 1 \end{pmatrix}$$

and initial probability vector (β,β_{n+1}), where β is a row n-vector. Here S is a subestochastic matrix such that $Se+S^0=e$, and (I-S) is non-singular. The density of the time until absorption is given by

$$p_0=\beta_{n+1},$$
$$p_k=\beta S^{k-1}S^0,\ \text{for k} \geq 1$$

It will be denoted that $\{p_k\}$ follows a $PH_d(\beta,S)$.
We use the Kronecker product (see [1]).

3. Shock model

Suppose that a device is subjected to shocks according to the following assumptions.

1. Let $X^{(k)}$ be the interarrival times between the shocks kth and (k+1)th, k=0,1,... These random times follow distributions $PH(\beta^{(k)},S^{(k)})$ of order n_k.
2. The device can accumulate a maximum of N shocks, in such a way that it is replaced by a new one to the arrival of the N+1 shock. We denote by p_k the probability of failure of the device due to the arrival of the kth shock, k≥1. We assume that p_k follows a distribution $PH_d(\gamma,L)$ of order N+1.

This representation is given by γ=(1,0,…,0) and

$$L=\begin{pmatrix} 0 & l_1 & 0 & \dots & 0 \\ 0 & 0 & l_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & l_N \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix};L^0=\begin{pmatrix} 1-l_1 \\ 1-l_2 \\ \vdots \\ 1-l_N \\ 1 \end{pmatrix}$$

The entries l_k, k=1,...,N, denote the conditional probability that the system will survive to the kth shock, given that it has survived to the (k-1)th shock. These l_k are useful to find a representation to the distribution $\{p_k\}$. It is clear that

$$p_k=\gamma L^{k-1}L^0, k\geq 1$$

The survival probability to the kth shock is

$$\bar{P}_k=\sum_{k=\nu+1}^{N+1} p_\nu=\gamma\left(L^k-L^{N+1}\right)e,$$
$$k=1,\dots,N;\bar{P}_0=1 \qquad (2)$$

Under the assumptions the survival function (1) follows a PH-distribution that will be calculated

4. Markov model

Under these assumptions, the probabilistic model that governs the system is a Markov process. The exponential occupied states by the device will be denoted by (k,i), k being the number of cumulated shocks, i the phase of the random variable $X^{(k)}$. We group these states in sets, named macro-states, that will be denoted by k, k=0,1,...,N. The number of exponential states of the macro-state k is n_{k+1}. The infinitesimal generator of the Markov process is built in terms of the transition between macro-states, and, consequently, it will be a generator formed by blocks.

We denote by T_k the lifetime of the device when the failure occurs at the arrival of the (k+1)th shock. It is clear that

$$T_k=\sum_{i=0}^{k} X^{(i)}, 1\leq k\leq N+1 \qquad (3)$$

This random variable is the sum of independent random variables PH-distributed and follows a distribution $PH(g^{(k)},G^{(k)})$. We calculate this representation.
If the device fails at the (k+1)th shock, it has survived to the first k shocks. Thus, the transitions between the up macro-states to the occurrence of the failure are 0→1→2→· · · →k. These transitions j→j+1, j=

0,1,...,k, occur when a non-fatal shock arrives being the device in the macro-state j, and these are governed by the absorption vector $S^{0(j)}$. Then, the new interarrival period initiates following the initial vector $\beta^{(j)}$. Thus, the matrix $G^{(k)}$ is

$$G^{(k)} = \begin{pmatrix} S^{(0)} & S^{0(0)}\beta^{(1)} & & \\ & S^{(1)} & S^{0(1)}\beta^{(2)} & \\ & & \ddots & \\ & & & S^{(k)} \end{pmatrix}$$

We assume that the device initiates with 0 shocks, so the initial vector $g^{(k)}$ is given by

$$g^{(k)} = \left(\beta^{(0)}, 0, \ldots, 0\right), k = 0,1,\ldots,N.$$

Denoting by T the lifetime of the system, we have

$$R(t) = P(T > t) = \sum_{k=0}^{N} p_{k+1} P(T_k > t), t \geq 0 \qquad (4)$$

We determine the distribution of the random variable T as follows. This is the distribution of a finite mixture of PH-distributions, it is a PH-distribution with well-known representation (see [4]) given by (v,V), with,

$$v = \begin{pmatrix} p_0 g^{(0)}, & p_1 g^{(1)}, & \ldots, & p_N g^{(N)} \end{pmatrix}$$

$$V = \begin{pmatrix} G^{(0)} & & & \\ & G^{(1)} & & \\ & & \ddots & \\ & & & G^{(N)} \end{pmatrix} \qquad (5)$$

The analytic expression of the survival function of T is:

$$P(T > t) = v \exp\{Vt\} e = \sum_{k=0}^{N} p_{k+1} g^{(k)} \exp\left(G^{(k)} t\right) e, t \geq 0 \qquad (6)$$

5. Numerical application

We illustrate by a numerical example the calculations throughout the paper. For the shocks, we assume that they arrive erratically, formalized by means of hyperexponential distributions as interarrival times.

We will assume that the device can undergo a maximum of three shocks, that is, it is replaced when the third shock arrives. The failure probabilities to the failures are given by $p_0 = 0$, $p_1 = 0.1$, $p_2 = 0.6$, $p_3 = 0.3$. It will be assumed that the interarrival times between shocks follow the PH-distributions:

- $X^{(0)} \to PH\left(\beta^{(0)}, S^{(0)}\right)$ with

$$\beta^{(0)} = \left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right), S^{(0)} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -2 \end{pmatrix}$$

- $X^{(1)} \to PH\left(\beta^{(1)}, S^{(1)}\right)$ with

$$\beta^{(1)} = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right), S^{(1)} = \begin{pmatrix} -3 & & \\ & -3 & \\ & & -3 \end{pmatrix}$$

- $X^{(2)} \to PH\left(\beta^{(2)}, S^{(2)}\right)$ with

$$\beta^{(2)} = \left(\frac{3}{10}, \frac{7}{10}\right), S^{(2)} = \begin{pmatrix} -1 & \\ & -9 \end{pmatrix}$$

From these, we obtain the lifetime distributions for the device due to shocks, the random variables T_k:

- $T_1 = X^{(0)} \to PH\left(\beta^{(0)}, S^{(0)}\right)$,
- $T_2 = X^{(0)} + X^{(1)} \to PH\left(g^{(1)}, G^{(1)}\right)$ with

$g^{(1)} = \begin{pmatrix} \beta^{(0)}, & 0, & 0, & 0 \end{pmatrix}$ and

$$G^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0.4 & 0.2 & 0.4 \\ & -1 & 0 & 0.4 & 0.2 & 0.4 \\ & & -2 & 0.8 & 0.4 & 0.8 \\ & & & -3 & 0 & 0 \\ & & & & -3 & 0 \\ & & & & & -3 \end{pmatrix}$$

- $T_3 = X^{(0)} + X^{(1)} + X^{(2)} \rightarrow PH\left(g^{(2)}, G^{(2)}\right)$ with $g^{(2)} = \left(\beta^{(0)}, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0,\right)$ and

$$G^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ & -1 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ & & -2 & 0.8 & 0.4 & 0.8 & 0 & 0 \\ & & & -3 & 0 & 0 & 0.9 & 2.1 \\ & & & & -3 & 0 & 0.9 & 2.1 \\ & & & & & -3 & 0.9 & 2.1 \\ & & & & & & -1 & 0 \\ & & & & & & & -9 \end{pmatrix}$$

In *Figures 1* and *Figure 2* we plot the survival and the failure rate functions, respectively. The survival function decreases quickly. The failure rate function increases quickly, reach a maximum, and then, decreases slowly and becomes almost constant. So, there is a high risk of failure when it starts, and once a short interval of time has passed, it tends to be constant.

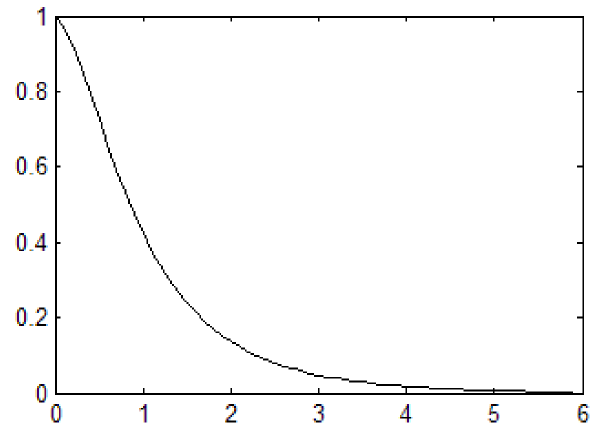

Figure 1. Survival function for the shock model

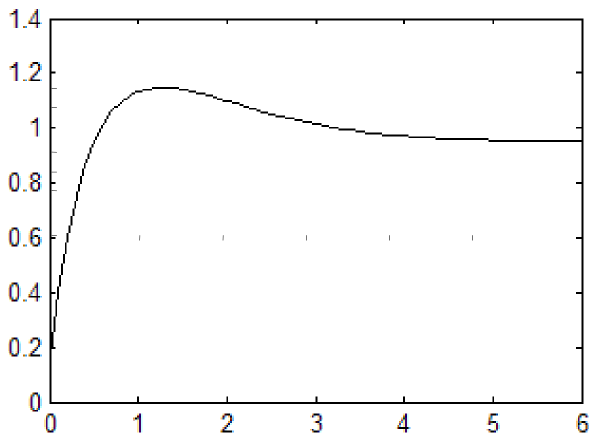

Figure 2. Failure rate function for the shock model

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